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ON COUPLED AND TRIPLED FIXED POINT THEORY OF MULTI-VALUED CONTRACTION MAPPINGS IN PARTIALLY ORDERED METRIC SPACES

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ABSTRACT. In this paper, we present a new and simple approach to coupled fixed point theory of multi-valued maps. By using our method, we first give a very simple proof of the recent coupled fixed point theorems established by Samet and Vetro [B. Samet, C. Vetro, Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces, Nonlinear Analysis **74**(2011), 4260-4268.] Then, we use our technique to present a new tripled fixed point theorem in a general setting.

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1. Introduction and Preliminaries

Existence of a fixed point for contraction type mappings in partially ordered metric spaces has been considered recently by many authors [1–7]. In [3] and [7], the authors proved some coupled fixed point theorems and noted their results can be used to investigate a large class of problems.

Recently, Samet and Vetro [10] introduced the concept of a coupled fixed point for a multi-valued mapping and obtained some existence theorems for contractive type multi-valued mappings in partially ordered metric spaces.

In this paper, we present a new and simple approach to coupled fixed point theory of multi-valued maps. By using our technique, we first show that the main results of Samet and Vetro [10] are immediate consequences of the well-known fixed point theorems of Ćirić [11]. Next, motivated by the work of Borcut and Berinde [12], we

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introduce the concept of a tripled fixed point of a multi-valued map and present a new tripled fixed point result for multi-valued contraction maps in a very general setting.

Let (X, d) be a metric space. We denote by $CL(X)$ the collection of all nonempty closed subsets of X. For $A \subseteq X$ and $x \in X$, let $d(x, A) = \inf_{a \in A} d(x, a)$.

The following theorems follow from Theorem 2.1 and Theorem 2.2 in [11] and Theorem 2.1 in [13] and will be used in the next sections.

Theorem 1.1 ([11], Theorem 2.1)**.** *Let (X, d) be a complete metric space, $\emptyset \neq Q \subseteq X$ and let $T : X \to CL(X)$ be a multi-valued map with $T(Q) \subseteq Q$. Suppose that the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, Tx), x \in X$ is lower semicontinuous and that there exists a function $\varphi : [0, \infty) \to [a, 1)$, $0 < a < 1$, satisfying*

$$\limsup_{r \to t^+} \varphi(r) < 1 \text{ for each } t \in [0, \infty).$$

Assume that for any $x \in Q$ there is a $y \in Tx$ satisfying the following conditions:

$$\sqrt{\varphi(f(x))}d(x, y) \leq f(x) \text{ and } f(y) \leq \varphi(f(x))d(x, y).$$

Then T has a fixed point.

Theorem 1.2 ([11], Theorem 2.2)**.** *Let (X, d) be a complete metric space, $\emptyset \neq Q \subseteq X$ and let $T : X \to CL(X)$ be a multi-valued map with $T(Q) \subseteq Q$. Suppose that the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, T(x)), x \in X$ is lower semicontinuous and that there exists a function $\varphi : [0, \infty) \to [a, 1)$, $0 < a < 1$, satisfying*

$$\limsup_{r \to t^+} \varphi(r) < 1 \text{ for each } t \in [0, \infty).$$

Assume that for any $x \in Q$ there is a $y \in Tx$ satisfying the following conditions:

$$\sqrt{\varphi(d(x, y))}d(x, y) \leq f(x) \text{ and } f(y) \leq \varphi(d(x, y))d(x, y).$$

Then T has a fixed point.

Theorem 1.3 ([13], Theorem 2.1)**.** *Let (X, d) be a complete metric space, $\emptyset \neq Q \subset X$ and let $T : X \to CL(X)$ be a multi-valued map with $T(Q) \subseteq Q$. Suppose that the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, T(x)), x \in X$, is lower semicontinuous and assume that for each $x \in Q$ there exists a $y \in Tx$ satisfying*

$$\alpha(f(x))d(x, y) \leq f(x) \text{ and } f(y) \leq \beta(f(x))d(x, y),$$

where $\alpha : [0, diam(X)) \to (0, 1]$ and $\beta : [0, diam(X)) \to [0, 1)$ satisfy that

$$\liminf_{r \to t^+} \alpha(r) > 0, \ \limsup_{r \to t^+} \frac{\beta(r)}{\alpha(r)} < 1, \ \forall\, t \in [0, diam(X)),$$

and one of α and β is nondecreasing. Then, T has a fixed point.

