

**MAXIMUM PRINCIPLE AND NONLINEAR THREE POINT
SINGULAR BOUNDARY VALUE PROBLEMS ARISING DUE TO
SPHERICAL SYMMETRY**

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ABSTRACT. We consider the following class of nonlinear three point singular boundary value problems (SBVPs)

$$\begin{aligned} -y''(x) - \frac{2}{x}y'(x) &= f(x, y), \quad 0 < x < 1, \\ y'(0) = 0, \quad y(1) &= \delta y(\eta), \end{aligned}$$

where $\delta > 0$ and $0 < \eta < 1$. We establish some new maximum principles. Further using these maximum principles and monotone iterative technique in the presence of upper and lower solution we prove existence of solutions for the above class of nonlinear three point SBVPs. Here the nonlinear term is one sided Lipschitz continuous in its domain, also $x = 0$ is regular singular point of the above differential equation.

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1. INTRODUCTION

Singular differential equations are of great importance, and the behavior of a physical system modeled by differential equation frequently is most interesting in the neighborhood of a singular point [1]. Many problems in applied mathematics and engineering lead to singular boundary value problems of the form

$$-y'' - \frac{\alpha}{x}y' = f(x, y), \quad 0 < x < 1, \tag{1.1}$$

$$y'(0) = 0, \quad y(1) = A, \tag{1.2}$$

where A is a finite constant and $\alpha \geq 1$. Existence and uniqueness of solutions of (1.1)–(1.2) has been studied by several researchers, e.g., [2]–[8].

Recently lot of activity is noted on the upper and lower solution techniques. Zhang [9] in his work justified that this technique is most promising specially for singular boundary value problems.

Three point variation of the two point SBVPs (1.1)–(1.2) in spherical symmetry can be written as

$$-y''(x) - \frac{2}{x}y'(x) = f(x, y), \quad 0 < x < 1, \quad (1.3)$$

$$y'(0) = 0, \quad y(1) = \delta y(\eta). \quad (1.4)$$

where $f(I \times R, R)$, $I = [0, 1]$, $0 < \eta < 1$, $\delta > 0$. The singular three point BVPs (1.3)–(1.4) are motivated by the mathematical model of heat generated in a chemical reaction [2] and equilibrium of charged gas in a spherical shaped container [3]. Equations (1.3)–(1.4) model the thermal balance [2] between the heat generated by the chemical reaction and that conducted away in spherical vessel. The boundary condition $y(1) = \delta y(\eta)$ represents the relation between temperature on the outer surface and a surface lies on a sphere concentric with the vessel and radius less than container. Similarly equilibrium of a charged gas in a spherical container [3] can be extended for three point boundary value problems of the type (1.3)–(1.4).

Lots of results are available based on different analytical techniques for three point nonlinear BVPs [10]–[19]. But when existing theory is applied to three point nonlinear SBVPs lot of complications arise and in the current work we have made an honest effort to address some of these issues. In this work we consider three point nonlinear SBVP (1.3)–(1.4) which represents some physical phenomenon occurring in spherical geometry. We use monotone iterative technique which is analytical but computational in nature. It is not easy to establish Maximum principle for the corresponding linear case for three point BVPs. As to achieve that we need to validate some inequalities which are nonlinear in nature.

Iterative technique goes way back to the time of Picard [20]. In this work we propose the following iterative scheme which is similar to the one considered in [4] and [5]

$$\begin{aligned} -y''_{n+1} - \frac{2}{x}y'_{n+1} - \lambda y_{n+1} &= f(x, y_n) - \lambda y_n, & y'_{n+1}(0) &= 0, \\ y_{n+1}(1) &= \delta y_{n+1}(\eta). \end{aligned} \quad (1.5)$$

We allow $\sup \left(\frac{\partial f}{\partial y} \right)$ to take both negative and positive values.

Under quite general conditions we show that a range for values of λ on both side of real line can be found so that the above iterative scheme produces convergent monotonic sequences which are solutions of the iterative scheme. These sequences converge uniformly to the solution of the nonlinear three point boundary value problem (1.3)–(1.4). To start the iteration and to produce monotonic sequences we need some initial guess in terms of differential inequalities. These inequalities provide initial guess as well as upper and lower bound for above discussed sequences of solutions.

This paper is organized in following sections. Section 2 we use Lommel's transformation to find out two linearly independent solutions in terms of Spherical Bessel Functions. Using these two linearly independent solutions Green's function is constructed in section 3 and Section 4 states Maximum principle. Finally all these results are used to establish existence theorems. The sufficient conditions derived in this paper are verified for 4 examples.

LOMMEL'S TRANSFORMATION. This section is devoted to the corresponding linear model of the nonlinear three point SBVPs (1.3)–(1.4). We consider the following class of three point linear BVPs,

$$-(x^2y'(x))' - \lambda x^2y(x) = x^2h(x), \quad 0 < x < 1, \quad (1.6)$$

$$y'(0) = 0, \quad y(1) = \delta y(\eta) + b, \quad (1.7)$$

where $h \in C(I)$ and b is any constant.

The corresponding homogeneous system is given by

$$-(x^2y'(x))' - \lambda x^2y(x) = 0, \quad 0 < x < 1, \quad (1.8)$$

$$y'(0) = 0, \quad y(1) = \delta y(\eta). \quad (1.9)$$

Consider the differential equation (1.8) written in the form

$$x^2y''(x) + 2xy'(x) + \lambda x^2y(x) = 0. \quad (1.10)$$

Using Lommel's transformation (§cf [5, 21])

$$z = x\sqrt{\lambda}, \quad w = x^{\frac{1}{2}}y(x), \quad (1.11)$$

the standard Bessel's equation

$$z^2 \frac{d^2w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2)w = 0, \quad (1.12)$$

is transformed into (1.10). Now, if $w_1(z)$ and $w_2(z)$ are two linearly independent solutions of Bessel's equation (1.12), then the two linearly independent solution of (1.10) are given by

$$y_1(x) = x^{-\frac{1}{2}}w_1(x\sqrt{\lambda}), \quad y_2(x) = x^{-\frac{1}{2}}w_2(x\sqrt{\lambda}). \quad (1.13)$$

