Communications in Applied Analysis **19** (2015), 353–368

MONOTONE METHOD FOR MULTI-ORDER 2-SYSTEMS OF RIEMANN-LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we develop the monotone method for nonlinear multi-order 2-systems of Riemann-Liouville fractional differential equations. That is, a hybrid system of nonlinear equations of orders q_1 and q_2 where $0 < q_1, q_2 < 1$. In the development of this method we recall any needed existence results along with any necessary changes; including results from needed linear theory. Further we prove a comparison result paramount for the discussion of fractional multi-order inequalities that utilizes lower and upper solutions of the system. The monotone method is then developed via the construction of sequences of linear systems based on the upper and lower solutions, and are used to approximate the solution of the original nonlinear multi-order system.

AMS (MOS) Subject Classification. 34A08, 34A34, 34A45, 34A38.

1. INTRODUCTION

Fractional differential equations have various applications in widespread fields of science, such as in engineering [4], chemistry [5, 11, 12], physics [1, 2, 6], and others [7, 8]. In the majority of the literature existence results for Riemann-Liouville fractional differential equations are proven by a fixed point method. Initially we will recall existence by lower and upper solution method, which will be useful to developing our main results. Despite there being a number of existence theorems for nonlinear fractional differential equations, much as in the integer order case, this does not necessarily imply that calculating a solution explicitly will be routine, or even possible. Therefore, it may be necessary to employ an iterative technique to numerically approximate a needed solution. In this paper we construct such a method.

Specifically, we construct a technique to approximate solutions to the nonlinear Riemann-Liouville (R-L) fractional differential multi-order 2-system. A multi-order system is of the type where the equation in each component is of unique order. That

Received October 20, 2014

is, a fractional system of the type

$$D^{q_1}x_1 = f_1(t, x_1, x_2)$$
$$D^{q_2}x_2 = f_2(t, x_1, x_2).$$

This is a generalization of normal R-L systems and yields a type of hybrid system of a fractional type. We note that various complications arise from systems of this type as many known properties used in the study of fractional differential equations require modification, but at the same time multi-order systems present far more possibilities for applications. For example, consider allowing each species in a population model to have their own order of derivative. Though we will not consider any specific applications in this study, we hope this will add to the groundwork of future studies.

The iterative technique we construct will be a generalization of the monotone method for multi-order R-L 2-systems of order q_1, q_2, where $0 < q_1,\ q_2 < 1$. The monotone method, in broad terms, is a technique in which sequences are constructed from the unique solutions of linear differential equations, and initially based off of lower and upper solutions of the original nonlinear equation. These sequences converge uniformly and monotonically, from above and below, to maximal and minimal solutions of the nonlinear equation. If the nonlinear DE considered has a unique solution then both sequences will converge uniformly to that unique solution. The advantage of the monotone method is that it allows us to approximate solutions to nonlinear DEs using linear DEs; further using upper and lower solutions guarantee the interval of existence. For more information on the monotone method for ordinary DEs see [9].

There are notable complications that arise when developing the monotone method for multi-order systems. First of all, as seen in previous work involving the R-L case in these methods, the sequences we construct, $\{v_n\}, \{w_n\}$ do not converge uniformly to extremal solutions, but the weighted sequences $\{t^{1-q_i}v_{n,i}\}$, $\{t^{1-q_i}w_{n,i}\}$ converge uniformly to $t^{1-q_i}v_i$ and $t^{1-q_i}w_i$ respectively, where $i \in \{1, 2\}$ and v, w are maximal and minimal solutions of the original equation. Another complication unique to multi-order systems is that various properties do not carry over simply. For example, a well known result for the R-L derivative is that the fractional derivative of the weighted Mittag-Leffer function, which we define below in Section 2, is itself. That is

$$D_t^q t^{q-1} E_{q,q}(t^q) = t^{q-1} E_{q,q}(t^q).$$

This property is dependent on the order of q used, and therefore the weighted Mittag-Leffler function of order q_1 will not have this property with the derivative of order q_2. This issue is present in the proof of Theorem 2.8 especially and will be detailed there.

For our main method we consider the case where the nonlinear function f is quasimonotone nondecreasing in x, and briefly discuss the case where f has mixed quasimonotone properties. We note that the monotone method has been established for the standard nonlinear Riemann-Liouville fractional differential N-systems of order q in [3], and that this study acts as a generalization of that work for $N = 2$. We hope to extend this method to the more general multi-order N-system case in the near future.

2. PRELIMINARY RESULTS

In this section, we will first consider basic results regarding scalar Riemann-Liouville differential equations of order q, $0 < q < 1$. We will recall basic definitions and results in this case for simplicity, and we note that many of these results carry over naturally to the multi-order case. Then we will consider existence and comparison results for multi-order systems of order $0 < q_1,\ q_2 < 1$ which will be used in our main result. In the next section, we will apply these preliminary results to develop the monotone method for these multi-order R-L systems. Note, for simplicity we only consider results on the interval $J = (0, T]$, where $T > 0$. Further, we will let $J_0 = [0, T]$, that is $J_0 = \bar{J}$.

