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MATCHING METHOD FOR NODAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS

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Dedicated to Professor Allan Peterson for his outstanding accomplishments in mathematics

ABSTRACT. In this paper, we study the nonlinear boundary value problem consisting of the equation $y'' + \int_a^b w(t,\tau)f(y,\tau)\,d\zeta(\tau) = 0$ on $[a,b]$ and a double Riemann-Stieltjes integral boundary condition. We establish the existence of various nodal solutions of this problem by matching the solutions of two boundary value problems, each of which involves one separated boundary condition and open Riemann-Stieltjes integral boundary condition, at some point in (a,b). We also obtain the conditions for nonexistence of nodal solutions of this boundary value problem.

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1. INTRODUCTION

We study the nonlinear boundary value problem (BVP) consisting of the equation

$$y'' + \int_a^b w(t,\tau)f(y,\tau)\,d\zeta(\tau) = 0, \quad t \in (a,b), \tag{1.1}$$

and the boundary condition (BC)

$$y(a) - \int_a^b y(s)\,d\eta(s) = 0, \quad y(b) - \int_a^b y(s)\,d\xi(s) = 0, \tag{1.2}$$

where $a, b \in \mathbb{R}$ with $a < b$, the integrals in Eq. (1.1) and BC (1.2) are Riemann-Stieltjes integrals with respect to $\zeta(\tau)$, $\eta(s)$, and $\xi(s)$, respectively, with $\zeta(\tau)$ being a nondecreasing function and $\eta(s)$ and $\xi(s)$ being functions of bounded variation. We comment that the intervals for integration in the equation and in the boundary

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conditions may be different; however, with the Riemann-Stieltjes integration, they can always be unified into the same interval.

Since the functions $\xi(s)$ and $\eta(s)$ in BC (1.2) are of bounded variation on $[a,b]$, then there are two nondecreasing functions $\xi_i(s)$ and $\eta_i(s)$, $i=1,2$, such that

$$\xi(s)=\xi_1(s)-\xi_2(s) \text{ and } \eta(s)=\eta_1(s)-\eta_2(s), \quad s\in[a,b], \tag{1.3}$$

Note in the case that $\zeta(\tau)=\tau$, $\eta(s)=s$, and $\xi(s)=s$, the Riemann-Stieltjes integrals in BVP (1.1), (1.2) reduce to the Riemann integrals. In the case that $\zeta(\tau)=\sum_{j=1}^{d}\chi(\tau-r_j)$, $\eta(s)=\sum_{j=1}^{l}h_j\chi(s-\eta_j)$, and $\xi(s)=\sum_{i=1}^{m}k_i\chi(s-\xi_i)$, where $d,l,m\geq 1$, and $\{r_j\}_{j=1}^{d}$, $\{\eta_j\}_{j=1}^{l}$, $\{\xi_i\}_{i=1}^{m}$ are strictly increasing sequences of distinct points in (a,b), and $\chi(s)$ is the characteristic function on $[0,\infty)$, i.e.,

$$\chi(s)=\begin{cases}1, & s\geq 0,\\ 0, & s<0.\end{cases}$$

BVP (1.1), (1.2) reduces to the BVP consisting of the equation

$$y''+\sum_{j=1}^{p}w_j(t)f_j(y)=0, \quad t\in(a,b), \tag{1.4}$$

and the boundary condition

$$y(a)-\sum_{j=1}^{l}h_jy(\eta_j)=0, \quad y(b)-\sum_{i=1}^{m}k_iy(\xi_i)=0. \tag{1.5}$$

where $w_j(t):=w(t,r_j)$ and $f_j(t):=f(y,r_j)$.

We assume throughout, and without further mention, that the following conditions hold:

(H1) $w(t,\tau)\in C^1([a,b]\times[a,b])$ and $w(t,\tau)>0$ on $[a,b]\times[a,b]$;

(H2) $f\in C(\mathbb{R}\times[a,b])$, $f(y,\tau)$ is locally Lipschitz in y on $(-\infty,0)\cup(0,\infty)$, and $yf(y,\tau)>0$ and $f(-y,\tau)=-f(y,\tau)$ for all $y>0$;

(H3) there exist extended measurable functions $f_0(\tau),f_\infty(\tau):[a,b]\to[0,\infty]$ such that

$$f_0(\tau)=\lim_{y\to 0}f(y,\tau)/y \quad\text{and}\quad f_\infty(\tau)=\lim_{|y|\to\infty}f(y,\tau)/y.$$

The existence of nodal solutions of BVPs with multi-point BCs has been studied extensively; see [3, 4, 5, 6, 13, 14, 18, 21] and the references therein. More specifically, in recent years, researchers have drawn their attention towards the existence of nodal solutions, solutions with a specific zero-counting property in (a,b), of nonlinear boundary value problems (BVPs) with nonlocal BCs. We shall draw the reader's attention to such results obtained for BVPs which involve special cases of Eq. (1.1) and BC (1.2).

Ma [15], Ma and O'Regan [16], Rynne [17], Xu [19], and Xu et al. [20] studied the special BVP consisting of the equation

$$y'' + f(y) = 0, \quad t \in (a, b), \tag{1.6}$$

and the multi-point BC

$$y(0) = 0, \quad y(1) - \sum_{i=1}^{m} k_i y(\eta_i) = 0. \tag{1.7}$$

Ma and O'Regan [16] and Rynne [17] used a standard global bifurcation method to establish the existence of nodal solutions of BVP (1.6), (1.7) by relating it to the eigenvalues of the corresponding linear Sturm-Liouville problem (SLP) with the multi-point BC (1.7). However, the establishment of these results rely on direct computation of the eigenvalues and eigenfunctions of the SLP associated with BVP (1.6), (1.7). Thus, these results cannot be extended to a general BVP with variable coefficient functions.

Motivated by these results, Kong, Kong, and Wong [9] obtained results on the existence of nodal solutions of the BVP consisting of the equation

$$y'' + w(t) f(y) = 0, \quad t \in (a, b), \tag{1.8}$$

and the separated–multi-point boundary condition

$$\begin{cases} \cos\alpha \; y(a) - \sin\alpha \; y'(a) = 0, & \alpha \in [0, \pi) \\ y(b) - \sum_{i=1}^{m} k_i y(\xi_i) = 0, & \end{cases}$$

by relating it to the corresponding linear SLP with a two-point separated boundary condition. The shooting method and an energy function were key tools used. These results were a significant improvement since the eigenvalues of two-point linear self-adjoint SLPs are easy to compute by already developed algorithms; see [1] and the references therein. By a very similar method, these results were generalized by Chamberlain, Kong, Kong [2], where they studied the BVP consisting of Eq. (1.4) and the separated–Riemann-Stieltjes integral boundary condition

$$\begin{cases} \cos\alpha \; y(a) - \sin\alpha \; y'(a) = 0, & \alpha \in [0, \pi) \\ y(b) - \int_a^b y(s)\, d\xi(s) = 0. & \end{cases}$$

Recently, Kong and St. George [8] obtained the existence of nodal solutions of the multi-point BVP (1.8), (1.5). By matching the nodal solutions of BVPs with one of the separated–multi-point BCs

$$y'(c) = 0, \quad y(b) - \sum_{i=1}^{m} k_i y(\eta_i) = 0$$

and

$$y(a) - \sum_{j=1}^{l} h_j y(\xi_j) = 0, \quad y'(d) = 0$$

at some point $c = d \in (a, b)$, we established the existence of various nodal solutions of BVP (1.8), (1.5).