Definition 1.4 ([10])**.** Let X be a nonempty set and $F : X \times X \to 2^X$ be a given multi-valued map. We say that $(x, y) \in X \times X$ is a coupled fixed point of F if

$$x \in F(x, y) \text{ and } y \in F(y, x).$$

2. **Coupled Fixed Point Theory**

Let (X, d) be a complete metric space. Throughout this section, let $M = X \times X$ and let ρ be the metric on M which is defined by

$$\rho((x, y), (u, v)) = d(x, u) + d(y, v).$$

Then it is straightforward to show that (M, ρ) is a complete metric space.

Definition 2.1 ([10])**.** Let (X, d) be a metric space endowed with a partial order $\preceq$. Let $G : X \to X$ be a given mapping. Let

$$\Delta = \{(x, y) \in X \times X : G(x) \preceq G(y)\}.$$

Let $F : X \times X \to 2^X$ be a given multi-valued map. F is said to be a Δ-symmetric mapping if

$$(x, y) \in \Delta \Rightarrow F(x, y) \times F(y, x) \subseteq \Delta.$$

Now we give a very simple proof of the main results of Samet and Vetro [10].

Theorem 2.2 ([8], Theorem 2.1)**.** *Let (X, d) be a complete metric space endowed with a partial order $\preceq$ and we suppose that $\Delta \neq \emptyset$. Let $F : X \times X \to CL(X)$ be a Δ-symmetric mapping. Suppose that the function $f : X \times X \to [0, \infty)$ defined by*

$$f(x, y) = d(x, F(x, y)) + d(y, F(y, x)) \text{ for all } x, y \in X,$$

is lower semicontinuous and that there exists a function $\varphi : [0, \infty) \to [a, 1)$, $0 < a < 1$, satisfying

$$\limsup_{r \to t^+} \varphi(r) < 1 \text{ for each } t \in [0, \infty).$$

Assume that for any $(x, y) \in \Delta$ there exist a $u \in F(x, y)$ and a $v \in F(y, x)$ satisfying

$$\sqrt{\varphi(f(x, y))}[d(x, u) + d(y, v)] \leq f(x, y)$$

and

$$f(u, v) \leq \varphi(f(x, y))[d(x, u) + d(y, v)].$$

Then F has a coupled fixed point.

Proof. Let $T : M \to M$ be defined by $T(x, y) = F(x, y) \times F(y, x)$, for each $(x, y) \in M$. Then (x, y) is a coupled fixed point of F if and only if (x, y) be a fixed point of T. Since F is Δ-symmetric then $T(\Delta) \subseteq \Delta$. From our assumptions, we have

$$f(x, y) = \rho((x, y), T(x, y)) = d(x, F(x, y)) + d(y, F(y, x)), \ (x, y) \in M$$

is lower semicontinuous and for any $(x, y) \in \Delta$ there exists $(u, v) \in T(x, y)$ such that

$$\sqrt{\varphi(f(x,y))}\rho((x,y),(u,v)) \leq f(x,y)$$

and

$$f(u,v) \leq \varphi(f(x,y))\rho((x,y),(u,v)).$$

Then by Theorem 1.1, T has a fixed point and so we are finished. □

Theorem 2.3 ([10], Theorem 2.2). *Let (X, d) be a complete metric space endowed with a partial order $\preceq$ and we suppose that $\Delta \neq \emptyset$. Let $F : X \times X \to CL(X)$ be a Δ-symmetric mapping. Suppose that the function $f : X \times X \to [0, \infty)$ defined by*

$$f(x,y) = d(x, F(x,y)) + d(y, F(y,x)) \text{ for all } x, y \in X,$$

is lower semicontinuous and that there exists a function $\varphi : [0, \infty) \to [a, 1)$, $0 < a < 1$, satisfying

$$\limsup_{r \to t^+} \varphi(r) < 1 \text{ for each } t \in [0, \infty).$$

Assume that for any $(x, y) \in \Delta$ there exist a $u \in F(x, y)$ and a $v \in F(y, x)$ satisfying

$$\sqrt{\varphi(d(x,u)+d(y,v))}[d(x,u)+d(y,v)] \leq f(x,y)$$

and

$$f(u,v) \leq \varphi(d(x,u)+d(y,v))[d(x,u)+d(y,v)].$$

Then F has a coupled fixed point.