Definition 2.1. Let $p = 1 - q$, a function $\phi(t) \in C(J, \mathbb{R})$ is a C_p function if $t^p\phi(t) \in C(J_0, \mathbb{R})$. The set of C_p functions is denoted $C_p(J, \mathbb{R})$. Further, given a function $\phi(t) \in C_p(J, \mathbb{R})$ we call the function $t^p\phi(t)$ the continuous extension of $\phi(t)$.

Now we define the R-L integral and derivative of order q on the interval J.

Definition 2.2. Let $\phi \in C_p(J, \mathbb{R})$, then $D_t^q\phi(t)$ is the q-th R-L derivative of ϕ with respect to $t \in J$ defined as

$$D_t^q\phi(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-s)^{-q}\phi(s)ds,$$

and $I_t^q\phi(t)$ is the q-th R-L integral of ϕ with respect to $t \in J$ defined as

$$I_t^q\phi(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\phi(s)ds.$$

Note that in cases where the initial value may be different or ambiguous, we will write out the definition explicitly. The next definition is related to the solution of linear R-L fractional differential equations and is also of great importance in the study of the R-L derivative.

Definition 2.3. The Mittag-Leffler function with parameters $\alpha, \beta \in \mathbb{R}$, denoted $E_{\alpha,\beta}$, is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

which is entire for $\alpha, \beta > 0$.

The next result gives us that the q-th R-L integral of a C_p continuous function is also a C_p continuous function. This result will give us that the solutions of R-L differential equations are also C_p continuous.

Lemma 2.4. *Let $f \in C_p(J, \mathbb{R})$, then $I_t^q f(t) \in C_p(J, \mathbb{R})$, i.e. the q-th integral of a C_p continuous function is C_p continuous.*

Note the proof of this lemma for $q \in \mathbb{R}^+$ can be found in [3]. Now we consider results for the nonhomogeneous linear R-L differential equation,

$$D_t^q x(t) = \lambda x(t) + z(t), \tag{2.1}$$

with initial condition

$$t^p x(t)\big|_{t=0} = x^0/\Gamma(q),$$

where x^0 and λ are constants, and $z \in C_p(J, \mathbb{R})$, which has unique solution

$$x(t) = x^0 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) z(s)\, ds.$$

Next, we recall a result we will utilize extensively in our proceeding comparison and existence results, and likewise in the construction of the monotone method. We note that this result is similar to the well known comparison result found in literature, as in [10], but we do not require the function to be Hölder continuous of order $\lambda > q$.

Lemma 2.5. *Let $m \in C_p(J, \mathbb{R})$ be such that for some $t_1 \in J$ we have $m(t_1) = 0$ and $m(t) \leq 0$ for $t \in (0, t_1]$. Then*

$$D_t^q m(t)\big|_{t=t_1} \geq 0.$$

The proof of this lemma can be found in [3], along with further discussion as to why and how we weaken the Hölder continuous requirement of this known comparison result. We use this lemma in the proof of the later main comparison result, which will be critical in the construction of the monotone method.

Now, we will turn our attention to results for the nonlinear R-L fractional multi-order systems, and in doing so we must discuss any changes. First, we will consider systems of orders q_1 and q_2, $0 \leq q_1, q_2 < 1$. For simplicity we will let $q = (q_1, q_2)$, and when we write inequalities $x \leq y$, we mean it is true for both components. Further, from this point on, we will use the subscript i which we will always assume is in $\{1, 2\}$. For defining C_p continuity for multi-order systems we define $p_i = 1 - q_i$ and for simplicity of notation we will define the function x_p such that $x_{p_i}(t) = t^{p_i} x_i(t)$ for $t \in J_0$. We also note that at times it will be convenient to emphasize the product of t^p, therefore we will define $t^p x(t) = x_p(t)$ for $t \in J_0$. Now, we define the set of C_p continuous functions as

$$C_p(J, \mathbb{R}^2) = \{x \in C(J, \mathbb{R}^2) \mid x_p \in C(J_0, \mathbb{R}^2)\}.$$

For the rest of our results we will be considering the nonlinear R-L fractional multi-order system

$$D^{q_i} x_i = f_i(t,x) \qquad (2.2)$$
$$x_{p_i}(0) = x_i^0/\Gamma(q_i)$$

where $f \in C(J_0 \times \mathbb{R}^2, \mathbb{R}^2)$, and x^0 is a constant. Note that just as in the scalar case, a solution $x \in C_p(J, \mathbb{R}^2)$ of (2.2) also satisfies the equivalent R-L integral equation

$$x_i(t) = \frac{x_i^0}{\Gamma(q_i)} t^{q_i-1} + \frac{1}{\Gamma(q_i)} \int_0^t (t-s)^{q_i-1} f_i(s, x(s)) ds. \qquad (2.3)$$

Thus, if $f \in C(J_0 \times \mathbb{R}^2, \mathbb{R}^2)$ then (2.2) is equivalent to (2.3). See [7, 10] for details. Now we will recall a Peano type existence theorem for equation (2.2).