In this paper, we generalize the results in [8] to BVP (1.1), (1.2) that involves Riemann-Stieltjes integrals in the equation and in the boundary conditions. To establish the existence of nodal solutions of BVP (1.1), (1.2), we first prove the existence of nodal solutions of BVPs consisting of Eq. (1.1) and one of the separated–Riemann-Stieltjes integral boundary conditions

$$y'(c) = 0, \quad y(b) - \int_a^b y(s)\, d\xi(s) = 0 \tag{1.9}$$

and

$$y(a) - \int_a^b y(s)\, d\eta(s) = 0, \quad y'(d) = 0, \tag{1.10}$$

respectively, and then match them at some point $c = d \in (a, b)$. We also derive the nonexistence of nodal solutions of BVP (1.1), (1.2).

2. MAIN RESULTS

We aim to study the solutions of BVP (1.1), (1.2) which belong to the class $\mathcal{T}_n^\gamma$.

Definition 2.1. A solution y of BVP (1.1), (1.2) is said to belong to class $\mathcal{T}_n^\gamma$ for $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $\gamma \in \{+, -\}$ if

(i) y and y' have only simple zeros in $[a, b]$,
(ii) y' has exactly $n + 1$ zeros in (a, b),
(iii) there is exactly one zero of y strictly between any two consecutive zeros of y',
(iv) $\gamma y(t) > 0$ in a right-neighborhood of a.

Remark 2.2. One can easily see that for $y \in \mathcal{T}_n^\gamma$ with $n \in \mathbb{N}_0$ and $\gamma \in \{+, -\}$, y may have n, $n + 1$, or $n + 2$ zeros in (a, b).

In this paper, we will use the notation $h_\pm(t, \tau) := \max\{0, \pm h(t, \tau)\}$ for any function h. Let $F(y, \tau) := \int_0^y f(\xi, \tau)\, d\xi$ for $y \in \mathbb{R}$ and $\tau \in [a, b]$. In addition, let

$$H(t, y) := \int_a^b w(t, \tau) F(y, \tau)\, d\zeta(\tau), \tag{2.1}$$

and

$$\gamma^- = \int_a^b l^-(t)\, dt, \quad \text{and} \quad \gamma^+ = \int_a^b l^+(t)\, dt,$$

where

$$l^{\pm}(t) := \max_{\tau\in[a,b]} \left\{ \frac{(w_t)_{\pm}(t,\tau)}{w(t,\tau)} \right\}. \tag{2.2}$$

By (H2), for any fixed $\tau \in [a,b]$, $F(y,\tau)$ is strictly increasing in y on $[0,\infty)$. Thus, for any fixed $t \in [a,b]$, $H(t,y)$ is strictly increasing in y on $[0,\infty)$, and hence, is invertible in y on $[0,\infty)$. We denote by $H_+^{-1}(t,y)$ its inverse. Similarly, $H(t,y)$ has an inverse $H_-^{-1}(t,y)$ in y on $(-\infty,0]$.

Note that assumption (H2) implies that F is even in y. Hence, for $t \in [a,b]$,

$$H^{-1}(t,y) := H_+^{-1}(t,y) = -H_-^{-1}(t,y), \quad y \in [0,\infty). \tag{2.3}$$

In addition, to simplify the notation we denote

$$\xi_+(s) := \xi_1(s) + \xi_2(s), \quad \text{and} \quad \eta_+(s) := \eta_1(s) + \eta_2(s), \quad s \in [a,b],$$

where ξ_i, η_i, $i = 1,2$, are given given by (1.3).

We now state the main results on the existence and nonexistence of nodal solutions of BVP (1.1), (1.2). The proofs of the main results are given in the subsequent section.

For $n \in \mathbb{N}_0$, let λ_n be the n-th eigenvalue of the SLP consisting of the equation

$$y'' + \lambda \left(\int_a^b w(t,\tau)\, d\zeta(\tau) \right) y = 0, \quad t \in (a,b), \tag{2.4}$$

and the Neumann BC

$$y'(a) = 0, \quad y'(b) = 0. \tag{2.5}$$

It is well-known that $\{\lambda_n\}_{n=0}^{\infty}$ satisfy that

$$0 = \lambda_0 < \lambda_1 < \cdots \lambda_n < \cdots, \text{ and } \lambda_n \to \infty,$$

and any eigenfunction associated with λ has exactly n simple zeros in (a,b) for $n \in \mathbb{N}_0$, see [22, Theorem 4.3.2].

Theorem 2.3. *Let $n \in \mathbb{N}_0$. Assume for $t \in [a,b]$, either*

$$\int_a^b w(t,\tau) \left(f_0(\tau) - \lambda_{\lfloor n/2 \rfloor} \right) d\zeta(\tau) \le 0 \quad \textit{and} \quad \int_a^b f_\infty(\tau)\, d\zeta(\tau) = \infty \tag{2.6}$$

or

$$\int_a^b w(t,\tau) \left(f_\infty(\tau) - \lambda_{\lfloor n/2 \rfloor} \right) d\zeta(\tau) \le 0 \quad \textit{and} \quad \int_a^b f_0(\tau)\, d\zeta(\tau) = \infty, \tag{2.7}$$

where $\lfloor n/2 \rfloor$ is the integer-part of $n/2$. Suppose that for any $r > 0$,

$$\int_a^b H^{-1}(s, re^{\gamma^-})\, d\xi_+(s) < H^{-1}(b,r), \tag{2.8}$$

and

$$\int_b^a H^{-1}(a+b-s, re^{\gamma^+})\, d\eta_+(a+b-s) < H^{-1}(a,r). \tag{2.9}$$

Furthermore, for $* = 0$ *or* ∞*, when* $\int_a^b f_*(\tau)d\zeta(\tau) > 0$*, we also require that*

$$\int_a^b \frac{e^{\gamma^-/2}}{\sqrt{\int_a^b f_*(\tau)w(s,\tau)\,d\zeta(\tau)}}\,d\xi_+(s) < \frac{1}{\sqrt{\int_a^b f_*(\tau)w(b,\tau)\,d\zeta(\tau)}} \tag{2.10}$$

and

$$\int_b^a \frac{e^{\gamma^+/2}}{\sqrt{\int_a^b f_*(\tau)w(a+b-s,\tau)\,d\zeta(\tau)}}\,d\eta_+(a+b-s) < \frac{1}{\sqrt{\int_a^b f_*(\tau)w(a,\tau)\,d\zeta(\tau)}}. \tag{2.11}$$

Then BVP (1.1)*,* (1.2) *has a solution* $y_n^\gamma \in \mathcal{T}_n^\gamma$ *for* $\gamma \in \{+,-\}$*.*

Remark 2.4. We comment that (2.8) and (2.9) imply that

$$\int_a^b d\xi_+(s) < 1 \quad \text{and} \quad \int_b^a d\eta_+(a+b-s) < 1. \tag{2.12}$$