Hence the two linearly independent solutions of (1.10) can be obtained in terms of $w_1(z)$ and $w_2(z)$. A solution of (1.12) which leads to say y_1 bounded in the neighborhood of the origin is $w_1 = J_{\frac{1}{2}}(z)$. Hence a solution of (1.10) which remains bounded in the neighborhood of the origin (except for a multiplicative constant) denoted as $y_1(x, \lambda)$ is given by

$$y_1(x, \lambda) = \begin{cases} x^{-\frac{1}{2}}J_{\frac{1}{2}}(x\sqrt{\lambda}), & \text{if } \lambda > 0; \\ (ix)^{-\frac{1}{2}}J_{\frac{1}{2}}(ix\sqrt{|\lambda|}), & \text{if } \lambda < 0. \end{cases} \quad (1.14)$$

2. GREEN'S FUNCTION

In this section we construct Green's function. We divide it into two cases.

Case I: $\lambda > 0$. Let us assume

$$(H_0) : 0 < \lambda \leq j_{-\frac{1}{2},1}^2, \quad 0 < \delta < 1, \quad \eta \cos \sqrt{\lambda} - \delta \cos \eta \sqrt{\lambda} \leq 0, \quad \eta \sin \sqrt{\lambda} - \delta \sin \eta \sqrt{\lambda} > 0$$

where $j_{-\frac{1}{2},1}$ is the first positive zero of $J_{-\frac{1}{2}}(x)$.

It is easy to see that (H_0) can be satisfied.

Lemma 2.1. *The Green's function for the following linear three point SBVPs*

$$(x^2 y'(x))' + \lambda x^2 y(x) = 0, \quad 0 < x < 1, \quad (2.1)$$

$$y'(0) = 0, \quad y(1) = \delta y(\eta), \quad (2.2)$$

is given by

$$G(x, t) = \begin{cases} \frac{\sin(x\sqrt{\lambda})(\eta \sin(\sqrt{\lambda}(t-1)) - \delta \sin(\sqrt{\lambda}(t-\eta)))}{x t \sqrt{\lambda} (\eta \sin(\sqrt{\lambda}) - \delta \sin(\eta \sqrt{\lambda}))}, & 0 \leq x \leq t \leq \eta; \\ \frac{\sin(t\sqrt{\lambda})(\eta \sin(\sqrt{\lambda}(x-1)) - \delta \sin(\sqrt{\lambda}(x-\eta)))}{x t \sqrt{\lambda} (\eta \sin(\sqrt{\lambda}) - \delta \sin(\eta \sqrt{\lambda}))}, & t \leq x, \quad t \leq \eta; \\ \frac{\eta \sin(\sqrt{\lambda}(t-1)) \sin(x\sqrt{\lambda})}{x t \sqrt{\lambda} (\eta \sin(\sqrt{\lambda}) - \delta \sin(\eta \sqrt{\lambda}))}, & x \leq t, \quad \eta \leq t; \\ \frac{(\delta \sin(\eta \sqrt{\lambda}) \sin(\sqrt{\lambda}(t-x)) + \eta \sin(t\sqrt{\lambda}) \sin(\sqrt{\lambda}(x-1)))}{x t \sqrt{\lambda} (\eta \sin(\sqrt{\lambda}) - \delta \sin(\eta \sqrt{\lambda}))}, & \eta \leq t \leq x \leq 1, \end{cases}$$

and if (H_0) holds then $G(x, t) \leq 0$.

Proof. Define the Green's function by the following equations

$$G(x, t) = \begin{cases} a_1 \frac{1}{\sqrt{x}} J_{\frac{1}{2}} \left(x\sqrt{\lambda} \right) + a_2 \frac{1}{\sqrt{x}} J_{-\frac{1}{2}} \left(x\sqrt{\lambda} \right), & 0 \leq x \leq t \leq \eta; \\ a_3 \frac{1}{\sqrt{x}} J_{\frac{1}{2}} \left(x\sqrt{\lambda} \right) + a_4 \frac{1}{\sqrt{x}} J_{-\frac{1}{2}} \left(x\sqrt{\lambda} \right), & t \leq x, \quad t \leq \eta; \\ a_5 \frac{1}{\sqrt{x}} J_{\frac{1}{2}} \left(x\sqrt{\lambda} \right) + a_6 \frac{1}{\sqrt{x}} J_{-\frac{1}{2}} \left(x\sqrt{\lambda} \right), & x \leq t, \quad \eta \leq t; \\ a_7 \frac{1}{\sqrt{x}} J_{\frac{1}{2}} \left(x\sqrt{\lambda} \right) + a_8 \frac{1}{\sqrt{x}} J_{-\frac{1}{2}} \left(x\sqrt{\lambda} \right), & \eta \leq t \leq x \leq 1. \end{cases}$$

According to the definition and properties of the Green's function, for any $t \in [0, \eta]$, we have

$$\begin{aligned} a_1 \frac{1}{\sqrt{t}} J_{\frac{1}{2}} \left(t\sqrt{\lambda} \right) + a_2 \frac{1}{\sqrt{t}} J_{-\frac{1}{2}} \left(t\sqrt{\lambda} \right) &= a_3 \frac{1}{\sqrt{t}} J_{\frac{1}{2}} \left(t\sqrt{\lambda} \right) + a_4 \frac{1}{\sqrt{t}} J_{-\frac{1}{2}} \left(t\sqrt{\lambda} \right), \\ \left(-a_1 \sqrt{\lambda} \frac{1}{\sqrt{t}} J_{\frac{3}{2}} \left(t\sqrt{\lambda} \right) + a_2 \sqrt{\lambda} \frac{1}{\sqrt{t}} J_{-\frac{3}{2}} \left(t\sqrt{\lambda} \right) \right) & \\ - \left(-a_3 \sqrt{\lambda} \frac{1}{\sqrt{t}} J_{\frac{3}{2}} \left(t\sqrt{\lambda} \right) + a_4 \sqrt{\lambda} \frac{1}{\sqrt{t}} J_{-\frac{3}{2}} \left(t\sqrt{\lambda} \right) \right) &= -\frac{1}{t^2}, \end{aligned}$$