Proof. Let $T : M \to M$ be defined by $T(x, y) = F(x, y) \times F(y, x)$, for each $(x, y) \in X \times X$. Since F is Δ-symmetric then $T(\Delta) \subseteq \Delta$. From our assumptions, we have

$$f(x,y) = \rho((x,y), T(x,y)) = d(x, F(x,y)) + d(y, F(y,x)),\ (x,y) \in M$$

is lower semicontinuous and for any $(x, y) \in \Delta$ there exists $(u, v) \in T(x, y)$ such that

$$\sqrt{\varphi(\rho((x,u),(y,v))}\rho((x,y),(u,v)) \leq \rho((x,y), T(x,y))$$

and

$$\rho((u,v), T(u,v)) \leq \varphi(\rho((x,y),(u,v))\rho((x,y),(u,v)).$$

Then by Theorem 1.2, T has a fixed point and so F has a coupled fixed point. □

Theorem 2.4. *Let (X, d) be a complete metric space, $\emptyset \neq Q_0 \subseteq X$ and let $F : X \times X \to CL(X)$ be a multi-valued map with $F(Q_0 \times Q_0) \subseteq Q_0$. Suppose that the function $f : X \times X \to \mathbb{R}$ defined by*

$$f(x,y) = d(x, F(x,y)) + d(y, F(y,x)) \text{ for all } x, y \in X,$$

is lower semicontinuous and assume that for each $(x, y) \in Q_0 \times Q_0$ there exist a $u \in F(x, y)$ and a $v \in F(y, x)$ satisfying

$$\alpha(f(x,y))[d(x,u)+d(y,v)] \leq f(x,y) \text{ and } f(u,v) \leq \beta(f(x,y))[d(x,u)+d(y,v)],$$

where $\alpha : [0, diam(X)) \to (0, 1]$ *and* $\beta : [0, diam(X)) \to [0, 1)$ *satisfy that*

$$\liminf_{r \to t^+} \alpha(r) > 0, \ \limsup_{r \to t^+} \frac{\beta(r)}{\alpha(r)} < 1, \ \forall \ t \in [0, diam(X)),$$

and one of α *and* β *is nondecreasing. Then,* F *has a coupled fixed point.*

Proof. Let $T : M \to M$ be defined by $T(x, y) = F(x, y) \times F(y, x)$, for each $(x, y) \in X \times X$. Let $Q = Q_0 \times Q_0$. Since $F(Q_0 \times Q_0) \subseteq Q_0$ then $T(Q) \subseteq Q$. From our assumptions, we have

$$f(x, y) = \rho((x, y), T(x, y)) = d(x, F(x, y)) + d(y, F(y, x)), \ (x, y) \in M$$

is lower semicontinuous and for any $(x, y) \in Q$ there exists a $(u, v) \in T(x, y)$ such that

$$\alpha(f(x, y))\rho((x, y), (u, v)) \leq f(x, y) \text{ and } f(u, v) \leq \beta(f(x, y))\rho((x, y), (u, v)).$$

Then by Theorem 1.3, T has a fixed point and so F has a coupled fixed point. □

3. Tripled Fixed Point Theory

Definition 3.1 ([14])**.** Let X be a nonempty set and $F : X \times X \times X \to 2^X$ be a multi-valued map. We say that $(x, y, z) \in X \times X \times X$ is a tripled fixed point of F if

$$x \in F(x, y, z), \ y \in F(y, z, x) \text{ and } z \in F(z, x, y).$$

Definition 3.2. Let $F : X \times X \times X \to 2^X$ be a given multi-map and $\emptyset \neq \overline{\Delta} \subseteq X \times X \times X$. We say that F is $\overline{\Delta}$-symmetric if

$$(x, y, z) \in \overline{\Delta} \Rightarrow F(x, y, z) \times F(y, z, x) \times F(z, x, y) \subseteq \overline{\Delta}.$$

Let (X, d) be a complete metric space. Throughout this section, let $M = X \times X \times X$ and let ρ be the metric on M which is defined by

$$\rho((x, y, z), (u, v, w)) = d(x, u) + d(y, v) + d(z, w).$$

Then it is straightforward to show that (M, ρ) is a complete metric space.