To show the first inequality in (2.12), for each fixed $\tau \in [a,b]$ we have $(w_-)_t(t,\tau) \geq -w_t(t,\tau)$. Thus,

$$\gamma^- = \int_a^b l^-(t)\,dt \geq \int_s^b \frac{-w_t(t,\tau)}{w(t,\tau)}\,dt = \ln\frac{w(s,\tau)}{w(b,\tau)},$$

for all $\tau \in [a,b]$. Hence, for each fixed $s \in [a,b]$ and for all $\tau \in [a,b]$, we have

$$\frac{w(s,\tau)}{w(b,\tau)} \leq w^* := \max_{\tau\in[a,b]}\left\{\frac{w(s,\tau)}{w(b,\tau)}\right\} \leq e^{\gamma^-}. \tag{2.13}$$

By definition of H and H^{-1} and from (2.13), we have

$$r = H(b, H^{-1}(b,r)) = \int_a^b w(b,\tau)F(H^{-1}(b,r),\tau)\,d\zeta(\tau)$$

and

$$r = \frac{rw^*}{w^*} = \frac{1}{w^*}H(s, H^{-1}(s,rw^*)) = \int_a^b \frac{w(s,\tau)}{w^*}F(H^{-1}(s,rw^*),\tau)\,d\zeta(\tau).$$

Combining these and (2.13) we have that for each fixed $s \in [a,b]$,

$$\begin{aligned} H(b,H^{-1}(b,r)) &= \int_a^b w(b,\tau)F(H^{-1}(b,r),\tau)\,d\zeta(\tau) \\ &= \int_a^b \frac{w(s,\tau)}{w^*}F(H^{-1}(s,rw^*),\tau)\,d\zeta(\tau) \\ &\leq \int_a^b w(b,\tau)F(H^{-1}(s,rw^*),\tau)\,d\zeta(\tau) = H(b,H^{-1}(s,rw^*)). \end{aligned}$$

Since $H(t,y)$ is strictly increasing in y on $[0,\infty)$,

$$H^{-1}(b,r) \leq H^{-1}(s,rw^*) \text{ for all } s \in [a,b].$$

It then follows that

$$\int_a^b H^{-1}(s, re^{\gamma^-})\,d\xi_+(s) \geq \int_a^b H^{-1}(s,rw^*)\,d\xi_+(s) \geq H^{-1}(b,r)\int_a^b d\xi_+(s).$$

Applying (2.8), we have the first inequality in (2.12) holds. The second inequality in (2.12) can be shown similarly.

Let $\{\lambda_n^{[1]}\}_{n=0}^{\infty}$ and $\{\lambda_n^{[2]}\}_{n=0}^{\infty}$ be the eigenvalues of SLPs consisting of Eq. (2.4) and the BCs

$$y(a) = 0 \quad y'(b) = 0$$

and

$$y'(a) = 0 \quad y(b) = 0,$$

respectively. The following is about the nonexistence of nodal solutions of BVP (1.1), (1.2).

Theorem 2.5. *Assume for some* $n \in \mathbb{N}_0$ *and* $i = 1$ *or* 2,

$$\int_a^b w(t,\tau)\left(\frac{f(y,\tau)}{y} - \lambda_n^{[i]}\right) d\zeta(\tau) < 0 \tag{2.14}$$

for all $t \in [a,b]$ *and* $y \neq 0$. *Then BVP* (1.1), (1.2) *has no solution in* $\mathcal{T}_i^{\gamma}$ *for all* $i \geq n+1$ *and* $\gamma \in \{+,-\}$.

Assume for some $n \in \mathbb{N}_0$ *and* $i = 1$ *or* 2,

$$\int_a^b w(t,\tau)\left(\frac{f(y,\tau)}{y} - \lambda_{n+1}^{[i]}\right) d\zeta(\tau) > 0 \tag{2.15}$$

for all $t \in [a,b]$ *and* $y \neq 0$. *Then BVP* (1.1), (1.2) *has no solution in* $\mathcal{T}_i^{\gamma}$ *for all* $i \leq n$ *and* $\gamma \in \{+,-\}$.

3. PROOFS OF THE MAIN RESULTS

In order to prove Theorem 2.3, we first consider the BVPs (1.1), (1.9) and (1.1), (1.10) where $c \in [a,b)$ and $d \in (a,b]$ are arbitrary. We classify the solutions of the above BVPs into the following classes, as extensions of Definition 2.1.

Definition 3.1. Let $n \in \mathbb{N}_0$.

(a) For any $c \in [a,b)$, a solution y of BVP (1.1), (1.9) is said to belong to class $\mathcal{T}_n^{\gamma}[c,b]$ for $\gamma \in \{+,-\}$ if
  (i) y and y' have only simple zeros in $[c,b]$,
  (ii) y' has exactly n zeros in (c,b),
  (iii) $\gamma y(c) > 0$.

(b) For any $d \in (a,b]$, a solution y of BVP (1.1), (1.10) is said to belong to class $\mathcal{T}_n^{\gamma}[a,d]$ for $\gamma \in \{+,-\}$ if
  (i) y and y' have only simple zeros in $[a,d]$,
  (ii) y' has exactly n zeros in (a,d),
  (iii) $\gamma y(d) > 0$.

For any $c \in [a, b)$ and $d \in (a, b]$, we let $\{\mu_n(c)\}_{n=0}^{\infty}$ and $\{\nu_n(d)\}_{n=0}^{\infty}$ be the eigenvalues of the SLPs consisting of Eq. (2.4) and the two-point BCs

$$y'(c) = 0, \quad y'(b) = 0 \tag{3.1}$$

and

$$y'(a) = 0, \quad y'(d) = 0, \tag{3.2}$$

respectively. It is well-known that $\{\mu_n(c)\}_{n=0}^{\infty}$ and $\{\nu_n(d)\}_{n=0}^{\infty}$ satisfy that

$$0 = \mu_0(c) < \mu_1(c) < \cdots \mu_n(c) < \cdots, \text{ and } \mu_n(c) \to \infty,$$

and

$$0 = \nu_0(d) < \nu_1(d) < \cdots \nu_n(d) < \cdots, \text{ and } \nu_n(d) \to \infty;$$

and any eigenfunction associated with $\mu_n(c)$ or $\nu_n(d)$ has exactly n simple zeros in (c, b) or (a, d), respectively, for $n \in \mathbb{N}_0$, see [22, Theorem 4.3.2].

This first lemma is a generalization of [7, Propositions 3.1 and 3.2 and Corollary 3.1], with essentially the same proof.

Lemma 3.2. *Any initial value problem associated with Eq.* (1.1) *has a unique solution which exists on the whole interval* $[a, b]$. *Consequently the solution depends continuously on the initial condition.*

As an immediate consequence of Lemma 3.2, we have the following corollary.

Corollary 3.3. *For any nontrivial solution y of Eq.* (1.1), *y and y' have only simple zeros in* $[a, b]$.

