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UNIQUENESS AND PARAMETER DEPENDENCE OF POSITIVE SOLUTIONS OF DISCRETE PERIODIC BOUNDARY VALUE PROBLEMS

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ABSTRACT. The authors study a type of second order nonlinear discrete periodic boundary value problems. The existence and uniqueness of positive solutions are discussed. The parametric dependence of the solutions is also investigated. Two examples are given as applications of the results.

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1. INTRODUCTION

In this paper, we consider the second-order nonlinear boundary value problem (BVP) consisting of the equation

$$-\Delta(p(t-1)\Delta u(t-1)) + q(t)u(t) = \lambda F(t, u(t)),\ t \in [1, T]_{\mathbb{Z}} \tag{1.1}$$

and the periodic boundary conditions (BCs)

$$u(0) = u(T), \quad p(0)\Delta u(0) = p(T)\Delta u(T) \tag{1.2}$$

where λ is a positive parameter, Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$, $[c, d]_{\mathbb{Z}}$ denotes the discrete interval $\{c, c+1, \ldots, d\}$ for any integers c and d with $c \leq d$, and $F : [1, T]_{\mathbb{Z}} \times (0, \infty) \to \mathbb{R}_+$ is continuous with $\mathbb{R}_+ = [0, \infty)$. We also assume that $p(t) > 0$ on $[0, T+1]_{\mathbb{Z}}$, $q(t) \geq 0$ and $q(t) \not\equiv 0$ on $[1, T]_{\mathbb{Z}}$. By a *positive solution* of BVP (1.1), (1.2) we mean a function $u : [0, T+1]_{\mathbb{Z}} \to \mathbb{R}$ that satisfies both (1.1) and (1.2), and $u(t) > 0$ for all $t \in [0, T+1]_{\mathbb{Z}}$.

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Nonlinear BVPs have been a focus of research for decades due to their special importance in theory and applications. Significant progress has been made on the existence, multiplicity, and nonexistence of solutions or positive solutions. For some work on differential equation periodic BVPs, see [3, 6, 7, 10, 11] and the references therein; for some results on discrete periodic BVPs, the reader is referred to [1, 2, 5, 12–15] and the references therein.

Recently, mixed monotone operator theory has been used by many authors to study the existence and uniqueness of positive solutions of BVPs; see, for example, [8,9,16]. In this paper, we apply this method to investigate the uniqueness of positive solutions of BVP (1.1), (1.2). Two theorems on the existence, uniqueness, and the parametric dependence of positive solutions of BVP (1.1), (1.2) are obtained by using mixed monotone operator theory. Our results reveal the relation between the solution and the parameter and provide a method to approximate the unique solutions by the solutions of the associated linear BVPs.

This paper is organized as follows: After this introduction, our main results are stated in Section 2. Two examples are given in Section 3. All the proofs are given in Section 4.

2. MAIN RESULTS

In this paper, we let $F(t,x) = f(t,x,x) + r(t,x)$, where $f \in C([1,T]_{\mathbb{Z}} \times (0,\infty) \times (0,\infty), \mathbb{R}_+)$ and $r \in C([1,T]_{\mathbb{Z}} \times (0,\infty), \mathbb{R}_+)$. The following assumptions will be needed:

(H1) $f(t,\cdot,y)$ is increasing for any fixed $(t,y) \in [1,T]_{\mathbb{Z}} \times (0,\infty)$, and $f(t,x,\cdot)$ is decreasing for any fixed $(t,x) \in [1,T]_{\mathbb{Z}} \times (0,\infty)$;

(H2) There exists $\alpha \in (0,1)$ such that $f(t,\kappa x,\kappa^{-1}y) \geq \kappa^{\alpha} f(t,x,y)$ for $t \in [1,T]_{\mathbb{Z}}$, $\kappa \in (0,1)$, $x \in (0,\infty)$, and $y \in (0,\infty)$;

(H3) $r(t,\cdot)$ is increasing for any fixed $t \in [1,T]_{\mathbb{Z}}$ and there exists a constant $w > 0$ such that $r(t,w) \not\equiv 0$ on $[1,T]_{\mathbb{Z}}$;

