Communications in Applied Analysis **19** (2015), 369–402

OSCILLATION CRITERIA FOR HIGHER ORDER NONLINEAR FUNCTIONAL DYNAMIC EQUATIONS WITH MIXED NONLINEARITIES

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ABSTRACT. In this paper, we will consider the higher-order functional dynamic equations with mixed nonlinearities of the form

$$\left\{r_{n-1}(t)\ \phi_{\alpha_{n-1}}\left[(r_{n-2}(t)(\cdots(r_1(t)\phi_{\alpha_1}[x^{\Delta}(t)])^{\Delta}\cdots)^{\Delta})^{\Delta}\right]\right\}^{\Delta}+\sum_{j=0}^{N}p_j(t)\phi_{\gamma_j}\left(x^{\sigma}(g_j(t))\right)=0,$$

on an above-unbounded time scale $\mathbb{T}$, where $n\geq 2$, and $\phi_\beta(u):=|u|^{\beta-1}u$, $\beta>0$. The funtions $g_j:\mathbb{T}\to\mathbb{T}$ are rd-continuous functions such that $\lim_{t\to\infty}g_j(t)=\infty$, $j=0,1,\ldots,N$. The results extend and improve some known results in the literature on higher order nonlinear dynamic equations.

AMS (MOS) Subject Classification. 34K11, 39A10, 39A99.

1. INTRODUCTION

In this paper we consider the oscillation of solutions of higher order dynamic equations with mixed nonlinearities of the form

$$\left\{r_{n-1}(t)\phi_{\alpha_{n-1}}\left[(r_{n-2}(t)(\cdots(r_1(t)\phi_{\alpha_1}[x^{\Delta}(t)])^{\Delta}\cdots)^{\Delta})^{\Delta}\right]\right\}^{\Delta}$$
$$+\sum_{j=0}^{N}p_j(t)\phi_{\gamma_j}\left(x^{\sigma}(g_j(t))\right)=0, \tag{1.1}$$

Received December 15, 2014

on an arbitrary time scale $\mathbb{T}$, where

(i) $n \geq 2$ and $\phi_\beta(u) := |u|^{\beta-1} u$, $\beta > 0$;
(ii) $r_i \in C_{rd}\left([t_0, \infty)_{\mathbb{T}}, (0, \infty)\right)$ for $i = 1, 2, \ldots, n-1$;
(iii) $\alpha_i > 0,\ i = 1, 2, \ldots, n-1$ and $\gamma_j > 0,\ j = 0, 1, \ldots, N$ are constants;
(iv) $p_j \in C_{rd}\left([t_0, \infty)_{\mathbb{T}}, [0, \infty)\right)$, $j = 0, 1, \ldots, N$ such that not all of the $p_j(t)$'s vanish in a neighborhood of infinity;
(v) $g_j : \mathbb{T} \to \mathbb{T}$ are rd-continuous functions such that $\lim_{t\to\infty} g_j(t) = \infty$ for $j = 0, 1, \ldots, N$, and $\tau(t) := \inf\{t, g_0(t), \ldots, g_N(t)\}$ is increasing function on $[t_0, \infty)_{\mathbb{T}}$.

Throughout this paper, we let

$$x^{[i]} := r_i\ \phi_{\alpha_i}\left[\left(x^{[i-1]}\right)^{\Delta}\right], \quad i = 1, 2, \ldots, n \text{ with } r_n = 1,\ \alpha_n = 1 \text{ and } x^{[0]} = x.$$

Recall that the knowledge and understanding of time scales and time scale notation is assumed. For an excellent introduction to the calculus on time scales, see Bohner and Peterson [6, 7]. By a solution of Eq. (1.1) we mean a nontrivial real–valued function $x \in C^1_{rd}[T_x, \infty)_{\mathbb{T}}$ for some $T_x \geq t_0$ such that $x^{[i]} \in C^1_{rd}[T_x, \infty)_{\mathbb{T}}$, $i = 1, 2, \ldots, n-1$ and $x(t)$ satisfies Eq. (1.1) on $[T_x, \infty)_{\mathbb{T}}$, where C_{rd} is the space of right-dense continuous functions.

In the last few years, there has been an increasing interest in the oscillation and nonoscillation of solutions of various dynamic equations. A large number of papers were devoted to second order linear and nonlinear dynamic equations on time scales. For example, Agarwal, Bohner, and Saker [1] discussed the linear delay dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x(g(t)) = 0;$$

Erbe, Peterson, and Saker [14], Saker [45], Agarwal, Regan, and Saker [2], and Hassan [33] investigated the pair of half-linear dynamic equations

$$(r(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} + p(t)x^{\alpha_1}(t) = 0,$$

and

$$(r(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} + p(t)x^{\alpha_1}(\sigma(t)) = 0;$$

Erbe, Hassan, Peterson, and Saker [12] and [13] studied the half-linear delay dynamic equation

$$(r(t)(x^{\Delta}(t))^{\alpha_1})^{\Delta} + p(t)x^{\alpha_1}(g(t)) = 0, \tag{1.2}$$

with $g(t) \leq t$ and

$$r^{\Delta}(t) \geq 0 \qquad \text{and} \qquad \int_{t_0}^{\infty} g^{\alpha_1}(t)p(t)\Delta t = \infty; \tag{1.3}$$

and Hassan [34] extended their results to the half-linear advanced dynamic equation (1.2) with $g(t) \geq t$.

Erbe, Hassan, Peterson [20] considered nonlinear dynamic equations with mixed nonlinearities of the form

$$\left(r\left(t\right)\left(x^{\Delta}\left(t\right)\right)^{\alpha_1}\right)^{\Delta}+\sum_{j=0}^{N}p_j\left(t\right)\phi_{\gamma_j}\left(x\left(g_j\left(t\right)\right)\right)=0.$$

Erbe, Peterson, and Saker [16, 17] and Yu and Wang [49] also derived oscillation criteria for the third order dynamic equations

$$\left(r_2(t)\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}+p(t)x(t)=0,$$

$$\left(r_2(t)\left[(r_1(t)x^{\Delta}(t))^{\Delta}\right]^{\alpha_2}\right)^{\Delta}+p\left(t\right)x^{\gamma}\left(t\right)=0,$$

and

$$\left(r_2(t)\left[(r_1(t)\left(x^{\Delta}(t)\right)^{\alpha_1})^{\Delta}\right]^{\alpha_2}\right)^{\Delta}+p\left(t\right)x\left(t\right)=0;$$

their work was further extended by Hassan [32] and Erbe, Hassan, and Peterson [18] to the equation with delay

$$\left(r_2(t)\left[(r_1(t)x^{\Delta}(t))^{\Delta}\right]^{\alpha_2}\right)^{\Delta}+p\left(t\right)x^{\gamma}\left(g\left(t\right)\right)=0;$$

also, Han, Li, Sun, and Zhang [31] discussed the third order delay dynamic equation

$$\left(r_2(t)\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}+p(t)x(g\left(t\right))=0,$$

where $g(t)\leq t$ and

$$r_1^{\Delta}(t)\leq 0 \qquad \text{and} \qquad \int_{t_0}^{\infty}g(t)p(t)\Delta t=\infty. \tag{1.4}$$

Also, Erbe, Hassan and Peterson [21] and Elabbasy and Hassan [9] studied third order nonlinear dynamic equation with mixed nonlinearities

$$\left(r_2(t)\left[(r_1(t)\left(x^{\Delta}(t)\right)^{\alpha_1})^{\Delta}\right]^{\alpha_2}\right)^{\Delta}+\sum_{j=0}^{N}p_j\left(t\right)\phi_{\gamma_j}\left(x\left(g_j\left(t\right)\right)\right)=0,$$

where α_1 and α_2 are the ratios of positive odd integers.