Let $c \in [a, b)$. For $\gamma \in \{+, -\}$, let $y(t, \rho)$ be the solution of the IVP consisting of the Eq. (1.1) and the initial conditions

$$y(c) = \gamma\rho \quad \text{and} \quad y'(c) = 0, \tag{3.3}$$

where $\rho > 0$ is a parameter. Let $\theta(t, \rho)$ be the Prüfer angle of $y(t, \rho)$, ie, $\theta(t, \rho)$ is a continuous function on $[a, b]$ such that

$$\tan\theta(t, \rho) = y(t, \rho)/y'(t, \rho) \quad \text{and} \quad \theta(c, \rho) = \pi/2.$$

By Lemma 3.2, $\theta(t, \rho)$ is continuous in ρ on $(0, \infty)$ for any $t \in [a, b]$. The following results are generalizations of [2, Lemmas 3.2 and 3.3].

Lemma 3.4. *(i) Assume that for some $n \in \mathbb{N}_0$ and for all $t \in [a, b]$,*

$$\int_a^b w(t, \tau)\,(f_0(\tau) - \mu_n(c))\, d\zeta(\tau) \le 0.$$

Then for all $\epsilon > 0$, there exists $\rho_* > 0$ such that $\theta(b, \rho) \le n\pi + \pi/2 + \epsilon$ for all $\rho \in (0, \rho_*]$.

(ii) Assume that for some $n \in \mathbb{N}_0$ and for all $t \in [a,b]$,

$$\int_a^b w(t,\tau)\,(\mu_n(c) - f_\infty(\tau))\,d\zeta(\tau) \le 0.$$

Then for all $\epsilon > 0$, there exists $\rho^* > 0$ such that $\theta(b,\rho) \ge n\pi + \pi/2 - \epsilon$ for all $\rho \in [\rho^*, \infty)$.

Lemma 3.5. *(i) Assume that for some $n \in \mathbb{N}_0$ and for all $t \in [a,b]$,*

$$\int_a^b w(t,\tau)\,(f_\infty(\tau) - \mu_n(c))\,d\zeta(\tau) \le 0.$$

Then for all $\epsilon > 0$, there exists $\rho^* > 0$ such that $\theta(b,\rho) \le n\pi + \pi/2 + \epsilon$ for all $\rho \in [\rho^*, \infty)$.

(ii) Assume that for some $n \in \mathbb{N}_0$ and for all $t \in [a,b]$,

$$\int_a^b w(t,\tau)\,(\mu_n(c) - f_0(\tau))\,d\zeta(\tau) \le 0.$$

Then for all $\epsilon > 0$, there exists $\rho_* > 0$, such that $\theta(b,\rho) \ge n\pi + \pi/2 - \epsilon$ for all $\rho \in (0, \rho_*]$.

Utilizing Lemmas 3.2, 3.4, and 3.5, we establish the following result which is an improvement of that in [2, Theorem 2.1].

Lemma 3.6. *Assume that for some $n \in \mathbb{N}_0$ and all $t \in [a,b]$, either*

(i) $\displaystyle\int_a^b w(t,\tau)\,(f_0(\tau) - \mu_n(c))\,d\zeta(\tau) \le 0$ *and*

$\displaystyle\int_a^b w(t,\tau)\,(\mu_{n+1}(c) - f_\infty(\tau))\,d\zeta(\tau) < 0;$

or

(ii) $\displaystyle\int_a^b w(t,\tau)\,(f_\infty(\tau) - \mu_n(c))\,d\zeta(\tau) \le 0$ *and*

$\displaystyle\int_a^b w(t,\tau)\,(\mu_{n+1}(c) - f_0(\tau))\,d\zeta(\tau) < 0.$

Suppose further that (2.8) *holds for any $r > 0$. Then BVP* (1.1), (1.9) *has a solution $y_n^\gamma \in \mathcal{T}_n^\gamma[c,b]$ for $\gamma \in \{+,-\}$.*

Proof. We first prove it under the assumption (i). Without loss of generality, we assume $\gamma = +$. The case when $\gamma = -$ can be proved similarly. Let $y(t,\rho)$ be the solution of Eq. (1.1) satisfying (3.3) with $\gamma = +$:

$$y(c) = \rho \;\text{ and }\; y'(c) = 0 \tag{3.4}$$

for $\rho > 0$ and $\theta(t,\rho)$ its Prüfer angle. By Lemmas 3.5, for any small $\epsilon > 0$, there exists $0 < \rho_* < \rho^* < \infty$ such that

$$\theta(b,\rho) \le n\pi + \pi/2 + \epsilon \text{ for all } \rho \in (0, \rho_*]$$

and

$$\theta(b,\rho) \geq n\pi + \pi/2 - \epsilon \text{ for all } \rho \in [\rho^*, \infty).$$

By the continuity of $\theta(t,\rho)$ in ρ, there exists $\rho_* \leq \rho_n < \rho_{n+1} \leq \rho^*$ such that

$$\theta(b,\rho_n) = n\pi + \pi/2 + \epsilon \quad \text{and} \quad \theta(b,\rho_{n+1}) = (n+1)\pi + \pi/2 - \epsilon, \tag{3.5}$$

and

$$\theta(b,\rho_n) < \theta(b,\rho) < \theta(b,\rho_{n+1}) \text{ for } \rho_n < \rho < \rho_{n+1}. \tag{3.6}$$

Then for $t \in [a,b]$ and $\rho > 0$, we define an energy function $E(t,\rho)$ for $y(t,\rho)$ by

$$E(t,\rho) = \frac{1}{2}[y'(t,\rho)]^2 + H(t, y(t,\rho)), \tag{3.7}$$

where $H(t,y)$ is given in (2.1). By (H1) and (H2), $F(y,\tau) \geq 0$ on $\mathbb{R} \times [a,b]$ and thus $E(t,\rho) \geq 0$ on $[a,b]$. By (1.1) and the definition of $l^-(t)$ in (2.2), we have

$$\begin{aligned} E'(t,\rho) &= \int_a^b w_t(t,\tau) F(y(t,\rho),\tau)\, d\zeta(\tau) \\ &\geq \int_a^b \frac{w_t(t,\tau)}{w(t,\tau)} w(t,\tau) F(y(t,\rho),\tau)\, d\zeta(\tau) \\ &\geq -l^-(t) H(t, y(t,\rho)) \geq -l^-(t) E(t,\rho). \end{aligned}$$

Thus, $E'(t,\rho) + l^-(t)E(t,\rho) \geq 0$ for all $t \in [a,b]$ and $\rho > 0$. By solving this inequality, we obtain

$$E(s,\rho) \leq E(b,\rho) e^{\int_s^b l^-(\tau)\, d\tau} \leq E(b,\rho) e^{\gamma^-}, \quad s \in [a,b]. \tag{3.8}$$

We observe from (3.7) that for $\rho = \rho_n$ and $\rho = \rho_{n+1}$,

$$E(s,\rho) \geq H(s, y(s,\rho)). \tag{3.9}$$

It is seen from (3.5) that as $\epsilon \to 0$,

$$y'(b,\rho) = o(1) \quad \text{and} \quad |y'(b,\rho)| = \rho + o(1),$$

and hence

$$E(b,\rho) = H(b, y(b,\rho)) + o(1) = H(b, y(b,\rho))[1 + o(1)].$$

Since H^{-1} is continuous, it follows that for $\rho = \rho_n$ and $\rho = \rho_{n+1}$,

$$|y(b,\rho)| = H^{-1}(b, E(b,\rho))(1 + o(1)) \quad \text{as } \epsilon \to 0. \tag{3.10}$$