Theorem 2.6. *Suppose $f \in C(R_0, \mathbb{R}^2)$ and $|f_i(t,x)| \leq M_i$ on R_0, where*

$$R_0 = \{(t,x) : |t^p x(t) - x^0| \leq \eta, t \in J_0\}$$

Then the solution of (2.2) exists on J.

This result is presented for the scalar case in [10], and in [3] it was proven that the solution can be extended to all of J. We note that for multi-order systems it is proved in much the same way. Next we will consider the main Comparison Theorem for multi-order 2-systems, which will be utilized extensively in our main results. For this result we will require f to satisfy the following definition.

Definition 2.7. A function $f(t, x_1, x_2)$ is said to be quasimonotone nondecreasing in x if f_1 is nondecreasing in x_2 and f_2 is nondecreasing in x_1. That is, if $x \leq y$ on J, then

$$f_1(t, x_1, x_2) \leq f_1(t, x_1, y_2),$$

$$f_2(t, x_1, x_2) \leq f_2(t, y_1, x_2).$$

For our comparison result below we will utilize that the Beta function

$$B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1} ds,$$

is decreasing in x and y for $x, y > 0$. To prove this suppose $x_1 \geq x_2 > 0$; then we have $s^{x_1-1} \leq s^{x_2-1}$ for $s \in (0,1)$. Therefore

$$B(x_1, y) \leq \int_0^1 s^{x_2-1}(1-s)^{y-1} ds = B(x_2, y).$$

Implying $B(x,y)$ is decreasing in x for $x > 0$, and by symmetry of the Beta function we have that B is also decreasing in y for $y > 0$.

Theorem 2.8. *Let $v, w \in C_p$ be lower and upper solutions of the nonlinear multiorder 2-system, i.e.*

$$\begin{aligned} D^{q_i} v_i &\leq f_i(t, v), \quad \Gamma(q_i) v_{p_i}(0) = v_i^0 \leq x_i^0 \\ D^{q_i} w_i &\geq f_i(t, w), \quad \Gamma(q_i) w_{p_i}(0) = w_i^0 \geq x_i^0. \end{aligned} \tag{2.4}$$

If f is quasimonotone nondecreasing and satisfies the following Lipschitz condition for $i = 1, 2$,

$$f_i(t, x) - f_i(t, y) \leq L_i\big[(x_1 - y_1) + (x_2 - y_2)\big], \tag{2.5}$$

for $x \geq y$, then $v(t) \leq w(t)$ on J provided $v^0 \leq w^0$.

Proof. First we will consider the case when one of the inequalities in (2.4) is strict. Without loss of generality suppose that

$$D^{q_i} w_i > f_i(t, w), \quad \textit{and} \quad w_i^0 > x_i^0,$$

then we claim that $v < w$ on J. To prove this, suppose that the conclusion is false, e.g. suppose the set

$$\omega = \{t \in J \,:\, w_1(t) \leq v_1(t)\} \cup \{t \in J \,:\, w_2(t) \leq v_2(t)\}$$

is nonempty. Let $\tau = \inf \omega > 0$. Now there exists an i such that $v_i(\tau) = w_i(\tau)$; without loss of generality suppose $i = 1$. Because $v^0 < w^0$, by the continuity of v and w we have that $v_1(t) - w_1(t) < 0$ on $(0, \tau)$, implying that $v_1(t) - w_1(t) \leq 0$ on $(0, \tau]$ and further $v_2(t) \leq w_2(t)$ for $t \in (0, \tau]$. Therefore, by Lemma 2.5 we have that

$$D_t^{q_1}[v_1(\tau) - w_1(\tau)] \geq 0.$$

Now utilizing this and the quasimonotonicity of f we have

$$\begin{aligned} f_1(\tau, v(\tau)) &\geq D_t^{q_1} v_1(\tau) \geq D_t^{q_1} w_1(\tau) \\ &> f_1(\tau, w_1(\tau), w_2(\tau)) \\ &= f_1(\tau, v_1(\tau), w_2(\tau)) \geq f_1(\tau, v(\tau)), \end{aligned}$$

which is a contradiction. Therefore, the result is true if one of the inequalities is strict. We will use this to prove the main result. To do so, consider the function $\widetilde{w}$ defined as

$$\widetilde{w}_i(t) = w_i(t) + \varepsilon\left(t^{q_i-1} E_{q_i,q_i}(2Lt^{q_i}) + \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{(2L)^{k+m-1}\, t^{kq_1+mq_2-1}}{\Gamma(kq_1 + mq_2)}\right), \tag{2.6}$$

for $\varepsilon > 0$ sufficiently small and where $L = \max\{L_1, L_2\}$. For simplicity, let $Z(t)$ denote the series in (2.6). Further, as the weighted Mittag-Leffler function defined in terms of powers q_1 and q_2 will be required throughout the remainder of the proof, for simplicity define the function E with a single whole number parameter as

$$E_i(t) = t^{q_i-1} E_{q_i,q_i}(2Lt^{q_i}).$$

Therefore $\widetilde{w}_i(t) = w_i(t) + \varepsilon[E_i(t) + Z(t)]$. Now we will show that the series Z converges uniformly. To do so, note that there exists a $K > 1$ such that for all $k > K$, $kq_1 - 1 > 0$. So, for any $k > K$ and $m > 1$,