Higher order dynamic equations have been studied by many authors. For instance, Grace, Agarwal, and Zafer [26] established oscillation and comparison criteria for the even order nonlinear dynamic equation

$$x^{\Delta^{2n}}\left(t\right)+p\left(t\right)\left(x^{\sigma}\left(t\right)\right)^{\gamma}=0,$$

and Grace [29] developed oscillation criteria for the even order dynamic equation

$$\left[r(t)\left(x^{\Delta^{n-1}}\left(t\right)\right)^{\alpha}\right]^{\Delta}+p\left(t\right)\left(x^{\sigma}\left(t\right)\right)^{\gamma}=0,$$

where α and γ are the ratios of positive odd integers. Recently, Grace and Hassan [28] establish oscillation criteria for more general higher order dynamic equation

$$x^{[n]}\left(t\right)+p\left(t\right)\phi_{\gamma}\left(x^{\sigma}\left(g\left(t\right)\right)\right)=0.$$

For more results on higher order dynamic equations, we refer the reader to the papers [10, 23, 43, 26, 47, 40, 29, 22, 27].

The purpose of this paper is to establish the asymptotic and oscillatory behavior of solutions of the nth order nonlinear dynamic equation (1.1) with mixed nonlinearities and without assuming the conditions (1.3) and (1.4). The results in this paper extend many results in the literature on the oscillation for second order, third order, and higher order nonlinear dynamic equations.

2. MAIN RESULTS

Before stating our main results, we begin with some preliminary lemmas which will play an important role in the proof of our main results.

The first one is cited from [28] and improves the well-known lemma due to Kiguradze.

Lemma 2.1. *Assume that*

$$\int_{t_0}^{\infty} r_i^{-1/\alpha_i}(s)\Delta s = \infty, \ \ i = 1, 2, \dots, n-1. \tag{2.1}$$

If Eq. (1.1) *has an eventually positive solution x, then there exists an integer $m \in [0, n]$ with $m + n$ odd such that*

$$m \geq 1 \quad \textit{implies} \quad x^{[k]} > 0 \quad \textit{for} \quad k = 0, \dots, m-1, \tag{2.2}$$

eventually, and

$$m \leq n \quad \textit{implies} \quad (-1)^{m+k} x^{[k]} > 0 \quad \textit{for} \quad k = m, \dots, n, \tag{2.3}$$

eventually.

The following lemma improves [46, Lemma 1] and also see [37, 35, 44].

Lemma 2.2. *Assume that*

$$\gamma_j > \gamma := \gamma_0, \ \ j = 1, 2, \dots, l; \quad \textit{and} \ \ \gamma_j < \gamma := \gamma_0, \ \ j = l+1, l+2, \dots, N. \tag{2.4}$$

Then, there exists an N-tuple $(\eta_1, \eta_2, \dots, \eta_N)$ with $\eta_j > 0$ satisfying

$$\sum_{j=1}^{N} \gamma_j \eta_j = \gamma \quad \textit{and} \quad \sum_{j=1}^{N} \eta_j = 1. \tag{2.5}$$

Lemma 2.3 ([25])**.** *Suppose that $|x|^{\Delta} > 0$ on $[t_0, \infty)_{\mathbb{T}}$, $\beta > 0$, and $\beta \neq 1$. Then*

$$\frac{|x|^{\Delta}}{(|x|^{\sigma})^{\beta}} \leq \frac{\left(|x|^{1-\beta}\right)^{\Delta}}{1-\beta} \leq \frac{|x|^{\Delta}}{(|x|)^{\beta}} \ \textit{on} \ [t_0, \infty)_{\mathbb{T}}.$$

We will use the following notations: For any $t, s \in \mathbb{T}$, define $\alpha[h,k] := \alpha_h \cdots \alpha_k$ for $1 \le h \le k \le n-1$ with $\alpha := \alpha[1,n]$ and for an integer $m \in \{0,\ldots,n-1\}$, define the functions $\hat{R}_i(v,u)$, $i = 0,\ldots,m-1$; $\bar{R}_i(v,u)$, $i = 0,\ldots,n-1$ and $P_i(t)$, $i = 0,\ldots,n-1$, by the following recurrence formulas:

$$\hat{R}_i(v,u) := \begin{cases} \left[\int_u^v \hat{R}_{i-1}(s,u)\,\Delta s \Big/ r_{m-i}(v)\right]^{1/\alpha_{m-i}}, & i = 1,\ldots,m-1, \\ r_m^{-1/\alpha_m}(v), & i = 0, \end{cases}$$

$$\bar{R}_i(v,u) := \begin{cases} \int_u^v \left[\bar{R}_{i-1}(v,s)/\, r_{n-i}(s)\right]^{1/\alpha_{n-i}}\,\Delta s, & i = 1,\ldots,n-1, \\ 1, & i = 0, \end{cases}$$

and

$$P_i(t) := \begin{cases} \left[\int_t^\infty P_{i-1}(s)\,\Delta s/\, r_{n-i}(t)\right]^{1/\alpha_{n-i}} & i = 1,\ldots,n-1, \\ p(t), & i = 0, \end{cases}$$

where $p(t) := p_0(t) + \prod_{j=1}^{N} [p_j(t)/\,\eta_j]^{\eta_j}$ and provided the improper integrals involved are convergent.

Lemma 2.4. *Assume Eq.* (1.1) *has an eventually positive solution* $x(t)$ *and* m *is given in Lemma* 2.1 *such that* $m \in \{1,\ldots,n-1\}$ *and* (2.2) *and* (2.3) *hold for* $t \ge t_1 \in [t_0,\infty)_{\mathbb{T}}$. *Then the following hold for* $v \ge u \ge t_1$*:*

(a) *For* $i = 0,\ldots,m-1$

$$\left(x^{[i]}(v)\right)^{\Delta} \ge \phi^{-1}_{\alpha[i+1,m]}\left(x^{[m]}(v)\right)\hat{R}_{m-i-1}(v,u),$$

and

$$x^{[i]}(v) \ge \phi^{-1}_{\alpha[i+1,m]}\left(x^{[m]}(v)\right)\int_u^v \hat{R}_{m-i-1}(s,u)\,\Delta s;$$

(b) *for* $i = m,\ldots,n-1$,

$$(-1)^{m+i}x^{[i]}(u) \ge \phi^{-1}_{\alpha[i+1,n]}\left(x^{[n-1]}(v)\right)\bar{R}_{n-i-1}(v,u).$$

Proof. (a) By using (2.2) and (2.3), we get for $v \ge u \ge t_1 \in [t_0,\infty)_{\mathbb{T}}$,

$$\left(x^{[m-1]}(v)\right)^{\Delta} = \phi^{-1}_{\alpha_m}\left(x^{[m]}(v)\right) r_m^{-1/\alpha_m}(v) = \phi^{-1}_{\alpha_m}\left(x^{[m]}(v)\right)\hat{R}_0(v,u).$$

Replacing v by s and integrating with respect to s from u to v, we get

$$\begin{aligned} x^{[m-1]}(v) &= x^{[m-1]}(u) + \int_u^v \phi^{-1}_{\alpha_m}\left(x^{[m]}(s)\right)\hat{R}_0(s,u)\,\Delta s \\ &\ge \phi^{-1}_{\alpha_m}\left(x^{[m]}(v)\right)\int_u^v \hat{R}_0(s,u)\,\Delta s, \end{aligned}$$

which yields

$$\begin{aligned} \left(x^{[m-2]}(v)\right)^{\Delta} &\ge \phi^{-1}_{\alpha[m-1,m]}\left(x^{[m]}(v)\right)\left[\int_u^v \hat{R}_0(s,u)\,\Delta s \Big/ r_{m-1}(v)\right]^{1/\alpha_{m-1}} \\ &= \phi^{-1}_{\alpha[m-1,m]}\left(x^{[m]}(v)\right)\hat{R}_1(v,u). \end{aligned}$$

