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MULTIPLE POSITIVE PERIODIC SOLUTIONS FOR A NONLINEAR FIRST ORDER FUNCTIONAL DIFFERENCE EQUATION WITH APPLICATION TO HEMATOPOIESIS MODEL

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ABSTRACT. Sufficient conditions are obtained for the existence of at least three positive T-periodic solutions for the first order functional difference equation

$$\Delta x(n) = -a(n)x(n) + f(n, x(h(n))).$$

The Leggett-Williams multiple fixed point theorem has been used to prove our results. We have applied our results to Hematopoiesis models in population dynamics and obtained an interesting result. The result is new in the literature.

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1. Introduction

Let R denote the set of real numbers, Z the set of integers, R_+ the set of positive reals, and $T \geq 1$ be an integer. Let $[a, b] = \{a, a+1, ..., b\}$ for $a < b, a, b \in Z$, $\prod_{n=a}^{b} u(n)$ denote the product of $u(n)$ from $n = a$ to $n = b$ with the understanding that $\prod_{n=a}^{b} u(n) = 1$ for all $a > b$.

In this paper, we investigate the existence of multiple positive periodic solutions for the first order functional difference equation

$$\Delta x(n) = -a(n)x(n) + f(n, x(h(n))) \tag{1.1}$$

where $a(n), b(n)$ and $h(n)$, $n \in Z$, are T-periodic positive sequences with $T \geq 1$, $0 < a(n) < 1, n \in Z$, $f(n, x)$ is T-periodic in n and is continuous in x for each $n \in Z$, and $\Delta x(n) = x(n+1) - x(n)$.

Equation (1.1) is the discrete analog of the first order scalar delay differential equation

$$x'(t) = -a(t)x(t) + f(t, x(h(t))). \tag{1.2}$$

Much attention have been given in recent years for the existence of positive periodic solutions of the Eqs. (1.2). One may refer the works in [3, 7, 10, 12, 13] and references cited there in. The results obtained in [3, 7, 10, 12, 13] deal with the existence of at

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least one or two positive periodic solutions of (1.2). In [1, 16, 17], attempts have been made by the authors to study the existence of three positive T-periodic solutions of (1.2) using the Leggett-Williams multiple fixed point theorem [11]. It has been observed that very little is known on the existence of positive periodic solutions of (1.1). It is only recently that attentions has been given to study the existence of positive periodic solutions of (1.1). The works of Ma and Yu [13], Raffoul [18] and Zeng [22] may be treated as a basis for the study of positive periodic solutions of (1.1). They used Krasnoselskii's fixed point theorem [2] to prove the results. Motivated by the work in Raffoul [18], Liu [12] obtained several sufficient conditions for the existence of at least one T-periodic solution for the functional difference equation

$$\Delta x(n) + a(n)x(n) = f(n, x(n), x(n-\tau_1(n)), \ldots, x(n-\tau_m(n)))$$

where $\{a(n); n \in Z\}$ and $\{\tau_i(n); n \in Z\}$, $i = 1, 2, \ldots, m$ are T-periodic sequences with $T \geq 1$, $f(n, u)$ is T-periodic in n for each $u = (x_0, \ldots, x_m, x_{m+1}) \in R^{m+2}$, and is continuous in u for each $n \in Z$.

In this paper, we obtain several sufficient conditions for the existence of at least three positive T-periodic solutions of (1.1) using the Leggett-Williams multiple fixed point theorem [11]. As dealt by the authors in the references, we shall obtain an equivalent summation series operator of (1.1) using a Green's kernel. Then applying the bounds on the Green's kernel, we shall prove that the operator satisfies the conditions of the Leggett-Williams multiple fixed point theorem. Our Corollaries 2.4 and 2.8 give a partial answer to an open problem proposed in [18, p. 07]. Some of the open problems in [18] has been proved in [13] using Krasnoselskii fixed point theorem.