and thus

$$a_1 - a_3 = -\frac{\pi J_{-\frac{1}{2}}(t\sqrt{\lambda})}{2\sqrt{t}},$$

$$a_2 - a_4 = \frac{\pi J_{\frac{1}{2}}(t\sqrt{\lambda})}{2\sqrt{t}}.$$

Using the boundary conditions, we have

$$a_2 = 0,$$

$$a_3 J_{\frac{1}{2}}(\sqrt{\lambda}) + a_4 J_{-\frac{1}{2}}(\sqrt{\lambda}) = \delta \left(a_3 \frac{1}{\sqrt{\eta}} J_{\frac{1}{2}}(\eta\sqrt{\lambda}) + a_4 \frac{1}{\sqrt{\eta}} J_{-\frac{1}{2}}(\eta\sqrt{\lambda}) \right).$$

Therefore

$$a_1 = \frac{\sqrt{\frac{\pi}{2}} \left(\eta \sin((t-1)\sqrt{\lambda}) - \delta \sin(\sqrt{\lambda}(t-\eta)) \right)}{t \sqrt[4]{\lambda} \left(\eta \sin(\sqrt{\lambda}) - \delta \sin(\eta\sqrt{\lambda}) \right)},$$

$$a_2 = 0,$$

$$a_3 = \frac{\sqrt{\frac{\pi}{2}} \sin(t\sqrt{\lambda}) \left(\eta \cos(\sqrt{\lambda}) - \delta \cos(\eta\sqrt{\lambda}) \right)}{t \sqrt[4]{\lambda} \left(\eta \sin(\sqrt{\lambda}) - \delta \sin(\eta\sqrt{\lambda}) \right)},$$

$$a_4 = -\frac{\sqrt{\frac{\pi}{2}} \sin(t\sqrt{\lambda})}{t \sqrt[4]{\lambda}}.$$

For any $t \in [\eta, 1]$, we have

$$a_5 \frac{1}{\sqrt{t}} J_{\frac{1}{2}}(t\sqrt{\lambda}) + a_6 \frac{1}{\sqrt{t}} J_{-\frac{1}{2}}(t\sqrt{\lambda}) = a_7 \frac{1}{\sqrt{t}} J_{\frac{1}{2}}(t\sqrt{\lambda}) + a_8 \frac{1}{\sqrt{t}} J_{-\frac{1}{2}}(t\sqrt{\lambda}),$$

$$\left(-a_5 \sqrt{\lambda} \frac{1}{\sqrt{t}} J_{\frac{3}{2}}(t\sqrt{\lambda}) + a_6 \sqrt{\lambda} \frac{1}{\sqrt{t}} J_{-\frac{3}{2}}(t\sqrt{\lambda}) \right)$$

$$- \left(-a_7 \sqrt{\lambda} \frac{1}{\sqrt{t}} J_{\frac{3}{2}}(t\sqrt{\lambda}) + a_8 \sqrt{\lambda} \frac{1}{\sqrt{t}} J_{-\frac{3}{2}}(t\sqrt{\lambda}) \right) = -\frac{1}{t^2},$$

and hence

$$a_5 - a_7 = -\frac{\pi J_{-\frac{1}{2}}(t\sqrt{\lambda})}{2\sqrt{t}},$$

$$a_6 - a_8 = \frac{\pi J_{\frac{1}{2}}(t\sqrt{\lambda})}{2\sqrt{t}}.$$

By using the boundary conditions, we have

$$a_6 = 0,$$

$$a_7 J_{\frac{1}{2}}(\sqrt{\lambda}) + a_8 J_{-\frac{1}{2}}(\sqrt{\lambda}) = \delta \left(a_5 \frac{1}{\sqrt{\eta}} J_{\frac{1}{2}}(\eta\sqrt{\lambda}) + a_6 \frac{1}{\sqrt{\eta}} J_{-\frac{1}{2}}(\eta\sqrt{\lambda}) \right).$$

Thus

$$a_5 = \frac{\sqrt{\frac{\pi}{2}} \eta \sin(\sqrt{\lambda}(t-1))}{t \sqrt[4]{\lambda} (\eta \sin(\sqrt{\lambda}) - \delta \sin(\eta\sqrt{\lambda}))},$$

$$a_6 = 0,$$

$$a_7 = \frac{\sqrt{\frac{\pi}{2}} (\eta \cos(\sqrt{\lambda}) \sin(\sqrt{\lambda}t) - \delta \sin(\eta\sqrt{\lambda}) \cos(\sqrt{\lambda}t))}{t \sqrt[4]{\lambda} (\eta \sin(\sqrt{\lambda}) - \delta \sin(\eta\sqrt{\lambda}))},$$

$$a_8 = -\frac{\sqrt{\frac{\pi}{2}} \sin(\sqrt{\lambda}t)}{t \sqrt[4]{\lambda}},$$

which completes the construction of Green's function. Using (H_0) we can easily verify that $G(x, t) \leq 0$. \square

Lemma 2.2. *Let $y \in C^2(I)$ be a solution of nonhomogeneous linear three point SBVPs (1.6)–(1.7) then*

$$y(x) = \frac{b \eta \sin(x\sqrt{\lambda})}{x (\eta \sin(\sqrt{\lambda}) - \delta \sin(\eta\sqrt{\lambda}))} - \int_0^1 t^2 G(x, t) h(t) dt. \quad (2.3)$$

Proof. Suppose $G(x, t)$ is the Green's function of

$$(x^2 y'(x))' + \lambda x^2 y(x) = 0, \quad 0 < x < 1,$$

$$y'(0) = 0, \quad y(1) = \delta y(\eta),$$

and \bar{y} is solution of

$$(x^2 y'(x))' + \lambda x^2 y(x) = 0, \quad 0 < x < 1,$$

$$y'(0) = 0, \quad y(1) = \delta(\eta) + b,$$

then the boundary value problem (1.6)–(1.7) is equivalent to

$$y(t) = \bar{y} - \int_0^1 t^2 G(x, t) h(t) dt.$$

Suppose

$$\bar{y} = c_1 \frac{1}{\sqrt{x}} J_{\frac{1}{2}}(x\sqrt{\lambda}) + c_2 \frac{1}{\sqrt{x}} J_{-\frac{1}{2}}(x\sqrt{\lambda}).$$