Replacing v by s and integrating with respect to s from u to v, we get

$$x^{[m-2]}(v) \geq x^{[m-2]}(u) + \int_u^v \phi^{-1}_{\alpha[m-1,m]}\left(x^{[m]}(s)\right) \hat{R}_1(s,u)\,\Delta s$$
$$\geq \phi^{-1}_{\alpha[m-1,m]}\left(x^{[m]}(v)\right) \int_u^v \hat{R}_1(s,u)\,\Delta s,$$

which implies

$$\left(x^{[m-3]}(v)\right)^{\Delta} \geq \phi^{-1}_{\alpha[m-2,m]}\left(x^{[m]}(v)\right) \left[\int_u^v \hat{R}_1(s,u)\,\Delta s \Big/ r_{m-2}(v)\right]^{1/\alpha_{m-2}}$$
$$= \phi^{-1}_{\alpha[m-2,m]}\left(x^{[m]}(v)\right) \hat{R}_2(v,u).$$

Again replacing v by s and integrating with respect to s from u to v, we get

$$x^{[m-3]}(v) \geq \phi^{-1}_{\alpha[m-2,m]}\left(x^{[m]}(v)\right) \int_u^v \hat{R}_2(s,u)\,\Delta s.$$

Continuing this process, one can easily see, for $i = 1, \ldots, m$

$$\left(x^{[m-i]}(v)\right)^{\Delta} \geq \phi^{-1}_{\alpha[m-i+1,m]}\left(x^{[m]}(v)\right) \hat{R}_{i-1}(v,u),$$

and

$$x^{[m-i]}(v) \geq \phi^{-1}_{\alpha[m-i+1,m]}\left(x^{[m]}(v)\right) \int_u^v \hat{R}_{i-1}(s,u)\,\Delta s,$$

or for $i = 0, \ldots, m-1$

$$\left(x^{[i]}(v)\right)^{\Delta} \geq \phi^{-1}_{\alpha[i+1,m]}\left(x^{[m]}(v)\right) \hat{R}_{m-i-1}(v,u),$$

and

$$x^{[i]}(v) \geq \phi^{-1}_{\alpha[i+1,m]}\left(x^{[m]}(v)\right) \int_u^v \hat{R}_{m-i-1}(s,u)\,\Delta s.$$

(b) By the fact that $x^{[n-1]}$ is nonincreasing on $[t_1,\infty)_{\mathbb{T}}$, we get for $v \geq u \geq t_1$,

$$x^{[n-1]}(u) \geq x^{[n-1]}(v) = \phi^{-1}_{\alpha_n}\left(x^{[n-1]}(v)\right) \bar{R}_0(v,u),$$

which implies

$$\left(x^{[n-2]}(u)\right)^{\Delta} \geq \phi^{-1}_{\alpha[n-1,n]}\left(x^{[n-1]}(v)\right) \left[\bar{R}_0(v,u)/\, r_{n-1}(u)\right]^{1/\alpha_{n-1}}.$$

Replacing u by s and integrating with respect to s from $u \geq t_1$ to $v \in [u,\infty)_{\mathbb{T}}$ and using (2.3), we get

$$-x^{[n-2]}(u) \geq x^{[n-2]}(v) - x^{[n-2]}(u)$$
$$= \phi^{-1}_{\alpha[n-1,n]}\left(x^{[n-1]}(v)\right) \int_u^v \left[\bar{R}_0(v,s)/\, r_{n-1}(s)\right]^{1/\alpha_{n-1}} \Delta s$$
$$= \phi^{-1}_{\alpha[n-1,n]}\left(x^{[n-1]}(v)\right) \bar{R}_1(v,u),$$

which yields

$$-\left(x^{[n-3]}(u)\right)^{\Delta} \geq \phi^{-1}_{\alpha[n-2,n]}\left(x^{[n-1]}(v)\right) \left[\bar{R}_1(v,u)/\, r_{n-2}(u)\right]^{1/\alpha_{n-2}}.$$

Again replacing u by s and integrating with respect to s from u to v, we get

$$\begin{aligned} x^{[n-3]}(u) &\geq -x^{[n-3]}(v) + x^{[n-3]}(u) \\ &= \phi^{-1}_{\alpha[n-2,n]}\left(x^{[n-1]}(v)\right) \int_u^v \left[\bar{R}_1(v,s)/r_{n-2}(s)\right]^{1/\alpha_{n-2}} \Delta s \\ &= \phi^{-1}_{\alpha[n-2,n]}\left(x^{[n-1]}(v)\right) \bar{R}_2(v,u). \end{aligned}$$

Continuing this process, we can see for $i = 1, \ldots, n-m$

$$(-1)^{n+m-i} x^{[n-i]}(u) \geq \phi^{-1}_{\alpha[n-i+1,n]}\left(x^{[n-1]}(v)\right) \bar{R}_{i-1}(v,u),$$

or for $i = m, \ldots, n-1$

$$(-1)^{m+i} x^{[i]}(u) \geq \phi^{-1}_{\alpha[i+1,n]}\left(x^{[n-1]}(v)\right) \bar{R}_{n-i-1}(v,u).$$

This completes the proof. □

In the following, we denote

$$P(t) := \left[\int_t^\infty p(s)\ \Delta s\right]^{1/\alpha}; \tag{2.6}$$

and for an integer $m \in \{0, \ldots, n-1\}$,

$$R_{m,1}(t,T) := \phi^{-1}_{\alpha[1,m]}\left(\bar{R}_{n-m-1}(t,\tau(t))\right) \int_T^{\tau(t)} \hat{R}_{m-1}(s,T)\ \Delta s; \tag{2.7}$$

and

$$R_{m,2}(t,T) := \phi^{-1}_{\alpha[1,m]}\left(\bar{R}_{n-m-1}(t,\tau(t))\right) \hat{R}_{m-1}(\tau(t),T). \tag{2.8}$$

3. OSCILLATION CRITERIA FOR EVEN ORDER EQUATIONS

In this section, we establish oscillation criteria for Eq. (1.1) when n is even. It follows from Lemma 2.1 that there exists an odd $m \in \{1, \ldots, n-1\}$ such that (2.2) and (2.3) hold eventually.

Theorem 3.1. *Assume that* (2.1) *holds and*

$$\sum_{j=0}^{N} \int_{t_0}^\infty p_j(s)\,\Delta s = \infty. \tag{3.1}$$

Then every solution of Eq. (1.1) *is oscillatory.*

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g_j(t)) > 0$, $j = 0, 1, 2, \ldots, N$ on $[t_0, \infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an odd integer $m \in \{1, \ldots, n-1\}$ such that (2.2) and (2.3) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$. This implies $x(t)$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. Then, for sufficiently large $t_2 \in [t_1, \infty)_{\mathbb{T}}$, we have $x^\sigma(g_j(t)) \geq l$ for $t \geq t_2$. It follows that

$$\phi_{\gamma_j}\left(x^\sigma\left(g_j(t)\right)\right) \geq l^{\gamma_j} \geq L \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}},$$

where $L := \inf_{0\leq j\leq N}\{l^{\gamma_j}\} > 0$. Eq. (1.1) becomes

$$-\left(x^{[n-1]}(t)\right)^{\Delta} = \sum_{j=0}^{N} p_j(t)\,\phi_{\gamma_j}\left(x^{\sigma}\left(g_j(t)\right)\right) \geq L\sum_{j=0}^{N} p_j(t) \quad \text{for } t\in[t_2,\infty)_{\mathbb{T}}. \tag{3.2}$$

Replacing t by s in (3.2)and integrating from t_2 to $t\in[t_2,\infty)_{\mathbb{T}}$ gives

$$-x^{[n-1]}(t) + x^{[n-1]}(t_2) \geq L\sum_{j=0}^{N}\int_{t_2}^{t} p_j(s)\,\Delta s.$$