$$\begin{aligned}\frac{(2L)^{k+m-1}\, T^{kq_1+mq_2-1}}{\Gamma(kq_1+mq_2)} &= B(kq_1, mq_2)\frac{(2L)^{k+m-1}\, t^{kq_1+mq_2-1}}{\Gamma(kq_1)\Gamma(mq_2)} \\ &\le B(q_1, q_2)\frac{(2L)^{k+m-1}\, T^{kq_1+mq_2-1}}{\Gamma(kq_1)\Gamma(mq_2)},\end{aligned}$$

which is obtained by the monotonicity of B. Now, note that the series

$$\begin{aligned}&B(q_1, q_2)\sum_{k=K}^{\infty}\sum_{m=1}^{\infty}\frac{(2L)^{k+m-1}\, T^{kq_1+mq_2-1}}{\Gamma(kq_1)\Gamma(mq_2)} \\ &\qquad = \frac{B(q_1,q_2)}{2L}\sum_{k=K}^{\infty}\frac{(2L)^k T^{kq_1}}{\Gamma(kq_1)}\sum_{m=1}^{\infty}\frac{(2L)^m T^{mq_2-1}}{\Gamma(mq_2)} \\ &\qquad \le 2L\, B(q_1,q_2) T^{q_1} E_{q_1,q_1}(2LT^{q_1}) E_2(T),\end{aligned}$$

which converges. So by the Weirstrass M-Test,

$$\sum_{k=K}^{\infty}\sum_{m=1}^{\infty}\frac{(2L)^{k+m-1}\, t^{kq_1+mq_2-1}}{\Gamma(kq_1+mq_2)}$$

converges uniformly on J_0. Therefore, $Z(t)$ is made up of a uniformly convergent series along with a finite number $(K-1)$ of weakly singular terms. With this result we may compute the q_1-th derivative of $Z(t)$ term by term, doing so we obtain

$$\begin{aligned}D_t^{q_1} Z(t) &= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{(2L)^{k+m-1}\, t^{(k-1)q_1+mq_2-1}}{\Gamma((k-1)q_1+mq_2)} \\ &= \sum_{m=1}^{\infty}\frac{(2L)^m\, t^{mq_2-1}}{\Gamma(mq_2)} + \sum_{k=2}^{\infty}\sum_{m=1}^{\infty}\frac{(2L)^{k+m-1}\, t^{(k-1)q_1+mq_2-1}}{\Gamma((k-1)q_1+mq_2)} \\ &= 2L\big(E_2(t) + Z(t)\big).\end{aligned}$$

Similarly, we can show that

$$D_t^{q_2} Z(t) = 2L\big(E_1(t) + Z(t)\big).$$

Using this result we note that

$$\begin{aligned}D_t^{q_1}\widetilde{w}_1(t) &= D_t^{q_1} w_1(t) + 2L\varepsilon\big(E_1(t) + E_2(t) + Z(t)\big) \\ &\ge f_1(t,\widetilde{w}) - L_1[(\widetilde{w}_1 - w_1) + (\widetilde{w}_2 - w_2)] + 2L\varepsilon\big(E_1(t) + E_2(t) + Z(t)\big) \\ &= f_1(t,\widetilde{w}) - L_1\varepsilon\big(E_1(t) + E_2(t) + 2Z(t)\big) + 2L\varepsilon\big(E_1(t) + E_2(t) + Z(t)\big) \\ &\ge f_1(t,\widetilde{w}) + L\varepsilon\big(E_1(t) + E_2(t)\big) \\ &> f_1(t,\widetilde{w}).\end{aligned}$$

Similarly, we can show that $D_t^{q_2}\widetilde{w}_2(t) > f_2(t,\widetilde{w})$.

Now we wish to show that $\widetilde{w}^0 > x^0$; to do so we note

$$t^{p_1}\widetilde{w}_1(t) = t^{p_1}w_1(t) + \varepsilon E_{q_1,q_1}(2Lt^{q_1}) + \varepsilon \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{(2L)^{k+m-1}\, t^{(k-1)q_1+mq_2}}{\Gamma(kq_1+mq_2)},$$

implying that

$$t^{p_1}\widetilde{w}_1(t)\big|_{t=0} = w_1^0/\Gamma(q_1) + \varepsilon/\Gamma(q_1) > x_1^0/\Gamma(q_1).$$

Therefore $\widetilde{w}_1^0 > x_1^0$, and similarly we can show that $\widetilde{w}_2^0 > x_2^0$. Now, from what was shown previously with strict inequalities we have that $v < \widetilde{w}$ on J, and letting $\varepsilon \to 0$ we get $v \leq w$ on J. □