Recall that, for fixed t, $H^{-1}(t,y)$ is strictly increasing in y on $[0,\infty)$. Thus from (3.9) and (2.3), we see that for $\rho = \rho_n$ and $\rho = \rho_{n+1}$ and $s \in [a,b]$,

$$|y(s,\rho)| \leq H^{-1}(s, E(s,\rho)). \tag{3.11}$$

Define

$$\Gamma(\rho) = y(b,\rho) - \int_a^b y(s,\rho)\, d\xi(s).$$

Let $n = 2k$ with $k \in \mathbb{N}_0$. Since $y(b, \rho_{2k}) > 0$ and $y(b, \rho_{2k+1}) < 0$, by (3.9)–(3.11) and (2.8) we have for $\epsilon > 0$ sufficiently small,

$$\begin{aligned}\Gamma(\rho_{2k}) &= y(b, \rho_{2k}) - \int_a^b y(s, \rho_{2k})\, d\xi(s)\\ &\geq H^{-1}(b, E(b, \rho_{2k})(1 + o(1)) - \int_a^b |y(s, \rho_{2k})|\, d\xi_+(s)\\ &\geq H^{-1}(b, E(b, \rho_{2k}) - \int_a^b H^{-1}(s, E(s, \rho_{2k}))\, d\xi_+(s) + o(1)\\ &\geq H^{-1}(b, E(b, \rho_{2k}) - \int_a^b H^{-1}(s, E(b, \rho_{2k})e^{\gamma^-})\, d\xi_+(s) + o(1) > 0\end{aligned}$$

and

$$\begin{aligned}\Gamma(\rho_{2k+1}) &= y(b, \rho_{2k+1}) - \int_a^b y(s, \rho_{2k+1})\, d\xi(s)\\ &\leq -H^{-1}(b, E(b, \rho_{2k+1}))(1 + o(1)) + \int_a^b |y(s, \rho_{2k+1})|\, d\xi_+(s)\\ &\leq -H^{-1}(b, E(b, \rho_{2k+1})) + \int_a^b H^{-1}(s, E(s, \rho_{2k+1}))\, d\xi_+(s) + o(1)\\ &\leq -H^{-1}(b, E(b, \rho_{2k+1})) + \int_a^b H^{-1}(s, E(b, \rho_{2k})e^{\gamma^-})\, d\xi_+(s) + o(1) < 0.\end{aligned}$$

By the continuity of $\Gamma(\rho)$, there exists $\bar\rho \in (\rho_{2k}, \rho_{2k+1})$ such that $\Gamma(\bar\rho) = 0$. Similarly, for $n = 2k + 1$ with $k \in \mathbb{N}_0$, there exists $\bar\rho \in (\rho_{2k+1}, \rho_{2k+2})$ such that $\Gamma(\bar\rho) = 0$. In both cases, since $\epsilon > 0$, we see that from (3.5) and (3.6)

$$n\pi + \pi/2 < \theta(b, \bar\rho) < (n + 1)\pi + \pi/2.$$

Note that for $t \in (a, b)$ with $y(t, \rho) \neq 0$, $\theta(t, \rho)$ satisfies the equation

$$\theta'(t, \rho) = \cos^2\theta(t, \rho) + \int_a^b w(t, \tau)\frac{f(y(t, \rho), \tau)}{y(t, \rho)} \sin^2\theta(t, \rho)\, d\zeta(\tau).$$

By (H1) and (H2), $\theta(\cdot, \rho)$ is strictly increasing on $[c, b]$. We note that $y(t) = 0$ if and only if $\theta(t, \rho) = 0 \pmod{\pi}$ and $y'(t) = 0$ if and only if $\theta(t, \rho) = \pi/2 \pmod{\pi}$. Thus, y' has exactly n zeros in (c, b) and y has exactly one zero strictly between any two consecutive zeros of y'. Initial condition (3.4) implies that $y(t, \bar\rho) > 0$ in a right neighborhood of c. Therefore, $y(t, \bar\rho) \in \mathcal{T}_n^+[c, b]$.

The proof under the assumption (ii) is essentially the same as above except that the discussion is based on Lemma 3.6 instead of Lemma 3.5. □

By using a transformation, we obtain a parallel result to Lemma 3.6 on the existence of nodal solutions of BVP (1.1), (1.10) with $d \in (a, b]$.

Lemma 3.7. *Assume that for some $n \in \mathbb{N}_0$ and all $t \in [a, b]$, either*

(i) $\int_a^b w(t,\tau)\,(f_0(\tau)-\nu_n(d))\;d\zeta(\tau)\le 0$ *and*

$\int_a^b (\nu_{n+1}(d)-f_\infty(\tau))\,w(t,\tau)\,d\zeta(\tau)<0;$

or

(i) $\int_a^b w(t,\tau)\,(f_\infty(\tau)-\nu_n(d))\;d\zeta(\tau)\le 0$ *and*

$\int_a^b w(t,\tau)\,(\nu_{n+1}(d)-f_0(\tau))\;d\zeta(\tau)<0.$

Suppose further that (2.9) *holds for any* $r>0$. *Then BVP* (1.1), (1.10) *has a solution* $y_n^\gamma\in\mathcal{T}_n^\gamma[a,d]$ *for* $\gamma\in\{+,-\}$.

Proof. Consider the following transformation: $t=a+b-s$, $d=a+b-c$. Then Eq. (1.1) becomes

$$\frac{d^2y}{ds^2}+\int_a^b w(a+b-s)f(y,\tau)\,d\zeta(\tau)=0,\quad \tau\in(a,b). \tag{3.12}$$

The boundary condition in BVP (1.1), (1.10) then becomes

$$\frac{dy}{ds}(c)=0,\quad y(b)-\int_b^a y(a+b-s)\,d\eta(a+b-s)=0. \tag{3.13}$$

Note that from (1.3), $\eta(a+b-s)$ is a difference of two decreasing functions. Hence, $-\eta(a+b-s)$ is a difference of two increasing functions and is similar to $\xi(s)$ in (1.3). Clearly $c\in[a,b)$. Note that for all $\tau\in[a,b]$, $[\frac{\partial}{\partial s}w(a+b-s,\tau)]_-=[\frac{\partial}{\partial s}w(a+b-s,\tau)]_+$. It follows that

$$\begin{aligned}\int_a^b l^-(a+b-s)\,ds&=\int_a^b \max_{\tau\in[a,b]}\left\{\frac{[w_s(a+b-s,\tau)]_-}{w(a+b-s,\tau)}\right\}ds\\ &=\int_a^b \max_{\tau\in[a,b]}\left\{\frac{[w_s(t,\tau)]_+}{w(t,\tau)}\right\}dt=\int_a^b l^+(t)\,dt=\gamma^+.\end{aligned}$$

Hence inequality (2.9) implies that inequality (2.8) holds for the transformed BVP (3.12), (3.13). Also note that $\{\nu_n\}_{n=0}^\infty$ are the eigenvalues of the SLP involving the equation

$$\frac{d^2y}{ds^2}+\lambda\left(\int_a^b w(a+b-s,\tau)\,d\zeta(\tau)\right)y=0,\quad s\in(a,b),$$

and BC (3.1). Thus the conclusion follows from Lemma 3.6. □

The lemmas below play critical roles in the proof of Theorem 2.3.