Since

$$\bar{y}'(0) = (0), \quad \text{and} \quad \bar{y}(1) = \delta \bar{y}(\eta) + b,$$

we get

$$c_1 = \frac{b}{J_{\frac{1}{2}}(\sqrt{\lambda}) - \frac{\delta}{\sqrt{\eta}} J_{\frac{1}{2}}(\eta\sqrt{\lambda})},$$

$$c_2 = 0.$$

Hence the three point linear SBVP (1.6)–(1.7) is equivalent to

$$y(x) = \frac{b \eta \sin(x\sqrt{\lambda})}{x \left(\eta \sin(\sqrt{\lambda}) - \delta \sin(\eta\sqrt{\lambda}) \right)} - \int_0^1 t^2 G(x, t) h(t) dt.$$

Namely $y \in C^2(I)$ is a solution of the boundary value problem (1.6)–(1.7) if and only if $y \in C(I)$ is a solution of the integral equation

$$y(x) = \frac{b \eta \sin(x\sqrt{\lambda})}{x \left(\eta \sin(\sqrt{\lambda}) - \delta \sin(\eta\sqrt{\lambda}) \right)} - \int_0^1 t^2 G(x, t) h(t) dt.$$

□

Case II: $\lambda < 0$. Assume that

$$(H'_0) \quad \lambda < 0, \delta > 0, \eta \cosh(\sqrt{|\lambda|}) - \delta \cosh(\eta\sqrt{|\lambda|}) \geq 0, \\ \eta \sinh(\sqrt{|\lambda|}) - \delta \sinh(\eta\sqrt{|\lambda|}) > 0.$$

It is easy to see that (H'_0) can be satisfied.

Lemma 2.3. *The Green's function for the following linear three point SBVPs*

$$(x^2 y'(x))' + \lambda x^2 y(x) = 0, \quad 0 < x < 1, \\ y'(0) = 0, \quad y(1) = \delta y(\eta)$$

for $\lambda < 0$ is given by

$$G(x, t) = \begin{cases} \frac{\sinh(\sqrt{|\lambda|x}) \left(\eta \sinh(\sqrt{|\lambda|(t-1)}) - \delta \sinh(\sqrt{|\lambda|(t-\eta)}) \right)}{x t \sqrt{|\lambda|} \left(\eta \sinh(\sqrt{|\lambda|}) - \delta \sinh(\eta\sqrt{|\lambda|}) \right)}, & 0 \leq x \leq t \leq \eta; \\ \frac{\sinh(\sqrt{|\lambda|t}) \left(\eta \sinh(\sqrt{|\lambda|(x-1)}) - \delta \sinh(\sqrt{|\lambda|(x-\eta)}) \right)}{x t \sqrt{|\lambda|} \left(\eta \sinh(\sqrt{|\lambda|}) - \delta \sinh(\eta\sqrt{|\lambda|}) \right)}, & t \leq x, t \leq \eta; \\ \frac{\eta \sinh(\sqrt{|\lambda|(t-1)}) \sinh(\sqrt{|\lambda|x})}{x t \sqrt{|\lambda|} \left(\eta \sinh(\sqrt{|\lambda|}) - \delta \sinh(\eta\sqrt{|\lambda|}) \right)}, & x \leq t, \eta \leq t; \\ \frac{\left(\delta \sinh(\eta\sqrt{|\lambda|}) \sinh(\sqrt{|\lambda|(t-x)}) + \eta \sinh(t\sqrt{|\lambda|}) \sinh((x-1)\sqrt{|\lambda|}) \right)}{x t \sqrt{|\lambda|} \left(\eta \sinh(\sqrt{|\lambda|}) - \delta \sinh(\eta\sqrt{|\lambda|}) \right)}, & \eta \leq t \leq x \leq 1. \end{cases}$$

and if (H'_0) holds then $G(x, t) \leq 0$.

Proof. Proof is same as given in Lemma 2.1. □

Lemma 2.4. *Let $y \in C^2(I)$ be a solution of nonhomogeneous linear three point SBVPs (1.6)–(1.7) then*

$$y(t) = \frac{b \eta \sinh(x\sqrt{|\lambda|})}{x \left(\eta \sinh(\sqrt{|\lambda|}) - \delta \sinh(\eta\sqrt{|\lambda|}) \right)} - \int_0^1 t^2 G(x, t) h(t) dt. \quad (2.4)$$

Proof. Proof is same as given in Lemma 2.2. □

3. MAXIMUM PRINCIPLE

We require two results. They are as follows.

Proposition 3.1. *Let (H_0) holds, $b \geq 0$ and $h(x) \in C[0, 1]$ is such that $h(x) \geq 0$, then $y(x)$ is non-negative for all $x \in [0, 1]$.*

Proposition 3.2. *Let (H'_0) holds, $b \geq 0$ and $h(x) \in C[0, 1]$ is such that $h(x) \geq 0$, then $y(x)$ is non-negative for all $x \in [0, 1]$.*

4. THE NONLINEAR THREE POINT SINGULAR BVP

In this section, we develop the theory of monotone iterative method for nonlinear three point SBVPs. We divide it into the following two subsections.

Case I: When $\lambda > 0$.