We note that if we were to generalize the series $Z(t)$ for an N-dimensional space, we could do so in the following way. Let

$$Z_N(t) = \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\cdots\sum_{k_N=1}^{\infty} \frac{(2L)^{\mathbf{1}\cdot\mathbf{k}-N+1}\, t^{\mathbf{q}\cdot\mathbf{k}-1}}{\Gamma(\mathbf{q}\cdot\mathbf{k})},$$

where $\mathbf{1}$ is the unit vector, $\mathbf{k} = (k_1, k_2, k_3, \ldots, k_N)$, and $\mathbf{q} = (q_1, q_2, q_3, \ldots, q_N)$. With this generalization, the function we used in the proof of Theorem 2.8 would be Z_2 and importantly,

$$Z_1(t) = 2Lt^{q_1-1}E_{q_1,q_1}(2Lt^{q_1}),$$

implying that Z_N is a generalization of the C_p weighted Mittag-Leffler function for multi-order N-systems. This function will be paramount when we turn our attention to N-systems at a later date.

Now, if we know of the existence of lower and upper solutions v and w such that $v \leq w$, we can prove the existence of a solution in the set

$$\Omega = \{(t, y) \,:\, v(t) \leq y \leq w(t),\, t \in J\}.$$

We consider this result in the following theorem.

Theorem 2.9. *Let $v, w \in C_p(J, \mathbb{R}^2)$ be lower and upper solutions of (2.2) such that $v(t) \leq w(t)$ on J and let $f \in C(\Omega, \mathbb{R})$, where Ω is defined as above. Then there exists a solution $x \in C_p(J, \mathbb{R}^2)$ of (2.2) such that $v(t) \leq x(t) \leq w(t)$ on J.*

This theorem is proved in the same way as seen in [3], with only minor additions to apply it to multi-order 2-systems.

3. MONOTONE METHOD

In this section we will develop the monotone iterative technique for nonlinear multi-order systems of the type (2.2).

Theorem 3.1. *Let $f \in C(J_0 \times \mathbb{R}^2, \mathbb{R}^2)$ be quasimonotone nondecreasing and let $v_0, w_0 \in C_p(J, \mathbb{R})$ be lower and upper solutions of system (2.2) such that $v_0 \leq w_0$ on J. Further suppose f satisfies a one-sided Lipschitz condition such that f_1 is Lipschitz in x_1 and f_2 is Lipschitz in x_2, that is*

$$f_1(t, z_1, x_2) - f_1(t, y_1, x_2) \geq -M_1(z_1 - y_1),$$
$$f_2(t, x_1, z_2) - f_2(t, x_1, y_2) \geq -M_2(z_2 - y_2),$$

whenever $v_0 \leq x \leq w_0$, and $v_0 \leq y \leq z \leq w_0$. Then there exist monotone sequences $\{v_n\}$ and $\{w_n\}$ such that

$$t^p v_n \to t^p v, \quad t^p w_n \to t^p w,$$

uniformly and monotonically on J_0, where v and w are minimal and maximal solutions of (2.2) on J provided $v_0^0 \leq x^0 \leq w_0^0$.

Proof. To begin we note that the sequences we wish to construct are defined as the unique solutions of the following linear multi-order fractional systems

$$D^{q_i} v_{n+1_i} = f_i(t, v_n) - M_i(v_{n+1_i} - v_{ni}) \tag{3.1}$$
$$D^{q_i} w_{n+1_i} = f_i(t, w_n) - M_i(w_{n+1_i} - w_{ni}),$$

where v_0 and w_0 are defined in our hypothesis. We would like to show that these sequences are monotone and that the weighted sequences converge uniformly. To do so we consider the more general multi-order system

$$D^{q_i} y_i = f_i(t, \xi) - M_i(y_i - \xi_i) \tag{3.2}$$
$$y_p(0) = x^0,$$

with $v_0 \leq \xi(t) \leq w_0$ on J. We note that since system (3.2) is linear that a unique solution exists in $C_p(J, \mathbb{R})$ for every particular choice of ξ. Therefore, we may construct a mapping F, such that $y = F[\xi]$ will output the unique solution of (3.2). With this mapping, we can define our sequences as

$$v_{n+1} = F[v_n], \quad w_{n+1} = F[w_n].$$

We claim that F is monotone nondecreasing. To prove this, suppose that $v_0 \leq \xi \leq \eta \leq w_0$ on J, and let $y = F[\xi]$ and $z = F[\eta]$. Now, using the quasimonotone property of f, along with the Lipschitz condition from our hypothesis we have that

$$\begin{aligned} D^{q_1} z_1 &\geq f_1(t, \eta_1, \xi_2) - M_1(z_1 - \eta_1) \\ &= f_1(t, \eta_1, \xi_2) + f_1(t, \xi_1, \xi_2) - f_1(t, \xi_1, \xi_2) - M_1(z_1 - \eta_1) \\ &\geq f_1(t, \xi_1, \xi_2) - M_1(z_1 - \xi_1). \end{aligned}$$

Similarly, we can show that $D^{q_2} z_2 \geq f_2(t,\xi) - M_2(z_2 - \xi_2)$. Now, since (3.2) is linear, it is Lipschitz of the form (2.5), thus $y \leq z$ on J by Theorem 2.8. This gives us that $F[\eta] \leq F[\xi]$ as we claimed.