Lemma 3.8. *Let* $c\in[a,b)$, $n\in\mathbb{N}_0$, *and* $\{\mu_n(c)\}_{n=0}^\infty$ *be the eigenvalues of SLP* (2.4), (3.1). *Assume for any* $r>0$ *and* $*=0,\infty$, (2.8)*and* (2.10) *hold.*

(i) Suppose

$$\int_a^b w(t,\tau)\,(f_0(\tau)-\mu_n(c))\;d\zeta(\tau)\le 0\quad\text{and}\quad\int_a^b f_\infty(\tau)\,d\zeta(\tau)=\infty, \tag{3.14}$$

and let $y_n(t;c) \in \mathcal{T}_n^+[c,b]$ *be the solution of BVP* (1.1)*,* (1.9) *given by Lemma* 3.6*. Then* $\lim_{c\to b^-} y_n(c;c) = \infty$*.*

(ii) Suppose

$$\int_a^b w(t,\tau)\,(f_\infty(\tau) - \mu_n(c))\,d\zeta(\tau) \le 0 \quad \textit{and} \quad \int_a^b f_0(\tau)\,d\zeta(\tau) = \infty, \tag{3.15}$$

and let $y_n(t;c) \in \mathcal{T}_n^+[c,b]$ *be the solution of BVP* (1.1)*,* (1.9) *given by Lemma* 3.6*. Then* $\lim_{c\to b^-} y_n(c;c) = 0$*.*

Proof. (i) Assume the contrary. There exists a sequence $\{c_k\}_{j=1}^\infty \subset [a,b)$ such that $c_k \to b^-$ and $y_n(c_k;c_k) \to l$ for some $l \in [0,\infty)$.

(a) Assume first $l \in (0,\infty)$. Let $\bar{y}(t)$ be the solution of Eq. (1.1) satisfying the IC

$$\bar{y}(b) = l \quad \text{and} \quad \bar{y}'(b) = 0.$$

Note that for $k \in \mathbb{N}$,

$$y_n(c_k;c_k) \to \bar{y}(b) \text{ as } c_k \to b^-$$

and

$$y_n'(c_k;c_k) = \bar{y}'(b) = 0.$$

By the continuous dependence of solutions of IVPs on the ICs and parameters, we have

$$\lim_{k\to\infty} y_n(t;c_k) = \bar{y}(t) \quad \text{uniformly for all } t \in [a,b].$$

Since for each k, $y_n(t;c_k)$ satisfies

$$y(b) - \int_a^b y(s)\,d\xi(s) = 0, \tag{3.16}$$

then $\bar{y}(t)$ satisfies (3.16). Define an energy function for $\bar{y}(t)$ by

$$E(t) = \frac{1}{2}[\bar{y}'(t)]^2 + H(t,\bar{y}(t)), \quad t \in [a,b],$$

where $H(t,y)$ is defined in (2.1). It follows that (3.8) holds with $E(\cdot,\rho)$ replaced by $E(\cdot)$ and so does (3.11). Additionally, with $\bar{y}'(b) = 0$ we have

$$E(b) = H(b,\bar{y}(b)),$$

and so

$$|\bar{y}(b)| = H^{-1}(b,E(b)). \tag{3.17}$$

Since $\bar{y}(b) = l > 0$, by (3.11), (3.17), (3.8), and (2.8) we have

$$\begin{aligned}\bar{y}(b) - \int_a^b \bar{y}(s)\,d\xi(s) &\ge |\bar{y}(b)| - \int_a^b |\bar{y}(s)|\,d\xi_+(s)\\ &\ge H^{-1}(b,E(b)) - \int_a^b H^{-1}(s,E(s))\,d\xi_+(s)\end{aligned}$$

$$\geq H^{-1}(b, E(b)) - \int_a^b H^{-1}(s, E(b)e^{\gamma^-})\, d\xi_+(s) > 0$$

However, this contradicts that $\bar{y}(t)$ satisfies (3.16).

(b) Then we assume $l = 0$. Since $y_n(c_k; c_k) \neq 0$, we may let

$$z_n(t; c_k) = y_n(t; c_k)/y_n(c_k; c_k).$$

It follows that $z_n(t; c_k)$ is a solution of

$$z'' + \int_a^b w(t,\tau) g_k(z,\tau) z\, d\zeta(\tau) = 0,$$

where

$$g_k(z,\tau) := \begin{cases} \dfrac{f(y_n(c_k;c_k)z,\tau)}{y_n(c_k;c_k)z}, & \text{for } z \neq 0, \\ f_0(\tau), & \text{for } z = 0, \end{cases}$$

and $g_k(z,\tau)$ is an integrable function on $\mathbb{R} \times [a,b]$ since (3.14) implies $f_0(\tau) < \infty$ a.e. on $[a,b]$. Note that as $k \to \infty$, $g_k(z,\tau) \to f_0(\tau)$. Also note that

$$z_n(c_k; c_k) = 1 \quad \text{and} \quad z_n'(c_k; c_k) = 0.$$

Let $\bar{z}(t)$ be the solution of the IVP

$$z'' + \int_a^b w(t,\tau) f_0(\tau) z\, d\zeta(\tau) = 0, \quad \bar{z}(b) = 1, \ \bar{z}'(b) = 0.$$

By the continuous dependence of solutions of IVPs on parameters, we see that

$$\lim_{k\to\infty} z_n(t; c_k) = \bar{z}(t) \quad \text{uniformly for all } t \in [a,b].$$

Since $y_n(t; c_k)$ satisfies (3.16) for each k, then $z_n(t; c_k)$ satisfies (3.16) for each k and so does $\bar{z}(t)$.

If $f_0(\tau) = 0$ a.e. $\tau \in [a,b]$, then $\bar{z}(t) \equiv 1$. It follows from (3.16) that $\int_a^b d\xi(s) = 1$. This contradicts (2.8) by Remark 2.4. Otherwise, define an energy function for $\bar{z}(t)$ by

$$E(t) = \frac{1}{2}[\bar{z}'(t)]^2 + \int_a^b \frac{f_0(\tau)}{2} w(t,\tau)[\bar{z}(t)]^2\, d\zeta(\tau), \quad t \in [a,b].$$

Then

$$\begin{aligned} E'(t) &= \int_a^b \frac{f_0(\tau)}{2} w_t(t,\tau)[\bar{z}(t)]^2\, d\zeta(\tau) = \int_a^b \frac{f_0(\tau)}{2} \frac{w_t(t,\tau)}{w(t,\tau)} w(t,\tau)[\bar{z}(t)]^2\, d\zeta(\tau) \\ &\geq -l^-(t)\left(\frac{1}{2}[\bar{z}'(t)]^2 + \int_a^b \frac{f_0(\tau)}{2} w(t,\tau)[\bar{z}(t)]^2\, d\zeta(\tau)\right) = -l^-(t)E(t). \end{aligned}$$

Thus $E'(t) + l^-(t)E(t) \geq 0$ for all $t \in [a,b]$. By solving this inequality, we obtain

$$E(s) \leq E(b) e^{\int_s^b l^-(\tau)\, d\tau} \leq E(b)e^{\gamma^-}, \quad s \in [a,b]. \tag{3.18}$$