Theorem 4.1. *Let there exist α_0, β_0 in $C^2[0, 1]$ such that $\beta_0 \geq \alpha_0$ and satisfy*

$$-(x^2\beta'_0(x))' \geq x^2f(x, \beta_0), \quad 0 < x < 1; \quad \beta'_0(0) = 0, \quad \beta_0(1) \geq \delta\beta_0(\eta), \quad (4.1)$$

and

$$-(x^2\alpha'_0(x))' \leq x^2f(x, \alpha_0), \quad 0 < x < 1; \quad \alpha'_0(0) = 0, \quad \alpha_0(1) \leq \delta\alpha_0(\eta). \quad (4.2)$$

If $f : D_0 \rightarrow R$ is continuous on $D_0 := \{(x, y) \in [0, 1] \times R : \alpha_0 \leq y \leq \beta_0\}$ and there exist $M \geq 0$ such that for all $(x, y), (x, w) \in D_0$

$$y \leq w \implies f(x, w) - f(x, y) \geq M(w - y),$$

then the three point nonlinear SBVP (1.3)–(1.4) has at least one solution in the region D_0 . If \exists a constant λ such that $M - \lambda \geq 0$ and (H_0) is satisfied then the sequences $\{\beta_n\}$ generated by

$$-(x^2\beta'_{n+1})' - \lambda x^2\beta_{n+1} = x^2F(x, \beta_n), \quad \beta'_{n+1}(0) = 0, \quad \beta_{n+1}(1) = \delta\beta_{n+1}(\eta), \quad (4.3)$$

$$-(x^2\alpha'_{n+1})' - \lambda x^2\alpha_{n+1} = x^2F(x, \alpha_n), \quad \alpha'_{n+1}(0) = 0, \quad \alpha_{n+1}(1) = \delta\alpha_{n+1}(\eta), \quad (4.4)$$

where $F(x, y) = f(x, y) - \lambda y$, with initial iterate β_0 converges monotonically (non-increasing) and uniformly towards a solution $\tilde{\beta}(x)$ of (1.3)–(1.4). Similarly using α_0 as an initial iterate leads to a non-decreasing sequences $\{\alpha_n\}$ converging to a solution $\tilde{\alpha}(x)$. Any solution $z(x)$ in D_0 must satisfy

$$\tilde{\alpha}(x) \leq z(x) \leq \tilde{\beta}(x).$$

Proof. From equation (4.1) and equation (4.3) (for $n = 0$)

$$\begin{aligned} -(x^2(\beta_0 - \beta_1)'(x))' - \lambda x^2(\beta_0 - \beta_1) &\geq 0, \\ (\beta_0 - \beta_1)'(0) = 0, \quad (\beta_0 - \beta_1)(1) &\geq (\beta_0 - \beta_1)(\eta). \end{aligned}$$

Since $h(x) \geq 0$ and $b \geq 0$, by using Proposition 3.1 we have $\beta_0 \geq \beta_1$.

In view of $M - \lambda \geq 0$, from equation (4.3) we get

$$-(x^2 \beta'_{n+1}(x))' \geq x^2 ((M - \lambda)(\beta_n - \beta_{n+1}) + f(x, \beta_{n+1}))$$

and if $(\beta_n \geq \beta_{n+1})$, then

$$-(x^2 \beta'_{n+1}(x))' \geq x^2 f(x, \beta_{n+1}). \quad (4.5)$$

Since $\beta_0 \geq \beta_1$, then from equation (4.5) (for $n = 0$) and (4.3) (for $n = 1$) we get

$$\begin{aligned} -(x^2(\beta_1 - \beta_2)'(x))' - \lambda x^2(\beta_1 - \beta_2) &\geq 0, \\ (\beta_1 - \beta_2)'(0) = 0, \quad (\beta_1 - \beta_2)(1) &\geq (\beta_1 - \beta_2)(\eta). \end{aligned}$$

From Proposition 3.1 we have $\beta_1 \geq \beta_2$.

Now from equations (4.2) and (4.3) (for $n = 0$)

$$\begin{aligned} -(x^2(\beta_1 - \alpha_0)'(x))' - \lambda x^2(\beta_1 - \alpha_0) &\geq 0, \\ (\beta_1 - \alpha_0)'(0) = 0 \quad (\beta_1 - \alpha_0)(1) &\geq \delta((\beta_1 - \alpha_0)(\eta)). \end{aligned}$$

Thus $\beta_1 \geq \alpha_0$ follows from Proposition 3.1.

Now assuming $\beta_n \geq \beta_{n+1}$, $\beta_{n+1} \geq \alpha_0$, we show that $\beta_{n+1} \geq \beta_{n+2}$ and $\beta_{n+2} \geq \alpha_0$ for all n . From equation (4.3) (for $n + 1$) and (4.5) we get

$$\begin{aligned} -(x^2(\beta_{n+1} - \beta_{n+2})'(x))' - \lambda x^2(\beta_{n+1} - \beta_{n+2}) &\geq 0, \\ (\beta_{n+1} - \beta_{n+2})'(0) = 0, \quad (\beta_{n+1} - \beta_{n+2})(1) &\geq \delta(\beta_{n+1} - \beta_{n+2})(\eta), \end{aligned}$$

and hence from Proposition 3.1 we have $\beta_{n+1} \geq \beta_{n+2}$.

From equation (4.3) (for $n + 1$) and (4.2) we get,

$$\begin{aligned} -(x^2(\beta_{n+2} - \alpha_0)'(x))' - x^2 \lambda(\beta_{n+2} - \alpha_0) &\geq 0, \\ (\beta_{n+2} - \alpha_0)'(0) = 0, \quad (\beta_{n+2} - \alpha_0)(1) &\geq \delta(\beta_{n+2} - \alpha_0)(\eta). \end{aligned}$$

Then from Proposition 3.1, $\beta_{n+2} \geq \alpha_0$ and hence we have

$$\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq \beta_{n+1} \geq \cdots \geq \alpha_0$$

and starting with α_0 it is easy to get

$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \alpha_{n+1} \leq \cdots \leq \beta_0.$$

Finally we show that $\beta_n \geq \alpha_n$ for all n . For this by assuming $\beta_n \geq \alpha_n$, we show that $\beta_{n+1} \geq \alpha_{n+1}$. From equation (4.3) it is easy to get

$$\begin{aligned} -(x^2(\beta_{n+1} - \alpha_{n+1})'(x))' - \lambda x^2(\beta_{n+1} - \alpha_{n+1}) &\geq 0, \\ (\beta_{n+1} - \alpha_{n+1})'(0) = 0, \quad (\beta_{n+1} - \alpha_{n+1})(1) &\geq \delta(\beta_{n+1} - \alpha_{n+1})(\eta). \end{aligned}$$

Hence from Proposition 3.1, $\beta_{n+1} \geq \alpha_{n+1}$. Thus we have

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \alpha_{n+1} \leq \cdots \leq \beta_{n+1} \leq \beta_n \leq \cdots \leq \beta_2 \leq \beta_1 \leq \beta_0.$$

So the sequences β_n and α_n are monotonically non-increasing and non-decreasing, respectively and are bounded by β_0 and α_0 . Hence by Dini's theorem they converges uniformly. Let $\tilde{\beta}(x) = \lim_{n \rightarrow \infty} \beta_n(x)$ and $\tilde{\alpha}(x) = \lim_{n \rightarrow \infty} \alpha_n(x)$.