Now, we are ready to prove our new tripled fixed point results for multi-maps.

Theorem 3.3. *Let* (X, d) *be a complete metric space and we assume that* $\emptyset \neq \overline{\Delta} \subseteq X \times X \times X$*. Let* $F : X \times X \times X \to CL(X)$ *be a* $\overline{\Delta}$*-symmetric mapping. Suppose that the function* $f : X \times X \times X \to [0, \infty)$ *defined by*

$$f(x, y, z) = d(x, F(x, y, z)) + d(y, F(y, z, x)) + d(z, F(z, x, y)) \ ,$$

$x, y, z \in X$*, is lower semicontinuous and that there exists a function* $\varphi : [0, \infty) \to [a, 1)$*,* $0 < a < 1$*, satisfying*

$$\limsup_{r \to t^+} \varphi(r) < 1 \textit{ for each } t \in [0, \infty).$$

Suppose that for each $(x,y,z) \in \overline{\Delta}$ *there exist* $u \in F(x,y,z)$, $v \in F(y,z,x)$ *and* $w \in F(z,x,y)$ *satisfying*

$$\alpha(f(x,y,z))[d(x,u)+d(y,v)+d(z,w)] \leq f(x,y,z)$$

and

$$f(u,v,w) \leq \beta(f(x,y,z))[d(x,u)+d(y,v)+d(z,w)],$$

where $\alpha : [0, diam(X)) \to (0,1]$ *and* $\beta : [0, diam(X)) \to [0,1)$ *satisfy that*

$$\liminf_{r\to t^+} \alpha(r) > 0, \ \limsup_{r\to t^+} \frac{\beta(r)}{\alpha(r)} < 1, \ \forall \ t \in [0, diam(X)),$$

and one of α *and* β *is nondecreasing. Then* T *has a fixed point.*

Proof. Let $T : M \to M$ be defined by

$$T(x,y,z) = F(x,y,z) \times F(y,z,x) \times F(z,x,y),$$

for each $(x,y,z) \in X \times X \times X$. Then (x,y,z) is a tripled fixed point of F if and only if (x,y,z) be a fixed point of T. Since F is $\overline{\Delta}$-symmetric then $T(\overline{\Delta}) \subseteq \overline{\Delta}$. From our assumptions, we have

$$\alpha(f(x,y,z))\rho((x,y,z),(u,v,w)) \leq f(x,y,z)$$

and

$$f(u,v,w) \leq \beta(f(x,y,z))\rho((x,y,z),(u,v,w)).$$

Then all of the assumptions of Theorem 1.3 are satisfied and then T has a fixed point. $\square$

From Theorem 1.2 and using our technique, we obtain the following result.

Theorem 3.4. *Let* (X,d) *be a complete metric space,* $\emptyset \neq Q_0 \subseteq X$ *and let* $F : X \times X \times X \to CL(X)$ *be a multimap with* $F(Q_0 \times Q_0 \times Q_0) \subseteq Q_0$. *Suppose that the function* $f : X \times X \times X \to [0,\infty)$ *defined by*

$$f(x,y,z) = d(x,F(x,y,z)) + d(y,F(y,z,x)) + d(z,F(z,x,y)) \ ,$$

$x,y,z \in X$, *is lower semicontinuous and that there exists a function* $\varphi : [0,\infty) \to [a,1)$, $0 < a < 1$, *satisfying*

$$\limsup_{r\to t^+} \varphi(r) < 1 \textit{ for each } t \in [0,\infty).$$

Suppose that for each $(x,y,z) \in Q_0 \times Q_0 \times Q_0$ *there exist* $u \in F(x,y,z)$, $v \in F(y,z,x)$ *and* $w \in F(z,x,y)$ *satisfying*

$$\sqrt{\varphi(d(x,u)+d(y,v)+d(z,w))}[d(x,u)+d(y,v)+d(z,w)] \leq f(x,y,z)$$

and

$$f(u,v,w) \leq \varphi(d(x,u)+d(y,v)+d(z,w))[d(x,u)+d(y,v)+d(z,w)].$$

Then F has a tripled fixed point.

Remark 3.5. By using our technique, coupled and tripled counterparts of well-known multi-valued fixed point theorems in metric spaces could be easily deduced.

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