Additionally,

$$E(s) \geq \int_a^b \frac{f_0(\tau)}{2} w(s,\tau)[\bar{z}(s)]^2 \, d\zeta(\tau), \quad s \in [a,b],$$

and

$$E(b) = \int_a^b \frac{f_0(\tau)}{2} w(b,\tau)[\bar{z}(b)]^2 \, d\zeta(\tau).$$

Hence,

$$|z(s)| \leq \sqrt{\frac{2E(s)}{\int_a^b f_0(\tau) w(s,\tau) \, d\zeta(\tau)}}, \quad s \in [a,b] \tag{3.19}$$

and

$$|\bar{z}(b)| = \sqrt{\frac{2E(b)}{\int_a^b f_0(\tau) w(b,\tau) \, d\zeta(\tau)}}. \tag{3.20}$$

From assumption (2.10) for $* = 0$, along with (3.18)–(3.20), we have

$$\begin{aligned}
\bar{z}(b) - \int_a^b \bar{z}(s) \, d\xi(s) &\geq |\bar{z}(b)| - \int_a^b |\bar{z}(s)| \, d\xi_+(s) \\
&\geq \sqrt{\frac{2E(b)}{\int_a^b f_0(\tau) w(b,\tau) \, d\zeta(\tau)}} - \int_a^b \sqrt{\frac{2E(s)}{\int_a^b f_0(\tau) w(s,\tau) \, d\zeta(\tau)}} \, d\xi_+(s) \\
&\geq \sqrt{2E(b)} \left(\frac{1}{\sqrt{\int_a^b f_0(\tau) w(b,\tau) \, d\zeta(\tau)}} - \int_a^b \frac{e^{\gamma^-/2}}{\sqrt{\int_a^b f_0(\tau) w(s,\tau) \, d\zeta(\tau)}} \, d\xi_+(s) \right) > 0,
\end{aligned}$$

contradicting that $\bar{z}(t)$ satisfies (3.16).

(ii) Assume the contrary. Then there exists $\{c_k\}_{k=1}^{\infty} \subset [a,b)$ such that $c_k \to b^-$ and $y_n(c_k; c_k) \to l$ for $l \in (0,\infty]$.

(a) Assume $l \in (0,\infty)$. Then the argument follows similarly to that in part (i), (a) above and is omitted.

(b) Assume $l = \infty$. Since $\int_a^b f_\infty(\tau) w(t,\tau) \, d\zeta(\tau) < \infty$, then by replacing $f_0(\tau)$ by $f_\infty(\tau)$ the argument follows similarly to that in part (i), (b) above and is omitted. □

The next lemma for BVP (1.1), (1.10) is a parallel result to Lemma 3.8 with a similar proof.

Lemma 3.9. *Let $d \in (a,b]$, $n \in \mathbb{N}_0$, and $\{\nu_n(d)\}_{n=0}^{\infty}$ be the eigenvalues of SLP* (2.4), (3.2). *Assume for any $r > 0$ and $* = 0, \infty$,* (2.9) *and* (2.11) *hold.*

(i) Suppose

$$\int_a^b w(t,\tau) \left(f_0(\tau) - \nu_n(d)\right) d\zeta(\tau) \leq 0 \text{ and } \int_a^b f_\infty(\tau) \, d\zeta(\tau) = \infty, \tag{3.21}$$

and let $y_n(t;d) \in \mathcal{T}_n^+[a,d]$ be the solution of BVP (1.1), (1.10) *given by Lemma* 3.7. *Then $\lim_{d \to a^+} y_n(d;d) = \infty$.*

(ii) Suppose

$$\int_a^b w(t,\tau)\,(f_\infty(\tau)-\nu_n(d))\,d\zeta(\tau)\le 0 \ \textit{and} \ \int_a^b f_0(\tau)\,d\zeta(\tau)=\infty, \tag{3.22}$$

and let $y_n(t;d)\in\mathcal{T}_n^+[a,d]$ *be the solution of BVP* (1.1), (1.10) *given by Lemma* 3.7. *Then* $\lim_{d\to a^+} y_n(d;d)=0$.

Remark 3.10. Lemmas 3.8 and 3.9 discuss the properties of nodal solutions for BVP (1.1), (1.9) and (1.1), (1.10) in the classes $\mathcal{T}_n^\gamma[c,b]$ and $\mathcal{T}_n^\gamma[a,d]$, respectively, with $\gamma=+$. Parallel results hold for $\gamma=-$.

Remark 3.11. (a) For $n\in\mathbb{N}_0$ and $c\in[a,b)$, Lemma 3.6 establishes the existence of a solution $y_n(t;c)$ of BVP (1.1), (1.9) in $\mathcal{T}_n^+[c,b]$. However, the uniqueness of such solutions are not guaranteed. As in [8], we can show that for each $n\in\mathbb{N}_0$, there is at least one continuous curve Λ_n^c in the $\rho-c$ plane which satisfies that

(i) for each $(\rho,c)\in\Lambda_n^c$, $c\in[a,b)$ and $\rho=y_n(c;c)$;

(ii) for each $c\in[a,b)$, there is at least one point $(\rho,c)\in\Lambda_n^c$.

Similarly for the solution $y_n(t;d)$ of BVP (1.1), (1.10) in $\mathcal{T}_n^+[a,d]$.

We now prove our main result, Theorem 2.3.

Proof of Theorem 2.3. Without loss of generality we consider the case where $\gamma=+$ and (2.6) holds. The other cases can be proved similarly. For any $c\in[a,b)$ and $d\in(a,b]$, let $\mu_n(c)$ be the n-th eigenvalue of SLP (2.4), (3.1) and $\nu_n(d)$ be the n-th eigenvalue of SLP (2.4), (3.2). We note that that $\mu_n(a)$ and $\nu_n(b)$ are the n-th eigenvalues of SLP (2.4), (2.5), and hence $\lambda_n=\mu_n(a)=\nu_n(b)$.

For $n\in\mathbb{N}_0$, let $i=\lfloor n/2\rfloor$, $j=n-i$. Clearly, $j\ge i$. From [12, Theorem 4.1] and [11, Theorem 2.2] we see that for $i,j\ge 1$ $\mu_i(c)$ is strictly increasing and $\lim_{c\to b^-}\mu_n(c)=\infty$, and $\nu_n(d)$ is strictly decreasing and $\lim_{d\to a^+}\nu_n(d)=\infty$. We note that $\mu_0(c)=\nu_0(d)=0$ for any $c\in[a,b)$ and $d\in(a,b]$. It follows from the assumptions that for any $c\in[a,b)$ and $d\in(a,b]$,

$$\int_a^b w(t,\tau)f_0(\tau)\,d\zeta(\tau)\le\mu_i(a)\int_a^b w(t,\tau)\,d\zeta(\tau)\le\mu_i(c)\int_a^b w(t,\tau)\,d\zeta(\tau)$$

and

$$\mu_{i+1}(a)\int_a^b w(t,\tau)\,d\zeta(\tau)\le\mu_{i+1}(c)\int_a^b w(t,\tau)\,d\zeta(\tau)<\int_a^b f_\infty(\tau)\,d\zeta(\tau),$$

along with

$$\int_a^b w(t,\tau)f_0(\tau)\,d\zeta(\tau)\le\nu_j(b)\int_a^b w(t,\tau)\,d\zeta(\tau)\le\nu_j(d)\int_a^b w(t,\tau)\,d\zeta(\tau)$$

and

$$\nu_{j+1}(b)\int_a^b w(t,\tau)\,d\zeta(\tau) \le \nu_{j+1}(d)\int_a^b w(t,\tau)\,d\zeta(\tau) < \int_a^b f_\infty(\tau)\,d\zeta(\tau).$$