Using Lemma 2.2, the solution β_n of (4.3) is given by

$$\beta_n(x) = \frac{b \eta \sin(x\sqrt{\lambda})}{x \left(\eta \sin(\sqrt{\lambda}) - \delta \sin(\eta\sqrt{\lambda}) \right)} - \int_0^1 G(x, t) t^2 (f(t, \beta_n) - \lambda \beta_n) dt.$$

Then by Lebesgue's dominated convergence theorem, taking the limit as n approaches to ∞ , we get

$$\tilde{\beta}(x) = \frac{b \eta \sin(x\sqrt{\lambda})}{x \left(\eta \sin(\sqrt{\lambda}) - \delta \sin(\eta\sqrt{\lambda}) \right)} - \int_0^1 G(x, t) t^2 (f(t, \tilde{\beta}) - \lambda \tilde{\beta}) dt$$

which is the solution of boundary value problem (1.3)–(1.4). Similar equation we can define for the sequence of lower solution also.

Any solution $z(x)$ in D can play the role of $\beta_0(x)$, hence $z(x) \geq \tilde{\alpha}(x)$ and similarly one concludes that $z(x) \leq \tilde{\beta}(x)$. \square

Case II: When $\lambda < 0$.

Theorem 4.2. *Let there exist α_0, β_0 in $C^2[0, 1]$ such that $\beta_0 \geq \alpha_0$ and satisfy*

$$-(x^2 \beta_0'(x))' \geq x^2 f(x, \beta_0), \quad 0 < x < 1; \quad \beta_0'(0) = 0, \quad \beta_0(1) \geq \delta \beta_0(\eta), \quad (4.6)$$

and

$$-(x^2 \alpha_0'(x))' \leq x^2 f(x, \alpha_0), \quad 0 < x < 1; \quad \alpha_0'(0) = 0, \quad \alpha_0(1) \leq \delta \alpha_0(\eta). \quad (4.7)$$

If $f : D_0 \rightarrow R$ is continuous on $D_0 := \{(x, y) \in [0, 1] \times R : \alpha_0 \leq y \leq \beta_0\}$ and there exist $M \geq 0$ such that for all $(x, \tilde{y}), (x, \tilde{w}) \in D_0$

$$\tilde{y} \leq \tilde{w} \implies f(x, \tilde{w}) - f(x, \tilde{y}) \geq -M(\tilde{w} - \tilde{y})$$

then the three point nonlinear SBVP (1.3)–(1.4) has at least one solution in the region D_0 . If \exists a constant λ such that $M + \lambda \leq 0$ and (H'_0) is satisfied then the sequences $\{\beta_n\}$ generated by

$$-(x^2 \beta_{n+1}')' - \lambda x^2 \beta_{n+1} = x^2 F(x, \beta_n), \quad \beta_{n+1}'(0) = 0, \quad \beta_{n+1}(1) = \delta \beta_{n+1}(\eta), \quad (4.8)$$

$$-(x^2 \alpha_{n+1}')' - \lambda x^2 \alpha_{n+1} = x^2 F(x, \alpha_n), \quad \alpha_{n+1}'(0) = 0, \quad \alpha_{n+1}(1) = \delta \alpha_{n+1}(\eta), \quad (4.9)$$

where $F(x, y) = f(x, y) - \lambda y$, with initial iterate β_0 converges monotonically (non-increasing) and uniformly towards a solution $\beta(x)$ of (1.3)–(1.4). Similarly using α_0 as an initial iterate leads to a non-decreasing sequences $\{\alpha_n\}$ converging to a solution $\alpha(x)$. Any solution $Z(x)$ in D_0 must satisfy

$$\alpha(x) \leq Z(x) \leq \beta(x).$$

Proof. Proof follows from the analysis of Theorem 4.1. □

5. NUMERICAL ILLUSTRATIONS

With the help of following examples, we verify our results and show that it is possible to choose a value of “ λ ” so that iterative scheme generates monotone sequences which converge to solution of nonlinear singular problem. Thus these examples validate sufficient conditions derived in the Theorem 4.1 and Theorem 4.2.

Example 5.1. Consider the boundary value problem

$$-y''(x) - \frac{2}{x}y'(x) = \frac{3}{4}e^{y(x)}, \tag{5.1}$$

$$y'(0) = 0, \quad y(1) = \frac{2}{5}y\left(\frac{1}{2}\right). \tag{5.2}$$

Here $f(x, y) = \frac{3}{4}e^y$, $\delta = \frac{2}{5}$, $\eta = \frac{1}{2}$. This problem has $\alpha_0 = 0$ and $\beta_0 = \frac{2-x^2}{3}$ as lower and upper solutions, and it is well ordered case. The nonlinear term is Lipschitz in y and continuous for all value of y , and Lipschitz constant is $M \leq \frac{3}{4}$. Now we can find out a subinterval $R_\lambda = (\xi_1, \xi_2)$ of $(0, j_{-\frac{1}{2},1}^2)$ such that the conditions $M - \lambda \geq 0$ and (H_0) (see Figure 1) are true.