(H4) $r(t,\kappa x) \geq \kappa r(t,x)$ for any $\kappa \in (0,1)$, $t \in [1,T]_{\mathbb{Z}}$, and $x \in (0,\infty)$;

(H5) There exists $\eta > 0$ such that $f(t,x,y) \geq \eta r(t,x)$ for $t \in [1,T]_{\mathbb{Z}}$, $x \in (0,\infty)$ and $y \in (0,\infty)$;

(H6) For α given in (H2), we have $\alpha \in (0,1/2)$ and

$$r(t,x) \geq \kappa^{\alpha} r(t,x) \text{ for } t \in [1,T]_{\mathbb{Z}}, \kappa \in (0,1), \text{ and } x \in (0,\infty).$$

Remark 2.1. We would like to make a few comments on the form of the nonlinear term f above. The analysis in this paper mainly relies on mixed monotone operator theory. To apply such theory, some authors write the nonlinearity as $f(t,x)$ and assume that $f(t,x)$ can be decomposed into $f(t,x) = g(t,x) + h(t,x)$, where g is continuous and nondecreasing in x, h is continuous and nonincreasing in x, and there exists $\alpha \in (0,1)$ such that

$$g(t, \kappa x) \geq \kappa^{\alpha} g(t, x) \tag{2.1}$$

and

$$h(t, \kappa^{-1} x) \geq \kappa^{\alpha} h(t, x) \tag{2.2}$$

for t in its domain, $\kappa \in (0,1)$, and $x \geq 0$. The reader may refer to [4] for a related discussion.

Here, the nonlinear term f is written as a function of three arguments. Then, to apply mixed monotone operator theory, we need to assume that the conditions (H2) and (H3) above are satisfied. By writing f this way, a larger class of functions can be covered. For instance, if $f(t,x,y) = \sqrt[3]{x}/\sqrt{y+1}$, then $f(t,x,y)$ cannot be decomposed into a summation of two functions g and h satisfying (2.1) and (2.2), but $f(t,x,y)$ does satisfy (H2) and (H3) with $\alpha = 5/6$.

For any $u : [0, T+1]_{\mathbb{Z}} \to \mathbb{R}$, let $\|u\| = \max_{t \in [0,T+1]_{\mathbb{Z}}} |u(t)|$. The following theorem is our main result.

Theorem 2.2. *Assume that (H1)–(H5) hold. Then*

(1) *for any $\lambda > 0$, BVP* (1.1), (1.2) *has a unique positive solution $u_\lambda(t)$;*

(2) *for any positive functions u_0, v_0 on $[0, T+1]_{\mathbb{Z}}$, i.e. $u_0(t) > 0$ and $v_0(t) > 0$ on $[0, T+1]_{\mathbb{Z}}$, let $\{u_n\}$ and $\{v_n\}$ be the solutions of the following linear periodic BVPs*

$$\begin{cases} -\Delta(p(t-1)\Delta u_n(t-1)) + q(t)u_n(t) \\ \qquad\qquad = \lambda[f(t, u_{n-1}(t), v_{n-1}(t)) + r(t, u_{n-1}(t))], \\ u_n(0) = u_n(T), \ p(0)\Delta u_n(0) = p(T)\Delta u_n(T), \end{cases} \tag{2.3}$$

and

$$\begin{cases} -\Delta(p(t-1)\Delta v_n(t-1)) + q(t)v_n(t) \\ \qquad\qquad = \lambda[f(t, v_{n-1}(t), u_{n-1}(t)) + r(t, v_{n-1}(t))], \\ v_n(0) = v_n(T), \ p(0)\Delta v_n(0) = p(T)\Delta v_n(T), \end{cases} \tag{2.4}$$

$n = 1, 2, \ldots$. *Then $\|u_n - u_\lambda\| \to 0$ and $\|v_n - u_\lambda\| \to 0$ as $n \to \infty$.*

(3) *If, in addition, (H6) holds, then the unique solution $u_\lambda(t)$ satisfies the following properties:*

 (a) *$u_\lambda(t)$ is strictly increasing in λ, i.e. $\lambda_1 > \lambda_2 > 0$ implies $u_{\lambda_1}(t) > u_{\lambda_2}(t)$, for $t \in [1, T]_{\mathbb{Z}}$;*

 (b) $\lim_{\lambda \to 0^+} \|u_\lambda\| = 0$ *and* $\lim_{\lambda \to \infty} \|u_\lambda\| = \infty$;

 (c) *$u_\lambda(t)$ is continuous in λ, i.e. $\lambda \to \lambda_0 > 0$ implies $\|u_\lambda - u_{\lambda_0}\| \to 0$.*

When $r(t,x) \equiv 0$ on $[1,T]_{\mathbb{Z}} \times (0, \infty)$, i.e., $F(t,x) = f(t,x,x)$, we obtain a similar result.