Since (2.8)–(2.11) hold with $* = 0, \infty$, by Lemmas 3.6 and 3.7 we have that BVPs (1.1), (1.9) and (1.1), (1.10) have solutions $y_i^{[1]} \in \mathcal{T}_i^+[c,b]$ and $y_j^{[2]} \in \mathcal{T}_j^+[a,d]$, respectively. Additionally, by Lemmas 3.8, (i) and 3.9, (i),

$$\lim_{c\to b^-} y_i^{[1]}(c;c) = \infty \quad \text{and} \quad \lim_{d\to a^+} y_j^{[2]}(d;d) = \infty.$$

Let $\rho_i^{[1]}(c) = y_i^{[1]}(c;c)$ such that $(\rho_i^{[1]}, c)$ is on the continuous curve Λ_i^c and $\rho_j^{[2]}(d) = y_j^{[2]}(d;d)$ such that $(\rho_j^{[2]}, d)$ is on the continuous curve Λ_j^d, as defined in Remark 3.11. Note that $y_i^{[1]}(a;a), y_j^{[2]}(b;b) \in (0,\infty)$. By the continuity of the curves Λ_i^c and Λ_j^d, there exists $c^* = d^* \in (a,b)$ such that $y_i^{[1]}(c^*;c^*) = y_j^{[2]}(d^*;d^*)$. Also note that $(y_i^{[1]})'(c^*;c^*) = 0$ and $(y_j^{[2]})'(d^*;d^*) = 0$. By the uniqueness of solutions of IVPs, we have $y_i^{[1]}(t;c^*) \equiv y_j^{[2]}(t;d^*)$ for $t \in [a,b]$. We denote $y_n(t) = y_i^{[1]}(t;c^*) = y_j^{[2]}(t;d^*)$ on $[a,b]$. Thus, we have that $y_n \in \mathcal{T}_i^+[c^*,b] \cap \mathcal{T}_j^+[a,d^*]$. Considering that $y_n'(c^*) = 0$, we see that y_n' has $n+1$ zeros in (a,b). It is easy to see from (H2) that $-y_n$ is also a solution of BVP (1.1), (1.2). Thus $-y_n'$ has $n+1$ zeros in (a,b). Clearly, condition (iv) in Definition 2.1 is satisfied by one of y_n and $-y_n$ for $\gamma = +$ and $\gamma = -$, respectively. Therefore, one of y_n and $-y_n$ is in $\mathcal{T}_n^+$ and the other is in $\mathcal{T}_n^-$. □

For $\alpha \in [0,\pi)$, let $\{\lambda_n^1(\alpha)\}_{n=0}^\infty$ denote the eigenvalues of the SLP consisting of Eq. (2.4) and the BC

$$\begin{cases} \cos\alpha\; y(a) - \sin\alpha\; y'(a) = 0, \quad \alpha \in [0,\pi), \\ y'(b) = 0. \end{cases}$$

We note that for $n \in \mathbb{N}_0$, $\lambda_n^1(0) = \lambda_n^1$, where λ_n^1 is the n-th eigenvalue of SLP (2.4), (2.5). From [10, Lemma 3.32] and [12, Theorem 4.2] , $\lambda_n^1(\alpha)$ is continuous and $\lambda_n^1(\alpha)$ is strictly decreasing in α on $[0,\pi)$; moreover,

$$\lim_{\alpha\to\pi^-} \lambda_0^1(\alpha) = -\infty \quad \text{and} \quad \lim_{\alpha\to\pi^-} \lambda_n^1(\alpha) = \lambda_{n-1}^1(0), \tag{3.23}$$

Consider the BVP consisting of Eq. (1.1) and the BC

$$\begin{cases} \cos\alpha\; y(a) - \sin\alpha\; y'(a) = 0, \quad \alpha \in [0,\pi), \\ y(b) - \displaystyle\int_a^b y(s)\,d\xi(s) = 0. \end{cases} \tag{3.24}$$

The following result is a generalization of [2, Theorem 2.2]. It plays a key role in the proof of Theorem 2.5.

Lemma 3.12. *(i) Assume that for some $n \in \mathbb{N}_0$ and $i = 1$, (2.14) holds for all $t \in [a,b]$ and $y \neq 0$. Then BVP (1.1), (3.24) has no solution with the derivative* *having $i+1$ zeros on (a,b) if $\alpha \in [0,\pi/2)$, and has no solution with the derivative having i zeros on (a,b) if $[\pi/2,\pi)$, for all $i \geq n$.*

(ii) Assume that for some $n \in \mathbb{N}_0$ and $i=1$, (2.15) holds for all $t \in [a,b]$ and $y \neq 0$. Then BVP (1.1), (3.24) has no solution with the derivative having $i+1$ zeros on (a,b) if $\alpha \in [0,\pi/2)$, and has no solution with the derivative having i zeros on (a,b) if $[\pi/2,\pi)$, for all $i \leq n$.

Proof of Theorem 2.5. Assume (2.14) holds for $i=1$. By contradiction, suppose BVP (1.1), (1.2) has a solution $y \in \mathbb{T}_i^\gamma$ for some $i \geq n+1$, $\gamma \in \{+,-\}$. Then there exists $\alpha^* \in [0,\pi)$ such that $\cos\alpha^*\, y(a) - \sin\alpha^*\, y'(a) = 0$. This means that $y(t)$ is a solution of BVP (1.1), (3.24) for $\alpha = \alpha^*$. From our assumptions, along with (3.23) and the fact that $\lambda_n^{[1]}(\alpha)$ is strictly decreasing in α on $[0,\pi)$, we have that for any $\alpha \in [0,\pi)$

$$\begin{aligned}&\int_a^b w(t,\tau)\frac{f(y,\tau)}{y}\,d\zeta(\tau) < \lambda_n^{[1]}\int_a^b w(t,\tau)\,d\zeta(\tau)\\ =&\lambda_n^{[1]}(0)\int_a^b w(t,\tau)\,d\zeta(\tau) < \lambda_{n+1}^{[1]}(\alpha)\int_a^b w(t,\tau)\,d\zeta(\tau).\end{aligned}$$

By Lemma 3.12, (i), BVP (1.1), (3.24) has no solution with the derivative having i or $i+1$ zeros, depending on α^*, on (a,b) for all $i \geq n+1$. We have reached a contradiction to $y \in \mathcal{T}_i^\gamma$.

The proof of the second part of Theorem 2.5 with $i=1$ is similar to above except that Lemma 3.12, (ii) is used instead of Lemma 3.12, (i). The proof for the case with $i=2$ is similar to the above and hence is omitted. □

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