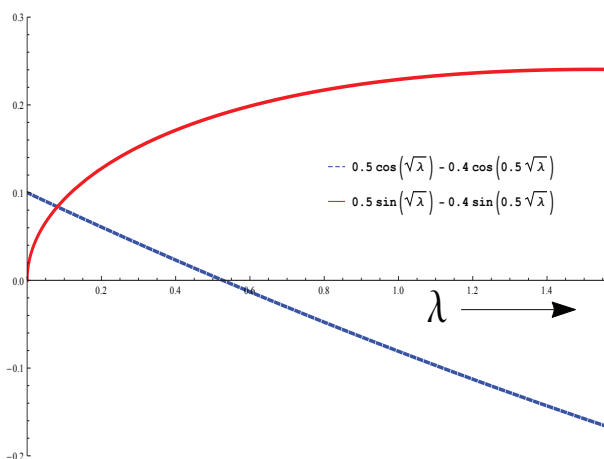


FIGURE 1. Plot of $\left(\eta \sin \sqrt{\lambda} - \delta \sin \sqrt{\lambda} \eta \right)$ and $\left(\eta \cos \sqrt{\lambda} - \delta \cos \sqrt{\lambda} \eta \right)$.

Example 5.2. Consider the boundary value problem

$$-y''(x) - \frac{2}{x}y'(x) = y(x) + 1, \tag{5.3}$$

$$y'(0) = 0, \quad y(1) = \frac{1}{2}y\left(\frac{3}{10}\right). \tag{5.4}$$

Here $f(x, y) = y + 1$, $\delta = \frac{1}{2}$, $\eta = \frac{3}{10}$. This problem has $\alpha_0 = 0$ and $\beta_0 = 2 - x^2$ as lower and upper solutions, and this is a well ordered case. The source term is linear,

Lipschitz in y and continuous for all value of y , and Lipschitz constant is $M = 1$. Now we can find out a subinterval $R_\lambda = (\xi_1, \xi_2)$ of $(0, j_{-\frac{1}{2},1}^2)$ such that the conditions $M - \lambda \geq 0$ and (H_0) (see Figure 2) are true.

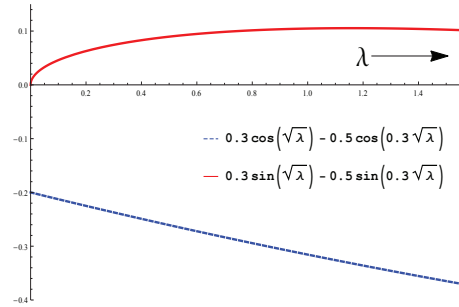


FIGURE 2. Plot of $\left(\eta \sin \sqrt{\lambda} - \delta \sin \sqrt{\lambda} \eta \right)$ and $\left(\eta \cos \sqrt{\lambda} - \delta \cos \sqrt{\lambda} \eta \right)$.

Example 5.3. Consider the boundary value problem

$$-y''(x) - \frac{2}{x}y'(x) = \frac{1}{36} \left[\frac{e^2}{5} - 2(y(x))^3 \right], \tag{5.5}$$

$$y'(0) = 0, \quad y(1) = \frac{7}{10}y \left(\frac{2}{5} \right). \tag{5.6}$$

Here $f(x, y) = \frac{1}{36} \left[\frac{e^2}{5} - 2(y(x))^3 \right]$, $\delta = \frac{7}{10}, \eta = \frac{2}{5}$. This problem has $\alpha_0 = -1$ and $\beta_0 = 1$ as lower and upper solutions, and this is a well ordered case. The nonlinear term is Lipschitz in y and continuous for all value of y , and Lipschitz constant is $M = \frac{1}{6}$. For some λ less than $(-\frac{1}{6})$, (H'_0) (see Figure 3) will be true. Using Mathematica 9.0 and iterative scheme (1.5) we compute upper and lower solutions (see Figure 4).

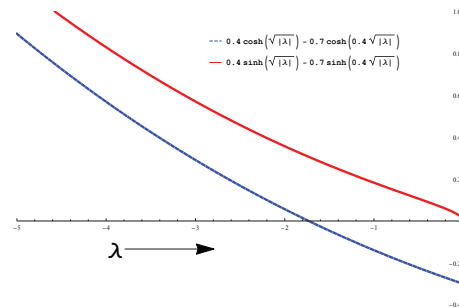


FIGURE 3. Plot of $\left(\eta \sinh \sqrt{|\lambda|} - \delta \sinh \eta \sqrt{|\lambda|} \right)$ and $\left(\eta \cosh \sqrt{|\lambda|} - \delta \cosh \eta \sqrt{|\lambda|} \right)$.

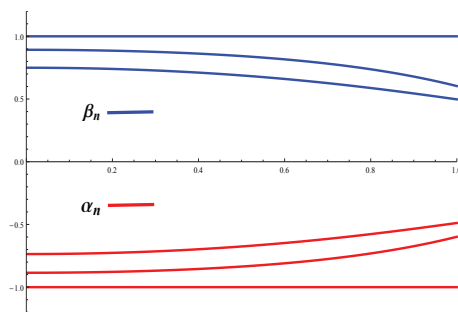


FIGURE 4. Plot of $\beta_n, \alpha_n, n = 0(1)2$ for $\lambda = -10$.

Example 5.4. Consider the boundary value problem

$$-y''(x) - \frac{2}{x}y'(x) = 1 - 2y(x), \tag{5.7}$$

$$y'(0) = 0, \quad y(1) = \frac{1}{10}y\left(\frac{2}{5}\right). \tag{5.8}$$

Here $f(x, y) = 1 - 2y, \delta = \frac{1}{10}, \eta = \frac{2}{5}$. This problem has $\alpha_0 = -1$ and $\beta_0 = 1$ as lower and upper solutions, and this is a well ordered case. The nonlinear term is Lipschitz in y and continuous for all value of y , and Lipschitz constant is $M = 2$. For $\lambda < -2$ we can see that (H'_0) (see Figure 5) will be true. Using Mathematica 9.0 and iterative scheme (1.5) we compute upper and lower solutions (see Figure 6).