Theorem 2.3. *Assume that (H1) and (H2) hold, and assume that there exists a constant $w > 0$ such that $f(t, w, w) \not\equiv 0$. Then*

(1) *for any $\lambda > 0$, BVP (1.1), (1.2) has a unique positive solution $u_\lambda(t)$.*
(2) *for any positive functions u_0, v_0 on $[0, T+1]_{\mathbb{Z}}$, let $\{u_n\}$ and $\{v_n\}$ be the solutions of the following linear periodic BVPs*

$$\begin{cases} -\Delta(p(t-1)\Delta u_n(t-1)) + q(t)u_n(t) = \lambda f(t, u_{n-1}(t), v_{n-1}(t)), \\ u_n(0) = u_n(T),\ p(0)\Delta u_n(0) = p(T)\Delta u_n(T), \end{cases} \tag{2.5}$$

and

$$\begin{cases} -\Delta(p(t-1)\Delta v_n(t-1)) + q(t)v_n(t) = \lambda f(t, v_{n-1}(t), u_{n-1}(t)), \\ v_n(0) = v_n(T),\ p(0)\Delta v_n(0) = p(T)\Delta v_n(T), \end{cases} \tag{2.6}$$

$n = 1, 2, \ldots$. Then $\|u_n - u_\lambda\| \to 0$ and $\|v_n - u_\lambda\| \to 0$ as $n \to \infty$.
(3) *If, in addition, $\alpha \in (0, 1/2)$, then the unique solution $u_\lambda(t)$ satisfies the three properties specified in conclusion (3) of Theorem 2.2.*

Remark 2.4. In Theorem 2.2 (2) and Theorem 2.3 (2), if we let $u_0 = v_0$, then it is easy to see that $u_n = v_n$ for any $n > 0$. Hence we only need to solve one linear BVP

$$\begin{cases} -\Delta(p(t-1)\Delta u_n(t-1)) + q(t)u_n(t) = \lambda F(t, u_{n-1}(t)), \\ u_n(0) = u_n(T),\ p(0)\Delta u_n(0) = p(T)\Delta u_n(T) \end{cases} \tag{2.7}$$

in each step. Since Theorem 2.2 (2) and Theorem 2.3 (2) guarantee the convergence of $\{u_n\}$, therefore, we may use this iteration to approximate the unique positive solution of BVP (1.1), (1.2).

3. EXAMPLES

In this section, we give two examples to demonstrate the applications of the results obtained in Section 2.

Example 3.1. Consider BVP (1.1), (1.2) with $p(t) = 3t + 6$, $q(t) = t^2$, and

$$F(t, u(t)) = \lambda[u^{1/3}(t) + u^{-1/2}(t) + \arctan(u(t)) + 1].$$

We claim that BVP (1.1), (1.2) has a unique positive solution for any $\lambda > 0$.

In fact, let

$$f(t, x, y) = x^{1/3} + y^{-1/2} + 1 \quad \text{and} \quad r(t, x) = \arctan(x). \tag{3.1}$$

It is easy to see that (H1)–(H3) and (H5) hold with $\alpha = 1/2$ and $\eta = 2/\pi$.

For $\kappa \in (0, 1)$ and $x > 0$, it is easy to see that

$$\frac{d}{dx}[r(\kappa x) - \kappa r(x)] = \frac{\kappa}{1 + \kappa^2 x^2} - \frac{\kappa}{1 + x^2} \geq 0.$$

Hence $r(\kappa x) \geq \kappa r(x)$ for $\kappa \in (0, 1)$ and $x > 0$, i.e., (H4) holds.

Thus by Theorem 2.2 (1), BVP (1.1), (1.2) has a unique positive solution u_λ.