Hence by (3.1), we have $\lim_{t\to\infty} x^{[n-1]}(t) = -\infty$, which contradicts the fact that $x^{[n-1]}(t) > 0$ eventually. This completes the proof. □

Theorem 3.2. *Assume that* (2.4) *and* (2.1) *hold and for sufficiently large* $T\in[t_0,\infty)_{\mathbb{T}}$ *and for every odd integer* $m\in\{1,\ldots,n-1\}$,

$$\begin{aligned} &\textstyle\int_{t_0}^{\infty} p(t)\,R_{m,1}^{\gamma}(t,T)\Delta t = \infty, && \text{if } \gamma<\alpha;\\ &\limsup_{t\to\infty} P(t)R_{m,1}(t,T) > 1, && \text{if } \gamma=\alpha;\\ &\textstyle\int_{t_0}^{\infty} \tau^{\Delta}(t)P(t)R_{m,2}(t,T)\Delta t = \infty, && \text{if } \gamma>\alpha. \end{aligned} \tag{3.3}$$

Then every solution of Eq. (1.1) *is oscillatory.*

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g_j(t)) > 0$, $j = 0,1,2,\ldots,N$, on $[t_0,\infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an odd integer $m\in\{1,\ldots,n-1\}$ such that (2.2) and (2.3) hold for $t\geq t_1\in[t_0,\infty)_{\mathbb{T}}$. Eq. (1.1) becomes

$$\begin{aligned} -x^{[n]}(t) &= \sum_{j=0}^{N} p_j(t)\,\phi_{\gamma_j}\left(x^{\sigma}\left(g_j(t)\right)\right) \geq \sum_{j=0}^{N} p_j(t)\,\phi_{\gamma_j}\left(x^{\sigma}\left(\tau(t)\right)\right)\\ &= \phi_{\gamma}\left(x^{\sigma}\left(\tau(t)\right)\right)\sum_{j=0}^{N} p_j(t)\left[x^{\sigma}\left(\tau(t)\right)\right]^{\gamma_j-\gamma}. \end{aligned} \tag{3.4}$$

By Lemma 2.2, we have there exists $\eta_1,\ldots,\eta_N$ with

$$\sum_{j=1}^{N}\gamma_j\eta_j - \gamma\sum_{j=1}^{N}\eta_j = 0.$$

Using the Arithmetic-geometric mean inequality, see [3, Page 17], we have

$$\sum_{j=1}^{N}\eta_j v_j \geq \prod_{j=1}^{N} v_j^{\eta_j}, \quad \text{for any } v_j\geq 0,\ j=1,\ldots,N.$$

Then for $t \geq t_1$

$$\begin{aligned}\sum_{j=0}^{N} p_j(t)\left[x^\sigma(\tau(t))\right]^{\gamma_j-\gamma} &= p_0(t)+\sum_{j=1}^{N}\eta_j\frac{p_j(t)}{\eta_j}\left[x^\sigma(\tau(t))\right]^{\gamma_j-\gamma}\\ &\geq p_0(t)+\prod_{j=1}^{N}\left[\frac{p_j(t)}{\eta_j}\right]^{\eta_j}\left[x^\sigma(\tau(t))\right]^{\eta_j(\gamma_j-\gamma)}\\ &= p_0(t)+\prod_{j=1}^{N}\left[\frac{p_j(t)}{\eta_j}\right]^{\eta_j}=p(t).\end{aligned}$$

This together with (3.4) shows that

$$-x^{[n]}(t)\geq p(t)\phi_\gamma\left(x^\sigma(\tau(t))\right),\quad \text{for } t\in[t_1,\infty)_{\mathbb{T}}. \tag{3.5}$$

Replacing t by s in (3.5), integrating from $t \geq t_1$ to $v \in [t,\infty)_{\mathbb{T}}$, and using the fact that τ is nondecreasing, we have

$$\begin{aligned}-x^{[n-1]}(v)+x^{[n-1]}(t) &\geq \int_t^v p(s)\,\phi_\gamma\left(x^\sigma(\tau(s))\right)\Delta s\\ &\geq \int_t^v p(s)\,\phi_\gamma\left(x^\sigma(\tau(s))\right)\Delta s\\ &\geq \phi_\gamma\left(x^\sigma(\tau(t))\right)\int_t^v p(s)\,\Delta s,\end{aligned}$$

and by (2.3), we see that $x^{[n-1]}(v) > 0$. Hence by taking limits as $v \to \infty$, we have

$$x^{[n-1]}(t)\geq \phi_\gamma\left(x^\sigma(\tau(t))\right)\int_t^\infty p(s)\,\Delta s. \tag{3.6}$$

Then by Lemma 2.4, Part (a) we have that for $i = 0$,

$$x(v)\geq \phi^{-1}_{\alpha[1,m]}\left(x^{[m]}(v)\right)\int_u^v \hat{R}_{m-1}(s,u)\,\Delta s, \tag{3.7}$$

and

$$x^{\Delta}(v)\geq \phi^{-1}_{\alpha[1,m]}\left(x^{[m]}(v)\right)\hat{R}_{m-1}(v,u). \tag{3.8}$$

Setting $v = \tau(t)$ and $u = t_1$ in (3.7) and (3.8), we have for $\tau(t) \in [t_1,\infty)_{\mathbb{T}}$,

$$x(\tau(t))\geq \phi^{-1}_{\alpha[1,m]}\left(x^{[m]}(\tau(t))\right)\int_{t_1}^{\tau(t)} \hat{R}_{m-1}(s,t_1)\,\Delta s, \tag{3.9}$$

and

$$x^{\Delta}(\tau(t))\geq \phi^{-1}_{\alpha[1,m]}\left(x^{[m]}(\tau(t))\right)\hat{R}_{m-1}(\tau(t),t_1). \tag{3.10}$$

Then by Lemma 2.4, Part (b) we have that for $i = m$,

$$x^{[m]}(u)\geq \phi^{-1}_{\alpha[m+1,n]}\left(x^{[n-1]}(v)\right)\bar{R}_{n-m-1}(v,u).$$

Setting$v = t$ and $u = \tau(t)$ gives

$$x^{[m]}(\tau(t))\geq \phi^{-1}_{\alpha[m+1,n]}\left(x^{[n-1]}(t)\right)\bar{R}_{n-m-1}(t,\tau(t))\quad \text{for } \tau(t)\in[t_1,\infty)_{\mathbb{T}}. \tag{3.11}$$

Pick $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $\tau(t) \in [t_1, \infty)_{\mathbb{T}}$ for $t \geq t_2$. Substituting (3.11) into (3.9) and (3.10), we get for $t \in [t_2, \infty)_{\mathbb{T}}$,

$$x(\tau(t)) \geq \phi_{\alpha}^{-1}\left(x^{[n-1]}(t)\right) \phi_{\alpha[1,m]}^{-1}\left(\bar{R}_{n-m-1}(t, \tau(t))\right) \int_{t_1}^{\tau(t)} \hat{R}_{m-1}(s, t_1)\, \Delta s$$
$$= \phi_{\alpha}^{-1}\left(x^{[n-1]}(t)\right) R_{m,1}(t, t_1), \tag{3.12}$$

and

$$x^{\Delta}(\tau(t)) \geq \phi_{\alpha}^{-1}\left(x^{[n-1]}(t)\right) \phi_{\alpha[1,m]}^{-1}\left(\bar{R}_{n-m-1}(t, \tau(t))\right) \hat{R}_{m-1}(\tau(t), t_1)$$
$$= \phi_{\alpha}^{-1}\left(x^{[n-1]}(t)\right) R_{m,2}(t, t_1), \tag{3.13}$$

where $\alpha = \alpha[1, n]$. We consider the following three cases:

(a) $\gamma < \alpha$. From (3.5) and (3.12) and using the fact that x is strictly increasing, we have

$$-x^{[n]}(t) \geq p(t)\, \phi_{\gamma}\left(x^{\sigma}(\tau(t))\right) \geq p(t)\, \phi_{\gamma}\left(x(\tau(t))\right) \geq p(t)\, \phi_{\gamma}\left(x(\tau(t))\right)$$
$$\geq p(t)\, R_{m,1}^{\gamma}(t, t_1)\ \left(x^{[n-1]}(t)\right)^{\gamma/\alpha}, \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}},$$

or

$$-\frac{\alpha}{\alpha-\gamma}\left(\left(x^{[n-1]}(t)\right)^{(\alpha-\gamma)/\alpha}\right)^{\Delta} \overset{\text{Lemma 2.3}}{\geq} -\frac{\left(x^{[n-1]}(t)\right)^{\Delta}}{\left(x^{[n-1]}(t)\right)^{\gamma/\alpha}} \geq p(t)\, R_{m,1}^{\gamma}(t, t_1)$$
$$\geq p(t)\, R_{m,1}^{\gamma}(t, t_2).$$

Integrating this inequality from t_2 to t, we get

$$\frac{\alpha}{\alpha-\gamma}\left(x^{[n-1]}(t_2)\right)^{(\alpha-\gamma)/\alpha} - \frac{\alpha}{\alpha-\gamma}\left(x^{[n-1]}(t)\right)^{(\alpha-\gamma)/\alpha} \geq \int_{t_2}^{t} p(s)\, R_{m,1}^{\gamma}(s, t_2)\Delta s.$$

Since $x^{[n-1]} > 0$ eventually and $\alpha > \gamma$, then

$$\frac{\alpha}{\alpha-\gamma}\left(x^{[n-1]}(t_2)\right)^{(\alpha-\gamma)/\alpha} \geq \int_{t_2}^{t} p(s)\, R_{m,1}^{\gamma}(s, t_2)\Delta s,$$

which contradicts (3.3) if $\gamma < \alpha$.

(b) $\gamma = \alpha$. Substituting (3.6) into (3.12) gives

$$x(\tau(t)) \geq P(t) R_{m,1}(t, t_1) x^{\sigma}(\tau(t))$$
$$\geq P(t) R_{m,1}(t, t_2) x(\tau(t))$$
$$\geq P(t) R_{m,1}(t, t_2) x(\tau(t)),$$

or

$$P(t) R_{m,1}(t, t_2) \leq 1,$$

which implies

$$\limsup_{t \to \infty} P(t) R_{m,1}(t, t_2) \leq 1.$$

This leads to a contradiction to (3.3) if $\gamma = \alpha$.

(c) $\gamma > \alpha$. Substituting (3.6) into (3.13) gives

$$x^{\Delta}(\tau(t)) \geq P(t)R_{m,2}(t,t_2) \ [x^{\sigma}(\tau(t))]^{\gamma/\alpha}$$

or

$$\frac{\alpha}{\alpha-\gamma}\left([x(\tau(t))]^{(\alpha-\gamma)/\alpha}\right)^{\Delta} \overset{\text{Lemma 2.3}}{\geq} \frac{(x(\tau(t)))^{\Delta}}{(x^{\sigma}(\tau(t)))^{\gamma/\alpha}} \geq P(t)R_{m,2}(t,t_1)\tau^{\Delta}(t)$$
$$\geq P(t)R_{m,2}(t,t_2)\tau^{\Delta}(t).$$

Integrating this inequality from t_2 to t, we get

$$\frac{\alpha}{\alpha-\gamma}\left[[x(\tau(t))]^{(\alpha-\gamma)/\alpha}\right]^{(\alpha-\gamma)/\alpha} - \frac{\alpha}{\alpha-\gamma}\left[[x(\tau(t_2))]^{(\alpha-\gamma)/\alpha}\right]^{(\alpha-\gamma)/\alpha}$$
$$\geq \int_{t_2}^{t} P(s)\tau^{\Delta}(s)R_{m,2}(s,t_2)\Delta s.$$

Since $x > 0$ eventually, then

$$\frac{\alpha}{\gamma-\alpha}\left[[x(\tau(t_2))]^{(\alpha-\gamma)/\alpha}\right]^{(\alpha-\gamma)/\alpha} \geq \int_{t_2}^{t} P(s)\tau^{\Delta}(s)R_{m,2}(s,t_2)\Delta s,$$

which contradicts (3.3) if $\gamma > \alpha$. This completes the proof. □

Theorem 3.3. *The conclusions of Theorem* 3.2 *hold if the third condition in* (3.3) *is replaced by*

$$\int_{T}^{\infty} \tau^{\Delta}(t)Q_m(t,T)\Delta t = \infty, \tag{3.14}$$

for sufficiently large $T \in [t_0,\infty)_{\mathbb{T}}$ *and for every odd integer* $m \in \{1,\ldots,n-1\}$, *where*

$$Q_m(t,T) := \phi^{-1}_{\alpha[1,m]}\left[\int_{s}^{\infty} P_{n-m-1}(u)\Delta u\right]\hat{R}_{m-1}(\tau(t),T).$$

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g_j(t)) > 0$, $j = 0,1,2,\ldots,N$, on $[t_0,\infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an odd integer $m \in \{1,\ldots,n-1\}$ such that (2.2) and (2.3) hold for $t \geq t_1 \in [t_0,\infty)_{\mathbb{T}}$. As seen in the proof of Theorem 3.2, we have

$$x^{[n-1]}(t) \geq \phi_{\gamma}(x^{\sigma}(\tau(t))) \int_{t}^{\infty} p(s)\Delta s$$
$$= \phi^{-1}_{\alpha[n,n]}(\phi_{\gamma}(x^{\sigma}(\tau(t)))) \int_{t}^{\infty} P_0(s)\Delta s.$$

Then

$$\left[x^{[n-2]}(t)\right]^{\Delta} \geq \phi^{-1}_{\alpha[n-1,n]}(\phi_{\gamma}(x^{\sigma}(\tau(t)))) \left[\int_{t}^{\infty} P_0(s)\Delta s \Big/ r_{n-1}(t)\right]^{1/\alpha_{n-1}}$$
$$= \phi^{-1}_{\alpha[n-1,n]}(\phi_{\gamma}(x^{\sigma}(\tau(t))))P_1(t).$$

Replacing t by s and integrating with respect to s from $t \geq T_1$ to $v \in [t, \infty)_{\mathbb{T}}$ and letting $v \to \infty$ and using (2.2) and (2.3), we get

$$-x^{[n-2]}(t) \geq \phi^{-1}_{\alpha[n-1,n]} \left[\phi_\gamma \left(x^\sigma \left(\tau \left(t\right)\right)\right)\right] \int_t^\infty P_1 \left(s\right) \Delta s.$$

Continuing this process $n - m - 2$-times, we get

$$x^{[m]}(t) \geq \phi^{-1}_{\alpha[m+1,n]} \left(\phi_\gamma \left(x^\sigma \left(\tau \left(t\right)\right)\right)\right) \int_t^\infty P_{n-m-1} \left(s\right) \Delta s.$$

Since $x^\Delta > 0$ and $\left(x^{[m]}\right)^\Delta < 0$ on $[t_1, \infty)_{\mathbb{T}}$, we have for sufficiently large $t_2 \in [t_1, \infty)_{\mathbb{T}}$

$$x^{[m]}(\tau(t)) \geq \phi^{-1}_{\alpha[m+1,n]} \left(\phi_\gamma \left(x^\sigma \left(\tau \left(t\right)\right)\right)\right) \int_t^\infty P_{n-m-1}(s) \Delta s \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}. \tag{3.15}$$