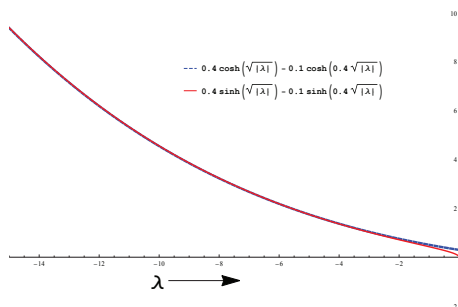


FIGURE 5. Plot of $\left(\eta \sinh \sqrt{|\lambda|} - \delta \sinh \eta \sqrt{|\lambda|}\right)$ and $\left(\eta \cosh \sqrt{|\lambda|} - \delta \cosh \eta \sqrt{|\lambda|}\right)$.

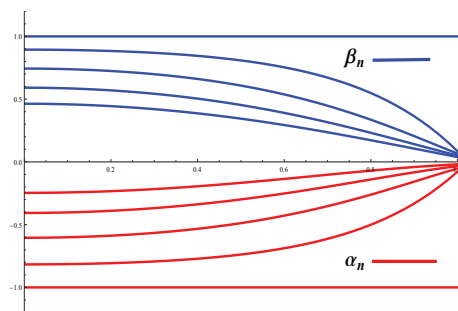


FIGURE 6. Plot of $\beta_n, \alpha_n, n = 0(1)4$ for $\lambda = -24$.

6. CONCLUSION

In this work we establish existence of solutions for a class of nonlinear singular three point boundary value problems. The BVPs of this kind can be considered as generalizations of problems of two point singular BVPs in spherical symmetry, e.g., [2], [3]. We allow the Lipschitz constant to take both positive and negative values. Due to Lack of uniform Anti Maximum principle reversed ordered upper and lower solutions case is not observed. We have used Mathematica to plot solutions for $\frac{\partial f}{\partial y} < 0$ but the same could not be achieved for $\frac{\partial f}{\partial y} > 0$. The work in this paper can further be generalized to a class of singular nonlinear differential equations, e.g.,

$$-(py')' = qf(x, y, py'), \quad 0 < x < 1, \quad p(0) = 0,$$

subject to different kind of multi point boundary conditions, which depend on the nature of p , q and f .

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REFERENCES

- [1] W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, Wiley, New York, 3rd ed., 1977.
- [2] P. L. Chamber, On the solution of the Poisson-Boltzmann equation with the application to the theory of thermal explosions, *J. Chem. Phys.*, **20** (1952) 1795–1797.
- [3] J. B. Keller, Electrohydrodynamics I. The equilibrium of a charged gas in a container, *J. Rational Mech. Anal.*, **5** (1956) 715–724.
- [4] R. D. Russell and L. F. Shampine, Numerical methods for singular boundary value problems, *SIAM J. Numer. Anal.*, **12** (1975) 13–36.
- [5] M. M. Chawla, P. N. Shivkumar, On the existence of solutions of a class of singular nonlinear two-point boundary value problems, *J. Comput. Appl. Math.*, **19** (1987) 379–388.
- [6] A. K. Verma, Monotone iterative method and zero's of Bessel functions for nonlinear singular derivative dependent BVP in the presence of upper and lower solutions, *Nonlinear Anal.*, **74** (14) (2011) 4709–4717.
- [7] A. K. Verma and R. P. Agarwal, Upper and lower solutions method for regular singular differential equations with quasi-derivative boundary conditions, *Communications in Nonlinear Science and Numerical Simulation*, **17**, Issue 12, 4551–4558.
- [8] A. K. Verma, R. P. Agarwal and L. Verma, Bessel functions and singular BVPs arising in physiology in the presence of upper and lower solutions in reverse order, *Journal of Applied Mathematics and Computing*, **39**, Numbers 1-2 (2012), 445–457.
- [9] Zhang, Y., Positive solutions of singular sublinear Dirichlet boundary value problems, *SIAM J. Math. Anal.*, **26** (1995), 329–339.
- [10] M. Gregus, F. Neumann, and F. M. Arscott, Three-point boundary value problems for differential equations, *J. London Math. Soc.*, **3**, (1971) 429–436.

- [11] A. R. Aftabizadeh, C. P. Gupta, and Jian-Ming Xu, Existence and uniqueness theorems for three-point boundary value problem, *SIAM J. Math. Anal.*, **20** (1989) 716–726.
- [12] C. P. Gupta, S. I. Trofimchuk, Existence of a solution of a three-point boundary value problem and spectral radius of a related linear operator, *Nonlinear Anal.*, **34** (1998) 489–507.
- [13] R. Ma, Existence of solutions of nonlinear m-point boundary-value problems, *J. Math. Anal. Appl.*, **256** (2001) 556–567.
- [14] J. Henderson, B. Karna and C. C. Tisdell, Existence of solutions for three-point boundary value problems for second order equations, *Proc. Amer. Math. Soc.*, **133** (2005) 1365–1369.
- [15] F. Li, M. Jia, X. Liu, C. Li, G. Li, Existence and uniqueness of solutions of second-order three-point boundary value problems with upper and lower solutions in the reversed order, *Nonlinear Anal.*, **68** (2008) 2381–2388.
- [16] J. Nieto, An abstract monotone iterative technique, *Nonlinear Analysis*, **28** (1997) 1923–1933.
- [17] D. Bai, H. Feng, Eigenvalue for a singular second order three-point boundary value problem, *Journal of Applied Mathematics and Computing*, **38** (2012) 443–452.
- [18] M. Singh and A. K. Verma, Picard type iterative scheme with initial iterates in reverse order for a class of nonlinear three point BVPs, *International Journal of Differential Equation*, **Volume 2013** (2013), Article ID 728149, 6 pages.
- [19] M. Singh and A. K. Verma, On a monotone iterative method for a class of three point nonlinear nonsingular BVPs with upper and lower solutions in reverse order, *Journal of Applied Mathematics and Computing*, **43** (2013) 99–114.
- [20] E. Picard, Sur l'application des méthodes d'approximations succesives à l'étude de certains équations différentielles ordinaires, *Journal de Mathématiques Pures et Appliquées*, **9** (1893) 217–271.
- [21] A. Erdryi, Ed., *Higher Transcendental Functions*, Vol. 1I, Bateman Manuscript Project, McGraw-Hill, New York, 1953.