From here we can show that the sequences in (3.1) are monotone. We will begin by showing that $v_0 \leq F[v_0]$ and $w_0 \geq F[w_0]$, to do so, let $v_1 = F[v_0]$, and then note that

$$D^{q_i} v_{1i} = f_i(t, v_0) - M_i(v_{1i} - v_{0i}),$$

and because

$$D^{q_i} v_{0i} \leq f_i(t, v_0) - M_i(v_{0i} - v_{0i})$$

we may apply Theorem 2.8 to show that $v_0 \leq v_1$ on J. Similarly, $w_1 \leq w_0$ on J. Next, by the monotonicity property of F we have that

$$v_1 = F[v_0] \leq F[w_0] = w_1.$$

Therefore, $v_0 \leq v_1 \leq w_1 \leq w_0$ on J. Using this as our inductive basis step suppose this is true for up to some $k \geq 1$, that is, $v_{k-1} \leq v_k \leq w_k \leq w_{k-1}$. Now, letting $v_{k+1} = F[v_k]$ and $w_{k+1} = F[w_k]$ and using the monotone property of F along with our induction hypothesis we have that

$$v_{k+1} = F[v_k] \geq F[v_{k-1}] = v_k,$$

and similarly we have that $w_{k+1} \leq w_k$ on J. Finally, we can also show that on J

$$v_{k+1} = F[v_k] \leq F[w_k] = w_{k+1}.$$

So, by induction we have that $v_0 \leq v_{n-1} \leq v_n \leq w_n \leq w_{n-1} \leq w_0$ for all $n \geq 1$ on J.

Now we wish to show that the weighted sequences $\{t^p v_n\}$ and $\{t^p w_n\}$ converge uniformly on J_0. To so we will apply the Arzelá-Ascoli Theorem; therefore we must show these sequences are uniformly bounded and equicontinuous. For any $n \geq 0$ we submit that

$$|t^{p_i} v_{ni}| \leq t^{p_i}\big(|v_{ni} - v_{0i}| + |v_{0i}|\big) \leq t^{p_i}\big(|w_{0i} - v_{0i}| + |v_{0i}|\big),$$

implying that the sequence $\{t^p v_n\}$ is uniformly bounded. Noting that we can show a similar result for $\{t^p w_n\}$ we conclude that both weighted sequences are uniformly bounded. Now using this we can show that our weighted sequences are equicontinuous. First, for simplicity let

$$\widetilde{f}_i(t, v_n) = f_i(t, v_{n-1}) - M_i(v_{ni} - v_{n-1i}),$$

for all $n \geq 1$, and noting that $\widetilde{f}$ is C_p continuous and that $\{t^p v_n\}$ is uniformly bounded, we can choose a $N \geq 0$ such that

$$t^{p_1} \widetilde{f}_1(t, v_n) \leq N_1$$

on J_0 for any $n \geq 1$. Now, choose t, τ such that $0 < t \leq \tau \leq T$. In the following proof of equicontinuity we use the fact that

$$\tau^{p_1}(\tau - s)^{q_1 - 1} - t^{p_1}(t - s)^{q_1 - 1} \leq 0$$

for $0 < s < t$. To show why this is true, consider the function $\phi(t) = t^{p_1}(t-s)^{q_1-1} = t^{p_1}(t-s)^{-p_1}$ and note that

$$\begin{aligned} \tfrac{d}{dt}\phi(t) &= p_1 t^{p_1 - 1}(t-s)^{-p_1} - p_1 t^{p_1}(t-s)^{-p_1-1} \\ &= -t^{p_1-1}(t-s)^{-p_1-1} p_1 s \leq 0. \end{aligned}$$

This implies that ϕ is nonincreasing, therefore $\phi(\tau) - \phi(t) \leq 0$. Now consider,

$$\begin{aligned} |\tau^{p_1} v_{n1}(\tau) - t^{p_1} v_{n1}(t)| &\leq \frac{1}{\Gamma(q_1)} \int_0^t |\tau^{p_1}(\tau - s)^{q_1-1} - t^{p_1}(t-s)^{q_1-1}||\widetilde{f}_1(t, v_n)| ds \\ &\quad + \frac{\tau^{p_1}}{\Gamma(q_1)} \int_t^\tau (\tau - s)^{q_1 - 1} |\widetilde{f}_1(t, v_n)| ds \\ &\leq \frac{N_1}{\Gamma(q_1)} \int_0^t \left[t^{p_1}(t-s)^{q_1-1} - \tau^{p_1}(\tau - s)^{q_1-1} \right] s^{q_1-1} ds \\ &\quad + \frac{N_1 \tau^{p_1}}{\Gamma(q_1)} \int_t^\tau (\tau - s)^{q_1-1} s^{q_1-1} ds \\ &\leq \frac{N_1 t^{p_1}}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} s^{q_1-1} ds - \frac{N_1 \tau^{p_1}}{\Gamma(q_1)} \int_0^\tau (\tau - s)^{q_1-1} s^{q_1-1} ds \\ &\quad + \frac{2N_1 \tau^{p_1}}{\Gamma(q_1) t^{p_1}} \int_t^\tau (\tau - s)^{q_1-1} ds \\ &= \frac{N_1 \Gamma(q_1)}{\Gamma(q_2)} (t^{q_1} - \tau^{q_1}) + \frac{2N_1 \tau^{p_1}}{\Gamma(q_1) t^{p_1}} \frac{1}{q_1} (\tau - t)^{q_1} \\ &\leq \frac{2N_1 \tau^{p_1}}{\Gamma(q_1 + 1) t^{p_1}} (\tau - t)^{q_1}. \end{aligned}$$