The following concept will be used in the statement of the Leggett-Williams fixed point theorem. Let X be a Banach space and K be a cone in X. A mapping ψ is said to be a concave nonnegative continuous functional on K if $\psi : K \to [0, \infty)$ is continuous and

$$\psi(\mu x + (1-\mu)y) \geq \mu\psi(x) + (1-\mu)\psi(y), \quad x, y \in K,\ \mu \in [0, 1].$$

Let c_1, c_2, c_3 be positive constants. With K and X as defined above, we define $K_{c_1} = \{y \in K : \|y\| < c_1\}$, $K(\psi, c_2, c_3) = \{y \in K : c_2 \leq \psi(y), \|y\| < c_3\}$.

Theorem 1.1 (Leggett-Williams fixed point theorem [11]). *Let $(X, \|\cdot\|)$ be a Banach space and $K \subset X$ a cone, and c_4 a positive constant. Suppose there exists a concave nonnegative continuous functional ψ on K with $\psi(u) \leq \|u\|$ for $u \in \bar{K}_{c_4}$ and let $A : \bar{K}_{c_4} \to \bar{K}_{c_4}$ be a completely continuous mapping. Assume that there are numbers c_1, c_2, c_3, c_4 with $0 < c_1 < c_2 < c_3 \leq c_4$ such that*

(i) *$\{u \in K(\psi, c_2, c_3) : \psi(u) > c_2\} \neq \phi$, and $\psi(Au) > c_2$ for all $u \in K(\psi, c_2, c_3)$;*

(ii) *$\|Au\| < c_1$ for all $u \in \bar{K}_{c_1}$;*

(iii) *$\psi(Au) > c_2$ for all $u \in K(\psi, c_2, c_4)$ with $\|Au\| > c_3$.*

Then A has at least three fixed points u_1, u_2, u_3 in $\bar{K}_{c_4}$. Furthermore, $\|u_1\| \le c_1 < \|u_2\|$, and $\psi(u_2) < c_2 < \psi(u_3)$.

In this article, let X be the set of all periodic sequences which forms a Banach space under the norm

$$\|x\| = \max_{n\in[0,T-1]} |x(n)|. \tag{1.3}$$

2. Main Results

One may observe that (1.1) is equivalent to

$$x(n) = \sum_{s=n}^{n+T-1} G(n,s)b(s)f(s,x(h(s))), \tag{2.1}$$

where $G(n,s) = \frac{\prod_{\theta=s+1}^{n+T-1}(1-a(\theta))}{1-\prod_{\theta=0}^{T-1}(1-a(\theta))}$, $n \le s \le n+T-1$, is the Green's kernel satisfying the property:

$$0 < \alpha = \frac{\delta}{1-\delta} \le G(n,s) \le \frac{1}{1-\delta} = \beta \tag{2.2}$$

where $0 < \delta = \frac{\alpha}{\beta} = \prod_{n=0}^{T-1}(1-a(n)) < 1$.

We consider the Banach space as defined in (1.3). Define an operator $A : X \to X$ by

$$(Ax)(n) = \sum_{s=n}^{n+T-1} G(n,s)b(s)f(s,x(h(s))). \tag{2.3}$$

Using (2.2) we obtain

$$\|Ax\| \le \beta \sum_{s=n}^{n+T-1} b(s)f(s,x(h(s)))$$

and hence

$$Ax \ge \alpha \sum_{s=n}^{n+T-1} b(s)f(s,x(h(s))) \ge \frac{\alpha}{\beta}\|Ax\|.$$

In view of the above inequality, we define a cone $K \subset X$ as

$$K = \{x \in X : x(n) \ge 0, n \in Z, x(n) \ge \delta\|x\|\}.$$

Then $A(K) \subset K$. The existence of a positive periodic solution of (1.1) is equivalent to the existence of a fixed point of A in K. Here we use the Leggett-Williams multiple fixed point theorem, that is, Theorem 1.1 to obtain the existence of three fixed points of A in K. A small exercise shows that $A : K \to K$ is completely continuous.