Note that for $\kappa \in (0,1)$ and $\alpha \in (0,1)$, $[r(\kappa x) - \kappa^\alpha r(x)]'|_{x=0} = \kappa - \kappa^\alpha < 0$. Hence

$$x^{1/3} + y^{-1/2} + \arctan(x) + 1$$

does not satisfy (H2), i.e., we cannot use this function as the function f needed in our theorems.

Numerical solutions of BVP (1.1), (1.2) with $\lambda = 1$ and $T = 30$ are computed by using (2.7) with $u_0 \equiv 1$ for 40 iterations. The maximum absolute errors $E_n = \|u_n - u_{n-1}\|$ between u_n and u_{n-1} are given in Table 1, which confirm Theorem 2.2 (2). The graphs of $\{u_n\}$, $n = 0, \ldots, 40$, are given in Figure 1.

n	1	5	9	14	19
E_n	0.9948	0.0016	4.7298e-05	6.1982e-07	8.0231e-09
i	23	27	31	36	40
E_n	2.4599e-10	7.5070e-12	2.2829e-13	2.8831e-15	8.8471e-17

TABLE 1. The maximum absolute error.

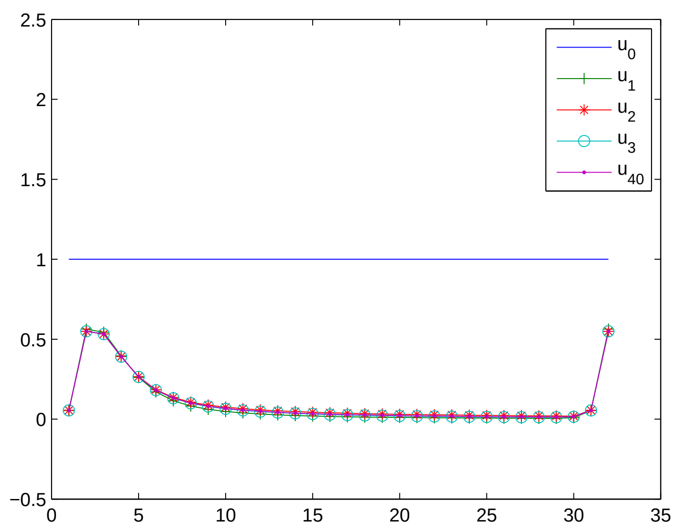

FIGURE 1. $\{u_n\}$, $n = 0, \ldots, 40$.

Example 3.2. Consider BVP (1.1), (1.2) with $p = t^2 + 1$, $q = \sin(t) + 5$, and

$$F(t, u(t)) = \frac{t^2 + \sqrt[7]{u(t)}}{\sqrt[3]{u(t)}}.$$

We claim that BVP (1.1), (1.2) has a unique positive solution for any $\lambda > 0$.

In fact, let

$$f(t, x, y) = y^{-1/3}(t^2 + x^{1/7}). \tag{3.2}$$

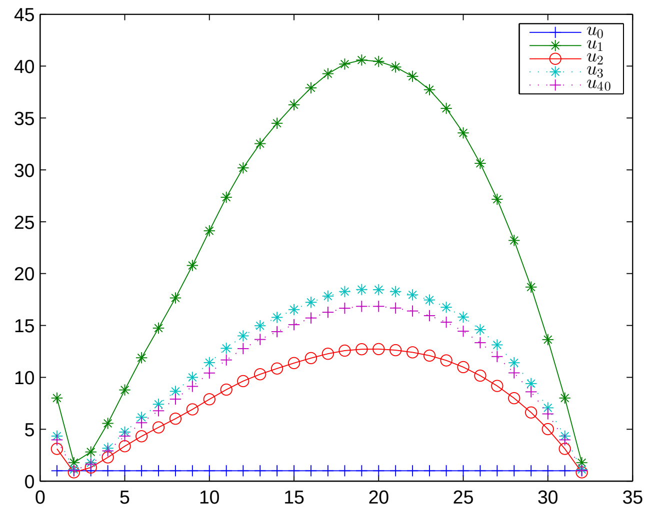

FIGURE 2. $\{u_n\}$, $n = 0, \ldots, 40$.