As shown in the proof of Theorem 3.2, we have

$$\begin{aligned} x^\Delta(\tau(t)) &\geq \phi^{-1}_{\alpha[1,m]} \left(x^{[m]} \left(\tau(t)\right)\right) \hat{R}_{m-1}(\tau(t), t_1) \\ &\geq \phi^{-1}_{\alpha[1,m]} \left(x^{[m]} \left(\tau(t)\right)\right) \hat{R}_{m-1}(\tau(t), t_2) \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we get

$$\begin{aligned} x^\Delta(\tau(t)) &\geq \phi^{-1}_\alpha \left[\phi_\gamma \left(x^\sigma \left(\tau \left(t\right)\right)\right)\right] \\ &\qquad \phi^{-1}_{\alpha[1,m]} \left[\int_s^\infty P_{n-m-1}(u) \Delta u\right] \hat{R}_{m-1}(\tau(t), t_2) \\ &= \phi^{-1}_\alpha \left[\phi_\gamma \left(x^\sigma \left(\tau \left(t\right)\right)\right)\right] Q_m(t, t_2), \end{aligned}$$

or

$$\frac{x^\Delta(\tau(t))\tau^\Delta(t)}{\left(x^\sigma \left(\tau \left(t\right)\right)\right)^{\gamma/\alpha}} \geq \tau^\Delta(t) Q_m(t, t_2).$$

We will consider only the case $\gamma > \alpha$ and the rest of the proof is similar to that of Theorem 3.2. In view of Lemma 2.3, we have

$$\frac{\alpha}{\alpha - \gamma} \left(\left[x(\tau(t))\right]^{(\alpha-\gamma)/\alpha}\right)^\Delta \overset{\text{Lemma 2.3}}{\geq} \frac{\left[x(\tau(t))\right]^\Delta}{\left(x^\sigma \left(\tau \left(t\right)\right)\right)^{\gamma/\alpha}} \geq \tau^\Delta(t) Q_m(t, t_2).$$

Integrating this inequality from t_2 to t, we get

$$\begin{aligned} \frac{\alpha}{\alpha - \gamma} \left[\left[x(\tau(t))\right]^{(\alpha-\gamma)/\alpha}\right]^{(\alpha-\gamma)/\alpha} &- \frac{\alpha}{\alpha - \gamma} \left[\left[x(\tau(t_2))\right]^{(\alpha-\gamma)/\alpha}\right]^{(\alpha-\gamma)/\alpha} \\ &\geq \int_{t_2}^t \tau^\Delta(s) Q_m(s, t_2) \Delta s. \end{aligned}$$

Since $x > 0$ eventually, then

$$\frac{\alpha}{\gamma - \alpha} \left[\left[x(\tau(t_2))\right]^{(\alpha-\gamma)/\alpha}\right]^{(\alpha-\gamma)/\alpha} \geq \int_{t_2}^t \tau^\Delta(s) Q_m(s, t_2) \Delta s,$$

which contradicts (3.14). This completes the proof. $\square$

Theorem 3.4. *Assume that* (2.4) *and* (2.1) *hold and for sufficiently large* $T \in [t_0, \infty)_{\mathbb{T}}$ *and for every odd integer* $m \in \{1, \ldots, n-1\}$,

$$\limsup_{t\to\infty} P(t) R_{m,1}(t, T) = \infty. \tag{3.17}$$

Then every bounded solution of Eq. (1.1) *is oscillatory.*

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(\tau_j(t)) > 0$, $j = 0, 1, 2, \ldots, N$, on $[t_0, \infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an odd integer $m \in \{1, \ldots, n-1\}$ such that (2.2) and (2.3) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$. As shown in the proof of Theorem 3.2, we have for sufficiently large $t_2 \in [t_1, \infty)_{\mathbb{T}}$

$$x(\tau(t)) \geq R_{m,1}(t, t_1) \phi_\alpha^{-1} \left(x^{[n-1]}(t) \right), \tag{3.18}$$

and

$$x^{[n-1]}(t) \geq \phi_\gamma \left(x^\sigma (\tau(t)) \right) \int_t^\infty p(s) \Delta s, \tag{3.19}$$

for $t \in [t_2, \infty)_{\mathbb{T}}$. Substituting (3.19) into (3.18) we obtain

$$x(\tau(t)) \geq P(t) R_{m,1}(t, t_1) \ \left[x^\sigma (\tau(t)) \right]^{\gamma/\alpha} \geq P(t) R_{m,1}(t, t_2) \ \left[x(\tau(t)) \right]^{\gamma/\alpha},$$

or

$$\left[x(\tau(t)) \right]^{1-\gamma/\alpha} \geq P(t) R_{m,1}(t, t_2),$$

which contradicts (3.17). This completes the proof. □

In the following, we denote

$$k_+ := \max\{k, 0\}, \quad k_- := \max\{-k, 0\} \text{ for any } k \in \mathbb{R},$$

and we employ the lemma below, see and using the inequality (see [30]).

Lemma 3.5. *If* X *and* Y *are nonnegative and* $\lambda > 1$, *then*

$$X^\lambda - \lambda X Y^{\lambda-1} + (\lambda - 1) Y^\lambda \geq 0, \tag{3.20}$$

where equality holds if and only if $X = Y$.

We are now ready to state and prove a Philos-type oscillation criteria for equation (1.1).

Theorem 3.6. *Assume that* (2.4) *and* (2.1) *hold and* $\tau \circ \sigma = \sigma \circ \tau$ *on* $[t_0, \infty)_{\mathbb{T}}$. *Furthermore, suppose that there exist functions* $\delta \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ *and* $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv \{(t, u) : t \geq u \geq t_0\}$ *such that*

$$H(t, t) = 0, \quad t \geq t_0, \ H(t, u) > 0, \quad t > u \geq t_0, \tag{3.21}$$

and H has a nonpositive continuous Δ-partial derivative $H^{\Delta_u}(t,u)$ with respect to the second variable and satisfies

$$H^{\Delta_u}(t,u) + H(t,u)\frac{\eta^{\Delta}(u)}{\delta^{\sigma}(u)} = -\frac{h(t,u)}{\delta^{\sigma}(u)}H^{\alpha/(\alpha+1)}(t,u), \tag{3.22}$$

and, for all sufficiently large T and for every odd integer $m \in \{1,\ldots,n-1\}$,

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\delta(u)p(u)H(t,u) - \frac{(\alpha/\gamma)^{\alpha}\left[h_-(t,u)\right]^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left[\delta(u)\tau^{\Delta}(u)A(u)R_{m,2}(u,T)\right]^{\alpha}}\right]\Delta u = \infty, \tag{3.23}$$

where

$$A(t) := \begin{cases} c_1, c_1 \text{ is any positive constant}, & \text{when } \gamma > \alpha; \\ 1, & \text{when } \gamma = \alpha; \\ c_2\ P^{\alpha/\gamma-1}(t), c_2 \text{ is any positive constant} & \text{when } \gamma < \alpha. \end{cases} \tag{3.24}$$