For the rest of the paper, we denote

$$f^\lambda = \limsup_{x\to\lambda} \max_{0\le n\le T-1} \frac{f(n,x)}{x}, \quad \lambda = 0, \infty.$$

Throughout the paper, we consider a nonnegative concave functional ψ on the cone K given by

$$\psi(x) = \min_{0 \le n \le T-1} x(n).$$

Assume that the following hold:

(H_1) There exists a constant $c_1 > 0$ such that

$$\beta \sum_{n=0}^{T-1} f(n, x(h(n))) < c_1 \text{ for } 0 \le x \le c_1.$$

(H_2) There exists a constant $c_2 > c_1 > 0$ such that

$$\alpha \sum_{n=0}^{T-1} f(n, x(h(n))) > c_2 \text{ for } c_2 \le x \le \frac{c_2}{\delta}.$$

(H_3) There exists a constant $c_4 > \frac{c_2}{\delta}$ such that

$$\beta \sum_{n=0}^{T-1} f(n, x(h(n))) \le c_4 \text{ for } 0 \le x \le c_4.$$

With the above defined concave functional ψ on K and the conditions $(H_1), (H_2)$ and (H_3), we observe that the conditions of the Theorem 1.1 are satisfied. Hence the operator Ax defined in (2.3) has at least three positive T-periodic solutions. This leads to the following theorem:

Theorem 2.1. *Under the above assumptions $(H_1), (H_2)$ and (H_3), Eq. (1.1) has at least three positive T-periodic solutions.*

Theorem 2.2. *Assume that there exists a constant $c_2 > 0$ such that (H_2) holds. Further, suppose that*

(H_4) $\beta \limsup_{x \to \infty} \sum_{n=0}^{T-1} \frac{f(n,x(h(n)))}{\|x\|} < 1$

and

(H_5) $\beta \limsup_{x \to 0} \sum_{n=0}^{T-1} \frac{f(n,x(h(n)))}{\|x\|} < 1.$

Then (1.1) has at least three positive T-periodic solutions.

Proof. By (H_4), there exists a constant $\epsilon \in (0, 1)$ and a real $\delta > 0$ such that

$$\beta \sum_{n=0}^{T-1} f(n, x(h(n))) < \epsilon \|x\| \text{ for } x \ge \delta.$$

Set

$$\gamma = \max_{0 \le x \le \delta} \beta \sum_{n=0}^{T-1} f(n, x(h(n))).$$

Then $\beta \sum_{n=0}^{T-1} f(n, x(h(n))) < \epsilon \|x\| + \gamma$ for $x \ge 0$. Choose

$$c_4 > \max\left\{\frac{\gamma}{1-\epsilon}, \frac{c_2}{\delta}\right\}.$$

Then for $x \in \overline{K}_{c_4}$,

$$\beta \sum_{n=0}^{T-1} f(n, x(h(n))) < \epsilon\|x\| + \gamma \leq \epsilon c_4 + \gamma \leq c_4$$

which implies (H_3).

Next from (H_5), it follows that for each $\epsilon_1 \in (0,1)$, there exists a $\delta_1 > 0$, $\delta \in (0, c_2)$ such that

$$\beta \sum_{n=0}^{T-1} f(n, x(h(n))) < \epsilon_1\|x\| \text{ for } x \leq \delta_1.$$

Choose $c_1 \in (0, \delta_1)$. Then $c_1 < c_2$ and for $x \in \overline{K}_{c_1}$,

$$\beta \sum_{n=0}^{T-1} f(n, x(h(n))) < \epsilon_1\|x\| < \|x\| \leq c_1,$$

which implies (H_1). Hence by Theorem 2.1, Eq.(1.1) has at least three positive T-periodic solutions. This completes the proof of the theorem. □