In the case that $t = 0$, we note that

$$|\tau^{p_1} v_{n1}(\tau) - x^0/\Gamma(q_1)| \leq \frac{N_1}{\Gamma(q_1)} \int_0^\tau (\tau - s)^{q_1-1} ds = \frac{N_1}{\Gamma(q_1+1)} \tau^{q_1}.$$

Now, we can choose $K_1 \geq 0$ such that

$$K_1 \geq \tfrac{2N_1}{\Gamma(q_1+1)} \tfrac{T^{p_1}}{t^{p_1}} \geq \tfrac{N_1}{\Gamma(q_1+1)},$$

which we note is not dependent on n. Therefore, we have that

$$|\tau^{p_1} v_{n1}(\tau) - t^{p_1} v_{n1}(t)| \leq K_1 |\tau - t|^{q_1},$$

for $0 \leq t \leq \tau \leq T$ and for all $n \geq 1$. Similarly, we can show that

$$|\tau^{p_2} v_{n2}(\tau) - t^{p_2} v_{n2}(t)| \leq K_2 |\tau - t|^{q_2},$$

for all $n \geq 1$. With this, it is now routine to show that $\{t^p v_n\}$ is equicontinuous. Likewise, $\{t^p w_n\}$ is also equicontinuous. Therefore, by the Arzelá-Ascoli Theorem there exist subsequences of both $\{t^p v_n\}$ and $\{t^p w_n\}$ that converge uniformly on J_0, and due to their monotonic nature the full sequences themselves also converge uniformly on J_0. Given this, suppose that $t^p v_n \to t^p v$ and $t^p w_n \to t^p w$ on J_0; we wish to show that v and w are extremal solutions of (2.2) on J. To do so, first note that $v_n \to v$ point wise on J, and due to the nature of $\widetilde{f}$ we have that

$$t^{p_i} v_{ni} = \frac{x_i^0}{\Gamma(q_i)} + \frac{t^{p_i}}{\Gamma(q_i)} \int_0^t (t-s)^{q_i-1} f_i(s, v_{n-1}) - M_i(v_{ni}(s) - v_{n-1_i}(s)) ds,$$

which converges uniformly on J_0 to

$$t^{p_i} v_i = \frac{x_i^0}{\Gamma(q_i)} + \frac{t^{p_i}}{\Gamma(q_i)} \int_0^t (t-s)^{q_i-1} f_i(s, v) ds,$$

implying that

$$v_i = \frac{x_i^0}{\Gamma(q_i)} t^{q_i-1} + \frac{1}{\Gamma(q_i)} \int_0^t (t-s)^{q_i-1} f_i(s, v) ds$$

on J, and thus that v is a solution to (2.2). By a similar argument w is also a solution to (2.2). We will use induction to show that v and w are minimal and maximal solutions. First, let x be a solution to (2.2), such that $v_0^0 \le x^0 \le w_0^0$. By Theorem 2.9 we know such a solution exists such that $v_0 \le x \le w_0$ on J. Given this, and using the same steps we used to prove the monotonicity of F we have that

$$D^{q_i} v_{1i} \le f_i(t, x) - M_i(v_{1i} - x_i) \quad \text{and} \quad D^{q_i} w_{1i} \ge f_i(t, x) - M_i(w_{1i} - x_i),$$

implying that $v_1 \le x \le w_1$ on J by Theorem 2.8. Using this as a basis step, we may use the same steps again to inductively show that $v_n \le x \le w_n$ on J for all $n \ge 0$, thus implying that $v \le x \le w$ on J. This gives us that v and w are extremal solutions and finishes the proof. □

We note that if f satisfies a full two-sided Lipshitz condition, then $v = x = w$ which will be the unique solution of (2.2).

Now, we can extend the result of Theorem 3.1 to a more general result. For the following we will assume that $f(t, x_1, x_2)$ satisfies a mixed quasimonotone property, which we define here.

Definition 3.2. A function $f(t, x_1, x_2)$ is said to possess a mixed quasimonotone property if f_1 is nonincreasing in x_2 and f_2 is nondecreasing in x_1. That is, if $x \le y$ on J, then

$$f_1(t, x_1, x_2) \le f_1(t, x_1, y_2),$$

$$f_2(t, x_1, x_2) \ge f_2(t, y_1, x_2).$$

With this generalization we can also consider a generalization of our lower and upper solutions to coupled lower and upper quasisolutions.