Then every solution of Eq. (1.1) *is oscillatory.*

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g_j(t)) > 0$, $j = 0,1,2,\ldots,N$, on $[t_0,\infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an odd integer $m \in \{1,\ldots,n-1\}$ such that (2.2) and (2.3) hold for $t \geq t_1 \in [t_0,\infty)_{\mathbb{T}}$. Define

$$w(t) := \frac{\delta(t)x^{[n-1]}(t)}{\phi_{\gamma}(x(\tau(t)))}. \tag{3.25}$$

By the product rule and the quotient rule, we get

$$\begin{aligned} w^{\Delta}(t) &= \frac{\delta(t)}{\phi_{\gamma}(x(\tau(t)))}\left(x^{[n-1]}(t)\right)^{\Delta} + \left[\frac{\delta(t)}{\phi_{\gamma}(x(\tau(t)))}\right]^{\Delta}\left(x^{[n-1]}(t)\right)^{\sigma} \\ &= \delta(t)\frac{\left(x^{[n-1]}(t)\right)^{\Delta}}{\phi_{\gamma}(x(\tau(t)))} \\ &\quad + \left[\frac{\delta^{\Delta}(t)}{\phi_{\gamma}(x(\tau^{\sigma}(t)))} - \frac{\delta(t)\left[\phi_{\gamma}(x(\tau(t)))\right]^{\Delta}}{\phi_{\gamma}(x(\tau(t)))\,\phi_{\gamma}(x(\tau^{\sigma}(t)))}\right]\left(x^{[n-1]}(t)\right)^{\sigma}. \end{aligned}$$

As shown in the proof of Theorem 3.2, we have for $t \in [t_1,\infty)_{\mathbb{T}}$

$$\left(x^{[n-1]}(t)\right)^{\Delta} \leq -p(t)\phi_{\gamma}\left(x^{\sigma}(\tau(t))\right),$$

which implies

$$\frac{\left(x^{[n-1]}(t)\right)^{\Delta}}{\phi_{\gamma}(x(\tau(t)))} \leq -p(t)\frac{\phi_{\gamma}\left(x^{\sigma}(\tau(t))\right)}{\phi_{\gamma}(x(\tau(t)))} \leq -p(t).$$

Therefore for $t \in [t_1, \infty)_{\mathbb{T}}$

$$w^{\Delta}(t) \leq -\delta(t)p(t) + \left[\frac{\delta^{\Delta}(t)}{\phi_{\gamma}\left(x\left(\tau^{\sigma}\left(t\right)\right)\right)} - \frac{\delta(t)\left[\phi_{\gamma}\left(x\left(\tau(t)\right)\right)\right]^{\Delta}}{\phi_{\gamma}\left(x\left(\tau\left(t\right)\right)\right)\phi_{\gamma}\left(x\left(\tau^{\sigma}\left(t\right)\right)\right)}\right]\left(x^{[n-1]}(t)\right)^{\sigma}$$
$$= -\delta(t)p(t) + \delta^{\Delta}(t)\left(\frac{w(t)}{\delta(t)}\right)^{\sigma} - \delta(t)\frac{\left[\phi_{\gamma}\left(x\left(\tau(t)\right)\right)\right]^{\Delta}}{\phi_{\gamma}\left(x\left(\tau\left(t\right)\right)\right)}\left(\frac{w(t)}{\delta(t)}\right)^{\sigma}. \tag{3.26}$$

Since x and τ are differentiable functions, and τ is a nondecreasing, we have $x \circ \tau$ is a differentiable function and $(x\left(\tau(t)\right))^{\Delta} = x^{\Delta}\left(\tau(t)\right)\tau^{\Delta}(t)$. Then, by the Pötzsche chain rule ([6, Theorem 1.90]), we obtain

$$\begin{aligned}
\left[x^{\gamma}\left(\tau(t)\right)\right]^{\Delta} &= \gamma\int_0^1 \left[x\left(\tau(t)\right) + h\mu(t)(x\left(\tau(t)\right))^{\Delta}\right]^{\gamma-1} dh\, \left(x\left(\tau(t)\right)\right)^{\Delta} \\
&= \gamma\int_0^1 \left[(1-h)\ x\left(\tau(t)\right) + h\ x(\tau^{\sigma}(t))\right]^{\gamma-1} dh\, x^{\Delta}\left(\tau(t)\right)\tau^{\Delta}(t) \\
&\geq \begin{cases} \gamma\left[x(\tau^{\sigma}(t))\right]^{\gamma-1} x^{\Delta}\left(\tau(t)\right)\tau^{\Delta}(t), & 0 < \gamma \leq 1; \\ \gamma\left[x(\tau(t))\right]^{\gamma-1} x^{\Delta}\left(\tau(t)\right)\tau^{\Delta}(t), & \gamma \geq 1. \end{cases}
\end{aligned}$$

If $0 < \gamma \leq 1$, we have that

$$w^{\Delta}(t) \leq -\delta(t)p(t) + \delta^{\Delta}(t)\left(\frac{w(t)}{\delta(t)}\right)^{\sigma} - \gamma\delta(t)\tau^{\Delta}(t)\left(\frac{w(t)}{\delta(t)}\right)^{\sigma}\frac{x^{\Delta}\left(\tau(t)\right)}{x\left(\tau^{\sigma}(t)\right)}\left[\frac{x\left(\tau^{\sigma}(t)\right)}{x\left(\tau(t)\right)}\right]^{\gamma},$$

whereas if $\gamma \geq 1$, we have that

$$w^{\Delta}(t) \leq -\delta(t)p(t) + \delta^{\Delta}(t)\left(\frac{w(t)}{\delta(t)}\right)^{\sigma} - \gamma\delta(t)\tau^{\Delta}(t)\left(\frac{w(t)}{\delta(t)}\right)^{\sigma}\frac{x^{\Delta}\left(\tau(t)\right)}{x\left(\tau^{\sigma}(t)\right)}\frac{x\left(\tau^{\sigma}(t)\right)}{x\left(\tau(t)\right)}.$$

Using the fact that $x^{\Delta}\left(t\right) > 0$ on $[t_1, \infty)_{\mathbb{T}}$, we get that, for $\gamma > 0$,

$$w^{\Delta}(t) \leq -\delta(t)p(t) + \delta^{\Delta}(t)\left(\frac{w(t)}{\delta(t)}\right)^{\sigma} - \gamma\delta(t)\tau^{\Delta}(t)\left(\frac{w(t)}{\delta(t)}\right)^{\sigma}\frac{x^{\Delta}\left(\tau(t)\right)}{x\left(\tau^{\sigma}(t)\right)}. \tag{3.27}$$

Now, by (3.13), there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $\tau(t) \in [t_1, \infty)_{\mathbb{T}}$ for $t \in [t_2, \infty)_{\mathbb{T}}$ and

$$\begin{aligned}
x^{\Delta}(\tau(t)) &\geq \phi_{\alpha}^{-1}\left[x^{[n-1]}\left(t\right)\right] R_{m,2}(t, t_1) \\
&\geq \phi_{\alpha}^{-1}\left[x^{[n-1]}\left(\sigma(t)\right)\right] R_{m,2}(t, t_1) \\
&= \left[\left(\frac{w(t)}{\delta(t)}\right)^{\sigma}\right]^{1/\alpha}\left[x\left(\tau^{\sigma}(t)\right)\right]^{\gamma/\alpha} R_{m,2}(t, t_1),
\end{aligned}$$

and so

$$\frac{x^{\Delta}\left(\tau(t)\right)}{x\left(\tau^{\sigma}(t)\right)} \geq \left[\left(\frac{w(t)}{\delta(t)}\right)^{\sigma}\right]^{1/\alpha}\left[x\left(\tau^{\sigma}(t)\right)\right]^{\gamma/\alpha-1} R_{m,2}(t, t_1).$$

Then (3.27) becomes for $t \in [t_2, \infty)_{\mathbb{T}}$

$$\begin{aligned} w^{\Delta}(t) &\le -\delta(t)p(t) + \delta^{\Delta}(t)\left(\frac{w(t)}{\delta(t)}\right)^{\sigma} \\ &\quad - \gamma\delta(t)\tau^{\Delta}(t)R_{m,2}(t,t_1)\left[\left(\frac{w(t)}{\delta(t)}\right)^{\sigma}\right]^{1+1/\alpha}\left[x\left(\tau^{\sigma}(t)\right)\right]^{\gamma/\alpha-1}. \end{aligned} \tag{3.28}$$