Theorem 2.3. *Let $f^\infty < \frac{1-\delta}{T}$ and $f^0 < \frac{1-\delta}{T}$ hold. Furthermore, suppose that there exists a constant $c_2 > 0$ such that*
(H_6) $\quad f(n,x) \geq \frac{1-\delta}{\delta T} c_2 \quad$ *for* $\quad c_2 \leq x \leq \frac{c_2}{\delta} \quad$ *and* $\quad 0 \leq n \leq T-1$.
Then (1.1) *has at least three positive T-periodic solutions.*

Proof. Clearly, $f^\infty < \frac{1-\delta}{T}$ and $f^0 < \frac{1-\delta}{T}$ implies (H_4) and (H_5) respectively. The condition (H_6) implies the condition (H_2). Consequently, by Theorem 2.2, (1.1) has at least three positive T-periodic solutions. The Theorem is proved. □

Remark 2.4. We observe Theorems 2.2–2.3 are different versions of Theorem 2.1.

Corollary 2.5. *Let $f^\infty = 0, f^0 = 0$ and assume that there exists a constant $c_2 > 0$ such that (H_6) holds. Then* (1.1) *has at least three positive T-periodic solutions.*

Remark 2.6. Corollary 2.5 answers the following open problem proposed in [18, p. 07]:

What can be said about Eq. (1.1) *when $f^\infty = 0$ and $f^0 = 0$.*

In fact, with an additional condition, we have shown that (1.1) has at least three positive T-periodic solutions. Raffoul [18] considered the equation

$$x(n+1) = a(n)x(n) + \lambda u(n) f(x(h - \tau(n))). \tag{2.4}$$

Equation (1.1) is equivalent to

$$x(n+1) = (1 - a(n))x(n) + f(n, x(h(n))). \tag{2.5}$$

Setting $b(n) = 1 - a(n)$ and $\lambda = 1$, we observe that (2.4) and (2.5) are equivalent. So far, we have obtained results if $0 < b(n) < 1$ holds. In the following, we give some analogous results for $b(n) > 1$. In this case, the Green's kernel is negative. Hence a proper adjustment on the bound of the Green's kernel is given. Equation (2.5) is equivalent to (2.1) where $G(n,s) = \frac{\prod_{\theta=s+1}^{n+T-1} b(\theta)}{1-\prod_{\theta=n}^{n+T-1} b(\theta)}$ is the Green's kernel satisfying the property

$$\alpha_1 = \frac{\prod_{s=0}^{T-1} b^{-1}(s)}{|1-\prod_{s=n}^{n+T-1} b(s)|} \leq G(n,s) \leq \frac{\prod_{s=0}^{T-1} b(s)}{|1-\prod_{s=n}^{n+T-1} b(s)|} = \beta_1$$

for all $s \in [n, n+T-1]$. Set $\sigma = \frac{\alpha_1}{\beta_1} = (\prod_{s=0}^{T-1} b^{-1}(s))^2$.

We consider the Banach space as defined in (1.3) and an operator as in (2.3). Define a cone K in X by

$$K = \{x \in X; x(n) \geq 0, x(n) \geq \sigma \|x\|\}.$$

Then it is easy to prove that $A(K) \subset K$ and is completely continuous. If we proceed as before in the proof of the earlier theorems given in the paper, we have the following:

Theorem 2.7. *Suppose that there exist constants* $0 < c_1 < c_2$ *and* $c_4 > 0$ *such that*
(H_7) $\quad \beta_1 \sum_{n=0}^{T-1} f(n, x(h(n))) < c_1$ *for* $0 \leq x \leq c_1$,
(H_8) $\quad \alpha_1 \sum_{n=0}^{T-1} f(n, x(h(n))) > c_2$ *for* $c_2 \leq x \leq \frac{c_2}{\sigma}$,
and
(H_9) $\quad \beta_1 \sum_{n=0}^{T-1} f(n, x(h(n))) \leq c_4$ *for* $0 \leq x \leq c_4$.
Then (1.1) *has at least three positive* T*-periodic solutions.*