It is easy to see that (H1) and (H2) hold with $\alpha = 10/21$. Furthermore, for any $w > 0$, we have that $f(t, w, w) > 0$. Therefore, all the conditions of Theorem 2.3 are satisfied. Thus for any $\lambda > 0$, BVP (1.1), (1.2) has a unique positive solution u_λ.

Numerical solutions of BVP (1.1), (1.2) with $\lambda = 1$ and $T = 30$ are computed by using (2.7) with $u_0 \equiv 1$ for 40 iterations. The results are given in Figure 2. The dependence of parameter is demonstrated in Figure 3 for different values of λ. These numerical results confirm our Theorem 2.3.

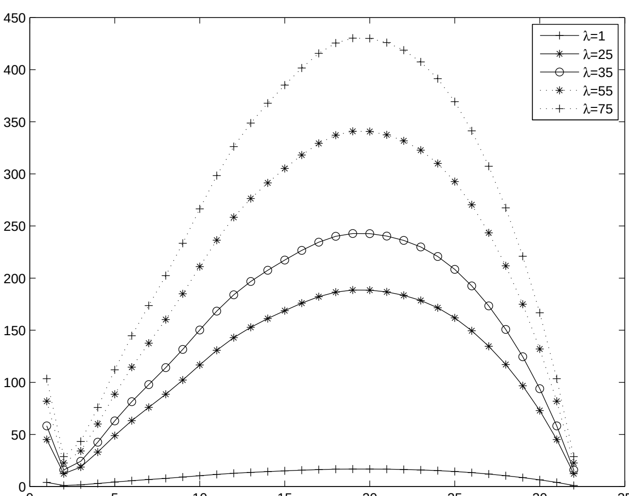

FIGURE 3. Dependence on λ

4. PROOFS

We will use mixed monotone operator theory to prove our theorems. The following definitions and lemma are needed. The reader is referred to [8, Lemma 2.1] for the details.

Definition 4.1. Let $(X, \|\cdot\|)$ be a Banach space and $\mathbf{0}$ be the zero element of X.

(a) A nonempty closed convex set $P \subset X$ is said to be a cone if it satisfies (i) $u \in P$ and $\lambda > 0 \Longrightarrow \lambda u \in P$; (ii) $u \in P$ and $-u \in P \Longrightarrow u = \mathbf{0}$.

(b) A cone P is said to be normal if there exists a constant $D > 0$ such that, for all $u, v \in X$, $\mathbf{0} \leq u \leq v \Longrightarrow \|u\| \leq D\|v\|$. The constant D is called the normality constant of P.

(c) The Banach space $(X, \|\cdot\|)$ is partially ordered by a normal cone $P \subset E$, i.e., $u \leq v$ if $v - u \in P$. If $u \leq v$ and $u \neq v$, then we write $u < v$ or $v > u$.

(d) For any $u, v \in X$, we use the notation $u \sim v$ to mean that there exist $\underline{d} > 0$ and $\overline{d} > 0$ such that $\underline{d}v \leq u \leq \overline{d}v$. Given $w > \mathbf{0}$, i.e., $w \geq \mathbf{0}$ and $w \neq \mathbf{0}$, we define $P_w = \{u \in X \mid u \sim w\}$. Clearly, $P_w \subset P$.

Definition 4.2. An operator $A : P_w \times P_w \to X$ is called mixed monotone if $A(x, y)$ is nondecreasing in x and non-increasing in y; i.e., for $x_1, x_2, y_1, y_2 \in P_w$, we have

$$x_1 \leq x_2 \text{ and } y_1 \geq y_2 \text{ imply } A(x_1, y_1) \leq A(x_2, y_2).$$

Definition 4.3. An element $u \in P_w$ is called a fixed point of A if $A(u, u) = u$.

Definition 4.4. An operator $B : P_w \to X$ is called sub-homogeneous if it satisfies

$$B(ku) \geq kB(u) \text{ for all } u \in P_w \text{ and } k \in (0, 1).$$

Definition 4.5. Let $\alpha \in [0, 1)$. An operator $B : P_w \to X$ is called α-concave if it satisfies

$$B(ku) \geq k^\alpha B(u) \text{ for all } u \in P_w \text{ and } k \in (0, 1).$$

Note that if an operator is α-concave, then clearly it is also sub-homogeneous.