Definition 3.3. Let $v, w \in C_p(J, \mathbb{R}^2)$; v and w are coupled lower and upper quasisolutions of (2.2) if

$$\begin{aligned} D^{q_1} v_1 &\le f_1(t, v_1, w_2), & D_t^{q_1} w_1 &\ge f_1(t, w_1, v_2),\\ D^{q_2} v_2 &\le f_2(t, v_1, v_2), & D_t^{q_2} w_2 &\ge f_2(t, w_1, w_2),\\ \Gamma(q_i) v_{p_i}(0) &= v_i^0 \le x_i^0, & \Gamma(q_i) w_{p_i}(0) &= w_i^0 \ge x_i^0. \end{aligned}$$

On the other hand, v and w are coupled quasisolutions of (2.2) if

$$\begin{aligned} D^{q_1} v_1 &= f_1(t, v_1, w_2), & D_t^{q_1} w_1 &= f_1(t, w_1, v_2),\\ D^{q_2} v_2 &= f_2(t, v_1, v_2), & D_t^{q_2} w_2 &= f_2(t, w_1, w_2),\\ v_{p_i}(0) &= x_i^0/\Gamma(q_i), & w_{p_i}(0) &= x_i^0/\Gamma(q_i). \end{aligned}$$

Further, one can define coupled extremal quasisolutions of (2.2) in the usual way.

Now, we will generalize the result of Theorem 3.1 to when f satisfies a mixed quasimonotone property. This case requires the use of coupled lower and upper quasisolutions and the construction will involve coupled quasisolutions.

Theorem 3.4. *Let $f \in C(J_0 \times \mathbb{R}^2, \mathbb{R}^2)$ possess the mixed quasimonotone property mentioned above. Let $v_0, w_0 \in C_p(J, \mathbb{R})$ be coupled lower and upper quasisolutions of system (2.2) such that $v_0 \le w_0$ on J. Suppose further that f satisfies the same Lipshitz condition of Theorem 3.1. Then there exist monotone sequences $\{v_n\}$ and $\{w_n\}$ such that*

$$t^p v_n \to t^p v, \quad t^p w_n \to t^p w,$$

uniformly and monotonically on J_0, where v and w are coupled minimal and maximal quasisolutions of (2.2) on J provided $v_0^0 \le x^0 \le w_0^0$.

Proof. The proof of this case follows the same process as Theorem 3.1, with the distinction of managing coupled quasisolutions. For example, the sequences we will construct are the unique solutions of linear systems of the form

$$\begin{aligned} D^{q_1} y_1 &= f_1(t, \eta_1, \xi_2) - M_1(y_1 - \eta_1),\\ D^{q_2} y_2 &= f_2(t, \eta_1, \eta_2) - M_2(y_2 - \eta_2),\\ y_{p_i}(0) &= x_i^0/\Gamma(q_i), \end{aligned} \tag{3.3}$$

for $v_0 \le \xi$, $\eta \le w_0$ on J. Similarly, for each ξ and η we have a unique solution y; therefore, as done previously, we define the mapping A to output this unique solution for each ξ and η. Therefore, $A[\eta, \xi] = y$. Now, we propose that A is monotone nondecreasing in its first variable and monotone nonincreasing in its second variable.

To prove this, suppose that $v_0 \le \xi \le \eta \le w_0$ and $v_0 \le \mu \le w_0$, and let $y = A[\xi, \mu]$ and $x = A[\eta, \mu]$. Then, using the Lipschitz condition of f we have

$$\begin{aligned} D^{q_1} x_1 &= f_1(t, \eta_1, \mu_2) - M_1(x_1 - \eta_1) + f_1(t, \xi_1, \mu_2) - f_1(t, \xi_1, \mu_2) \\ &\ge f_1(t, \xi_1, \mu_2) - M_1(x_1 - \xi_1). \end{aligned}$$

Further, using this process again along with quasimonotonic property of f we have that

$$D^{q_2} x_2 \ge f_2(t, \eta_1, \eta_2) - M_2(x_2 - \xi_2).$$

Therefore, by Theorem 2.8 we have that $x \ge y$ on J, implying that A is monotonic nondecreasing in its first variable, since $A[\xi, \mu] \le A[\eta, \mu]$. Similarly, we can show that $A[\mu, \xi] \ge A[\mu, \eta]$, hence proving our proposition. From here we define the sequences $\{v_n\}$ and $\{w_n\}$ as

$$v_{n+1} = A[v_n, w_n], \quad \text{and} \quad w_{n+1} = A[w_n, v_n].$$

The proof from here follows in the same manner as Theorem 3.1. □

We note that we can develop similar results if the mixed qasuimonotone property of f is reversed, that is if f_1 is nondecreasing in x_2 and f_2 is nonincreasing in x_1. The construction and proof of this case will follow in a similar fashion as Theorem 3.4.

In the future, we wish to turn our attention to multi-order systems of finite order N. Further, the construction of numerical applications of this type is quite unwieldy, but is something we would like to consider along with N-systems. From here, it would be compelling to study various physical models that would lend themselves to multi-order fractional systems. Our hope is that this initial study may open the doors to further results in multi-order systems beyond the use of the Caputo derivative.

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