Proceeding as in the lines of Theorem 2.2, we can prove the following theorem:

Theorem 2.8. *Suppose that there exists a constant* $c_2 > 0$ *such that* (H_8) *holds. Further assume that*
(H_4) $\quad \beta_1 \limsup_{x\to\infty} \sum_{n=0}^{T-1} \frac{f(n,x(h(n)))}{\|x\|} < 1$
and
(H_5) $\quad \beta_1 \limsup_{x\to 0} \sum_{n=0}^{T-1} \frac{f(n,x(h(n)))}{\|x\|} < 1$.
Then (1.1) *has at least three positive* T*-periodic solutions.*

Theorem 2.9. *Let* $f^\infty < \frac{1}{\beta_1 T}$ *and* $f^0 < \frac{1}{\beta_1 T}$ *hold, and assume that there exists a constant* $c_2 > 0$ *such that* (H_8) *holds. Then* (1.1) *has at least three positive* T*-periodic solutions.*

Corollary 2.10. *Let* $f^\infty = 0, f^0 = 0$ *and assume that there exists a constant* $c_2 > 0$ *such that* (H_8) *hold. Then* (1.1) *has at least three positive* T*-periodic solutions.*

Our Corollary 2.10 answers the open problem (3) proposed by Raffoul in [18].

3. Application to Hematopoiesis Model

As a particular case of (1.1), we have the scalar equation

$$\Delta u(n) = -a(n)u(n) + p(n)\frac{u^l(n-\tau(n))}{1+u^m(n-\tau(n))}, \tag{3.1}$$

which is a hematopoiesis model; it describes the production of red blood cells. In this model it is realistic to assume the periodicity of some parameters because of the periodic variations of the environment, which play an important role in many biological and ecological systems. Mackey and Glass [14] also used this equation, with a continuous function as an initial condition to describe some physiological control systems.

Here a, p, τ are positive periodic sequences with a common period T, and the constants m, l, T are positive. Equation (3.1) is the discrete analog of the differential equation

$$x'(t) = -a(t)x(t) + p(t)\frac{x^l(t-\tau(t))}{1+x^m(t-\tau(t))}. \tag{3.2}$$

Existence of a solution to (3.1) has been proved by Wan et al [19], while global attractivity has been studied by Wang and Li [21].

Zeng [22] shown that if $l = 1, 0 < a(n) < 1$, and $\max_{n\in[0,T-1]} p(n) > \frac{1-\delta}{\delta^2 T}$, then (3.1) has at least one positive T-periodic solution, where $\delta = \prod_{n=0}^{T-1}(1-a(n))^{-1}$. Now, we apply Theorem 2.1 to Eq. (3.1) to obtain a sufficient condition for the existence of three positive T-periodic solutions of Eq. (3.1).

Theorem 3.1. *Let* $1 < l < m$, $0 < a(n) < 1$, *and*

$$\sum_{n=0}^{T-1} p(n) \geq \frac{m(1-\delta)}{\delta^l(m-l+1)}\left[\frac{m-l+1}{l-1}\right]^{\frac{l-1}{m}} \tag{3.3}$$

hold. Then (3.1) *has at least three positive* T*-periodic solutions.*

Proof. Set $f(n,x) = p(n)\frac{x^l(n-\tau(n))}{1+x^m(n-\tau(n))}$. Since $1 < l < m$, it follows that $f^0 = 0$ and $f^\infty = 0$. To complete the proof of the theorem, it requires to show that there exist a $c_2 > 0$ such that (H_2) is satisfied, that is, there exists a positive constant $c_2 > 0$ such that

$$\sum_{n=0}^{T-1} p(n)\frac{x^l(n-\tau(n))}{1+x^m(n-\tau(n))} > (1-\delta)c_2 \text{ for } c_2 \leq \|x\| \leq \frac{c_2}{\delta}. \tag{3.4}$$