Lemma 4.6. *Let $\alpha \in (0, 1)$ and $A : P_w \times P_w \to X$ be a mixed monotone operator satisfying*

$$A(\kappa u, \kappa^{-1} v) \geq \kappa^\alpha A(u, v) \quad \text{for all} \quad u, v \in P_w \quad \text{and} \quad \kappa \in (0, 1). \tag{4.1}$$

(A) *Assume that $B : P_w \to X$ is an increasing sub-homogeneous operator and the following conditions hold:*

(i) *$A(w, w) \in P_w$ and $B(w) \in P_w$;*

(ii) *there exists a constant $\eta > 0$ such that $A(u, v) \geq \eta B(u)$ for all $u, v \in P_w$.*

Then:

(1) *for any $\lambda > 0$, $\lambda(A(u, u) + B(u)) = u$ has a unique solution u_λ in P_w.*

(2) *for any initial values $u_0, v_0 \in P_w$, consider the sequences u_n and v_n defined by*

$$u_n = \lambda(A(u_{n-1}, v_{n-1}) + B(u_{n-1}))$$

$$v_n = \lambda(A(v_{n-1}, u_{n-1}) + B(v_{n-1})),$$

for $n = 1, 2, \ldots$ Then, $\|u_n - u_\lambda\| \to 0$ and $\|v_n - u_\lambda\| \to 0$ as $n \to \infty$;

(3) *if we further assume that $\alpha \in (0, 1/2)$ and B is α-concave, then the unique solution u_λ satisfies the properties:*
   (a) *u_λ is strictly increasing in λ, that is, if $\lambda_1 > \lambda_2 > 0$, then $u_{\lambda_1} > u_{\lambda_2}$;*
   (b) *$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0$ and $\lim_{\lambda \to \infty} \|u_\lambda\| = \infty$;*
   (c) *u_λ is continuous in λ, that is, if $\lambda \to \lambda_0 > 0$, then $\|u_\lambda - u_{\lambda_0}\| \to 0$.*

(B) *Assume that $A(w, w) \in P_w$. Then:*
   (1) *for any $\lambda > 0$, $\lambda A(u, u) = u$ has a unique solution u_λ in P_w;*
   (2) *for any initial values $u_0, v_0 \in P_w$, consider the sequences u_n and v_n defined by*

$$u_n = \lambda A(u_{n-1}, v_{n-1}), v_n = \lambda A(v_{n-1}, u_{n-1}), n = 1, 2, \ldots$$

   Then, $\|u_n - u_\lambda\| \to 0$ and $\|v_n - u_\lambda\| \to 0$ as $n \to \infty$;
   (3) *if we further assume that $\alpha \in (0, 1/2)$, then the unique solution u_λ satisfies the three properties* (a), (b) *and* (c) *specified in* (3) *of part* (A).

The following lemma is excerpted from Atici and Guseinov [2, Theorems 2.1 and 2.2]. We will use it to construct the operators A and B.

Lemma 4.7. *Assume $p(t) > 0$ on $[0, T+1]_{\mathbb{Z}}$, $q(t) \geq 0$, and $q(t) \not\equiv 0$ on $[1, T]_{\mathbb{Z}}$. Then the BVP consisting of the equation*

$$-\Delta(p(t-1)\Delta u(t-1)) + q(t)u(t) = 0, \quad t \in [1, T]_{\mathbb{Z}} \tag{4.2}$$

and the BC (1.2) *has a Green's function $G : [0, T+1]_{\mathbb{Z}} \times [0, T+1]_{\mathbb{Z}} \to \mathbb{R}$ with*

$$G(t, s) > 0 \text{ for } (t, s) \in [0, T+1]_{\mathbb{Z}} \times [1, T]_{\mathbb{Z}}. \tag{4.3}$$

Remark 4.8. The reader is referred to [2, Theorem 2.1] for the detail of G.