Since $x \in K$, $c_2\delta \leq x(s-\tau(s)) \leq \|x\| \leq \frac{c_2}{\delta}$. Now,

$$\begin{aligned}\sum_{n=0}^{T-1} p(n)\frac{x^l(n-\tau(n))}{1+x^m(n-\tau(n))} &\geq \frac{\delta^l c_2^l}{1+(\frac{c_2}{\delta})^m}\sum_{n=0}^{T-1} p(n) \\ &\geq \frac{\delta^l\delta^m c_2^l}{\delta^m + c_2^m}\sum_{n=0}^{T-1} p(n).\end{aligned}$$

Hence (3.4) holds if

$$\sum_{n=0}^{T-1} p(n) > \frac{1-\delta}{\delta^{m+l}} \cdot \frac{\delta^m + c_2^m}{c_2^{l-1}}. \tag{3.5}$$

Set $c_2 = \delta(\frac{l-1}{m-l+1})^{\frac{1}{m}}$, which is the minimizer of $\frac{1-\delta}{\delta^{m+l}} \cdot \frac{\delta^m + c_2^m}{c_2^{l-1}}$. Then (3.4) follows from (3.3). This completes the proof of the theorem. □

Another particular case of Eq. (1.1) is the functional difference equation

$$\Delta x(n) = -a(n)x(n) + p(n)x^m(n-\tau(n))e^{-\gamma x(n-\tau(n))} \tag{3.6}$$

Zeng [22] showed that if $m = 1$, $\max_{0\le n\le T-1} p(n) > \frac{1-\delta}{\delta^2 T}$, and $0 < a(n) < 1$, then (1.1) has at least three positive T-periodic solutions. In the following theorem, we give a sufficient condition for the existence of three positive T-periodic solutions of (3.6).

Theorem 3.2. *Let $m > 1$, $0 < a(n) < 1$, and*

$$\sum_{n=0}^{T-1} p(n) \ge (1-\delta)\delta^{m-2}(\frac{\gamma l}{\delta^2(m-1)})^{m-1} \tag{3.7}$$

hold. Then (3.6) *has at least three positive T-periodic solutions.*

Proof. Set $f(n,x) = p(n)x^m(n-\tau(n))e^{-\gamma x(n-\tau(n))}$. Clearly $m > 1$ implies that $f^0 = 0$ and $f^\infty = 0$. Thus, in order to show that (3.6) has three positive T-periodic solutions, in view of Theorem 2.1, it remains to show that there exists a constant c_2 such that (H_2) holds, that is,

$$\alpha \sum_{n=0}^{T-1} p(n)x^m(n-\tau(n))e^{-\gamma x(n-\tau(n))} > c_2 mbox for c_2 \le \|x\| \le \frac{c_2}{\delta}. \tag{3.8}$$

Clearly

$$\sum_{n=0}^{T-1} p(n)x^m(n-\tau(n))e^{-\gamma x(n-\tau(n))} \ge c_2^m e^{-\gamma\frac{c_2}{\delta}} \sum_{n=0}^{T-1} p(n).$$

Set $c_2 = \frac{\delta(m-1)}{\gamma}$. It is easy to see that $c_2 = \frac{\delta(m-1)}{\gamma}$ is the minimizer of

$$\left(\frac{1}{\delta} - 1\right) c_2(\delta c_2)^{-m} e^{\gamma\frac{c_2}{\delta}}$$

which shows that the property

$$\sum_{n=0}^{T-1} p(n) \ge \frac{(1-\delta)}{\delta c_2^{m-1}} e^{\gamma\frac{c_2}{\delta}}$$

follows from (3.7). Consequently, (3.8) holds. Thus the theorem is proved. □

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