In the sequel, we define X as the Banach space of real-valued functions on $[0, T+1]_{\mathbb{Z}}$ with the norm $\|u\| = \max_{t \in [0, T+1]_{\mathbb{Z}}} |u(t)|$. Also, we define $P \subset X$ by

$$P = \{u \in X \mid u(t) \geq 0 \text{ on } [0, T+1]_{\mathbb{Z}}\}.$$

Clearly, P is a normal cone with normality constant $D = 1$. For w given in (H3), it is easy to see that $P_w = \{u \in P \mid u(t) > 0 \text{ on } [0, T+1]_{\mathbb{Z}}\}$ since for any $u \in P_w$,

$0 < \min_{t\in[0,T+1]_{\mathbb{Z}}} u(t) \le u(t) \le \|u\|$ on $[0,T+1]_{\mathbb{Z}}$. We define the operators $A : P_w \times P_w \to X$ and $B : P_w \to X$ as follows:

$$A(u,v)(t) = \sum_{s=1}^{T} G(t,s)f(s,u(s),v(s)) \tag{4.4}$$

and

$$B(u)(t) = \sum_{s=1}^{T} G(t,s)r(s,u(s)), \tag{4.5}$$

where G is the Green's function given in Lemma 4.7.

Remark 4.9. By Lemma 4.7, (4.4), and (4.5), it is easy to see that $u(t)$ is a solution of BVP (1.1), (1.2) if and only if $u = \lambda(A(u,u) + B(u))$.

Proof of Theorem 2.2. (1). By (H1), (4.3), and (4.4), it is easy to see that A is mixed monotone. Similarly, by (H3), (4.3), and (4.5), B is increasing.

Now, by (H2), (4.3), and (4.4), for $u, v \in P_w$, $\kappa \in (0,1)$, and $t \in [0,T+1]_{\mathbb{Z}}$, we have

$$\begin{aligned} A(\kappa u, \kappa^{-1}v)(t) &= \sum_{s=1}^{T} G(t,s)f(s,\kappa u(s),\kappa^{-1}v(s)) \\ &\ge \kappa^{\alpha} \sum_{s=1}^{T} G(t,s)f(s,u(s),v(s)) = \kappa^{\alpha} A(u,v)(t). \end{aligned}$$

So we have that (4.1) holds. Similarly, by (H4), (4.3), and (4.5), for $u \in P_w$, $\kappa \in (0,1)$, and $t \in [0,T+1]_{\mathbb{Z}}$,

$$B(\kappa u)(t) = \sum_{s=1}^{T} G(t,s)r(s,\kappa u(s)) \ge \kappa \sum_{s=1}^{T} G(t,s)r(s,u(s)) = \kappa B(u)(t).$$

Hence B is sub-homogeneous.

For $u, v \in P_w$, (H5) implies $f(t,u(t),v(t)) \ge \eta r(t,u(t))$ on $[0,T+1]_{\mathbb{Z}}$. Then by (4.3), (4.4), and (4.5), for $t \in [0,T+1]_{\mathbb{Z}}$,

$$\begin{aligned} A(u,v)(t) =& \sum_{s=1}^{T} G(t,s)f(s,u(s),v(s)) \\ \ge& \eta \sum_{s=1}^{T} G(t,s)r(s,u(s)) = \eta B(u)(t), \end{aligned}$$

i.e., condition (ii) of Lemma 4.6 (A) holds. In particular, we have

$$A(w,w)(t) \ge \eta B(w)(t) \quad \text{on} \quad [0,T+1]_{\mathbb{Z}}.$$

Since $r(t,w) \not\equiv 0$ on $[0,T+1]_{\mathbb{Z}}$, it is easy to see that $A(w,w)(t) > 0$ and $B(w)(t) > 0$ on $[0,T+1]_{\mathbb{Z}}$. Hence we have $A(w,w) \in P_w$, $B(w) \in P_w$, i.e., condition (i) of Lemma 4.6 (A) is satisfied.

Therefore, by applying Lemma 4.6 (A) (1), for any $\lambda > 0$, BVP (1.1), (1.2) has a unique solution u_λ in P_w, which is positive. On the other hand, by the definition of P_w, any positive solution of BVP (1.1), (1.2) must be in P_w. Hence BVP (1.1), (1.2) has a unique positive solution.

(2). By Lemma 4.7, for $n = 1, 2, \dots$ and $t \in [0, T+1]_{\mathbb{Z}}$,

$$u_n(t) = \lambda \sum_{s=1}^{T} G(t,s)[f(s, u_{n-1}(s), v_{n-1}(s)) + r(s, u_{n-1}(s))]$$

is the solution of BVP (2.3). Then by (4.4) and (4.5),

$$u_n = \lambda(A(u_{n-1}, v_{n-1}) + B(u_{n-1})).$$

Similarly, we can show that for $n = 1, 2, \dots$, $v_n = \lambda(A(v_{n-1}, u_{n-1}) + B(v_{n-1}))$ is the solution of BVP (2.4). Then the conclusion follows from Lemma 4.6 (A) (2).

(3). If (H6) holds, then $\alpha \in (0, 1/2)$ and

$$\begin{aligned} B(\kappa u)(t) = &\sum_{s=1}^{T} G(t,s) r(s, \kappa u(s)) \\ &\geq \kappa^\alpha \sum_{s=1}^{T} G(t,s) r(s, u(s)) = \kappa^\alpha B(u)(t), \end{aligned}$$

i.e., B is α-concave. Then Theorem 2.2 (3) follows from Lemma 4.6 (A) (3). □

The proof of Theorem 2.3 is in the same way by using Lemma 4.6 (B). We omit the detail.

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REFERENCES

[1] F. M. Atici and A. Cabada, Existence and uniqueness results for discrete second-order periodic boundary value problems, *Computers & Mathematics with Applications* **45** (2003), 1417–1427.

[2] F. M. Atici and G. Sh. Guiseinov, Positive periodic solutions for nonlinear difference equations with periodic coefficients, *J. Mathematical Analysis and App.* **232** (1999), 166–182.

[3] F. M. Atici and G. Sh. Guseinov, On the existence of solutions for nonlinear differential equations with periodic boundary conditions, *J. Comput. Applied Math.* **132** (2001), 341–356.

[4] A. Dogan, J. R. Graef, and L. Kong, Higher order singular multi-point boundary value problems on time scales, *Proc. Edinburgh Math. Soc.* **54** (2011), 345–361.

[5] T. He, Y. Lu, Y. Lei, and F. Yang, Nontrivial Periodic Solutions for Nonlinear Second-Order Difference Equations, *Discrete Dynamics in Nature and Society* **(2011)** Article ID 153082, 14 pages.

[6] J. R. Graef and L. Kong, Existence results for nonlinear periodic boundary value problems, *Proc. Edinburgh Math. Soc.* **52** (2009), 79–95.

[7] J. R. Graef, L. Kong, and H. Wang, Existence, multiplicity, and dependence on a parameter for a periodic boundary value problem, *J. Differential Equations* **245** (2008), 1185–1197.

[8] J. R. Graef, L. Kong, M. Wang, and B. Yang, Uniqueness and parameter dependence of positive solutions of a discrete fourth order problem, *J. Difference Equ. Appl.* **19** (2013), 1133–1146.

[9] L. Kong, Second order singular boundary value problems with integral boundary conditions, *Nonlinear Anal.* **72** (2010), 2628–2638.

[10] Q. Kong and M. Wang, Eigenvalue approach to even order system periodic boundary value problems, *Canad. Math. Bull.* **56** (2013), 102–115.

[11] Q. Kong and M. Wang, Positive solutions of even order system periodic boundary value problems, *Nonlinear Anal.* **72** (2010), 1778–1791.

[12] R. Ma, Y. Lu and T. Chen, Existence of one-signed solutions of discrete second-order periodic boundary value problems, *Abstract and Applied Analysis.* (2012) Article ID 437912, 13 pages.

[13] R. Ma and H. Ma, Positive solutions for nonlinear discrete periodic boundary value problems, *Computers & Mathematics with Applications* **59** (2010), 136–141.

[14] H. Xu, New fixed point theorems of mixed monotone operators and applications to singular boundary value problems on time scales, *Boundary Value Problems* **2011** Article ID 567054, 14 pages.

[15] C. Yuan, D. Jiang, and Y. Zhang, Existence and uniqueness of solutions for singular higher order continuous and discrete boundary value problems, *Boundary Value Problems* **2008** Article ID 123823, 11 pages.

[16] C. Zhai and L. Zhang, New fixed point theorems for mixed monotone operators and local existence-uniqueness of positive solutions for nonlinear boundary value problems, *J. Math. Anal. Appl.* **382** (2011), 594–614.