If $\gamma > \alpha$, then $x\left(\tau^{\sigma}(t)\right) \ge x(t_1)$ for $t \in [t_2, \infty)_{\mathbb{T}}$, and we have

$$\left[x\left(\tau^{\sigma}(t)\right)\right]^{\gamma/\alpha-1} \ge \left[x(t_1)\right]^{\gamma/\alpha-1} =: c_1 > 0.$$

If $\gamma = \alpha$, we have $\left[x\left(\tau^{\sigma}(t)\right)\right]^{\gamma/\alpha-1} = 1$ for $t \in [t_2, \infty)_{\mathbb{T}}$; whereas if $\gamma < \alpha$, then there exist $b > 0$ and $t_3 \ge t_2$ such that $x^{[n-1]}(t) \le b$ for all $t \ge t_3$, and hence from (3.6), we have

$$b \ge x^{[n-1]}(t) \ge \phi_{\gamma}\left(x^{\sigma}\left(\tau\left(t\right)\right)\right)\int_t^{\infty} p\left(s\right)\Delta s.$$

So

$$\begin{aligned} \left[x\left(\tau^{\sigma}(t)\right)\right]^{\gamma/\alpha-1} &= \left[\phi_{\gamma}\left(x\left(\tau^{\sigma}\left(t\right)\right)\right)\right]^{\frac{\gamma-\alpha}{\alpha\gamma}} \\ &= \left[\phi_{\gamma}\left(x^{\sigma}\left(\tau\left(t\right)\right)\right)\right]^{\frac{\gamma-\alpha}{\alpha\gamma}} \ge c_2\ \left[\int_t^{\infty} p\left(s\right)\Delta s\right]^{\frac{\alpha-\gamma}{\alpha\gamma}} = c_2\ P^{\alpha/\gamma-1}(t), \end{aligned}$$

where $c_2 := b^{\frac{\gamma-\alpha}{\alpha\gamma}} > 0$. Combining all these we see that

$$\left[x\left(\tau^{\sigma}(t)\right)\right]^{\gamma/\alpha-1} \ge A(t), \qquad \text{for } t \ge t_3. \tag{3.29}$$

From (3.28) and (3.29), we obtain for $t \in [t_3, \infty)_{\mathbb{T}}$

$$w^{\Delta}(t) \le -\delta(t)p(t) + \delta^{\Delta}(t)\left(\frac{w(t)}{\delta(t)}\right)^{\sigma} - \gamma\delta(t)\tau^{\Delta}(t)A(t)R_2(t,t_1)\left[\left(\frac{w(t)}{\delta(t)}\right)^{\sigma}\right]^{\lambda}, \tag{3.30}$$

where $\lambda := 1 + 1/\alpha > 1$. Multiplying both sides of (3.30), with t replaced by u, by $H\left(t,u\right)$ and integrating with respect to u from t_3 to $t \in [t_3, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} \int_{t_3}^{t} H\left(t,u\right)\delta(u)p(u)\Delta u &\le -\int_{t_3}^{t} H\left(t,u\right)w^{\Delta}\left(u\right)\Delta u + \int_{t_3}^{t} H\left(t,u\right)\delta^{\Delta}(u)\left(\frac{w(u)}{\delta(u)}\right)^{\sigma}\Delta u \\ &\quad - \int_{t_3}^{t} \gamma\delta(u)\tau^{\Delta}(u)A(u)R_2(u,t_1)H\left(t,u\right)\left[\left(\frac{w(u)}{\delta(u)}\right)^{\sigma}\right]^{\lambda}\Delta u. \end{aligned}$$

Integrating by parts and using (3.21) and (3.22), we obtain

$$\begin{aligned}\int_{t_3}^{t} H(t,u)\,\delta(u)p(u)\Delta u &\le H(t,t_3)\,w(t_3) + \int_{t_3}^{t} H^{\Delta_u}(t,u)\,w^{\sigma}(u)\,\Delta u \\ &\quad + \int_{t_3}^{t} H(t,u)\,\delta^{\Delta}(u)\left(\frac{w(u)}{\delta(u)}\right)^{\sigma}\Delta u \\ &\quad - \int_{t_3}^{t}\left[\gamma\delta(u)\tau^{\Delta}(u)A(u)R_2(u,t_1)H(t,u)\left[\left(\frac{w(u)}{\delta(u)}\right)^{\sigma}\right]^{\lambda}\right]\Delta u \\ &= H(t,t_3)\,w(t_3) + \int_{t_3}^{t}\left[h_-(t,u)\,(H(t,u))^{\frac{1}{\lambda}}\left(\frac{w(u)}{\delta(u)}\right)^{\sigma}\right. \\ &\quad \left. -\gamma\delta(u)\tau^{\Delta}(u)A(u)R_2(u,t_1)H(t,u)\left[\left(\frac{w(u)}{\delta(u)}\right)^{\sigma}\right]^{\lambda}\right]\Delta u. \end{aligned} \tag{3.31}$$

Defining $X \ge 0$ and $Y \ge 0$ by

$$X^{\lambda} := \gamma\delta(u)\tau^{\Delta}(u)A(u)R_2(u,t_1)H(t,u)\left[\left(\frac{w(u)}{\delta(u)}\right)^{\sigma}\right]^{\lambda},$$

$$Y^{\lambda-1} := \frac{h_-(t,u)}{\lambda\left(\gamma\delta(u)\tau^{\Delta}(u)A(u)R_2(u,t_1)\right)^{1/\lambda}},$$

and using Lemma 3.5, we have

$$\begin{aligned}&\gamma\delta(u)\tau^{\Delta}(u)A(u)R_2(u,t_1)H(t,u)\left[\left(\frac{w(u)}{\delta(u)}\right)^{\sigma}\right]^{\lambda} - h_-(t,u)\,(H(t,u))^{\frac{1}{\lambda}}\left(\frac{w(u)}{\delta(u)}\right)^{\sigma} \\ &\quad + \frac{(\alpha/\gamma)^{\alpha}\,h_-^{\alpha+1}(t,u)}{(\alpha+1)^{\alpha+1}\left[\delta(u)\tau^{\Delta}(u)A(u)R_2(u,t_1)\right]^{\alpha}} \ge 0.\end{aligned}$$

From this last inequality and (3.31), we have

$$\int_{t_3}^{t}\left[\delta(u)p(u)H(t,u) - \frac{(\alpha/\gamma)^{\alpha}\,h_-^{\alpha+1}(t,u)}{(\alpha+1)^{\alpha+1}\left[\delta(u)\tau^{\Delta}(u)A(u)R_2(u,t_1)\right]^{\alpha}}\right]\Delta u \le H(t,t_3)\,w(t_3),$$

and this implies that

$$\frac{1}{H(t,t_3)}\int_{t_3}^{t}\left[\delta(u)p(u)H(t,u) - \frac{(\alpha/\gamma)^{\alpha}\,h_-^{\alpha+1}(t,u)}{(\alpha+1)^{\alpha+1}\left[\delta(u)\tau^{\Delta}(u)A(u)R_2(u,t_3)\right]^{\alpha}}\right]\Delta u \le w(t_3),$$

which contradicts assumption (3.23). This completes the proof. □

We assume $\alpha \ge 1$ in the following theorem.

Theorem 3.7. *Assume that* (2.4) *and* (2.1) *hold and* $\alpha \ge 1$ *and* $\tau \circ \sigma = \sigma \circ \tau$ *on* $[t_0,\infty)_{\mathbb{T}}$. *Furthermore, suppose that there exist functions* $\delta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ *and* $H, h \in C_{rd}(\mathbb{D},\mathbb{R})$, *where* $\mathbb{D} \equiv \{(t,u) : t \ge u \ge t_0\}$ *such that* (3.21) *and* H *has a nonpositive continuous* Δ*-partial derivative* $H^{\Delta_u}(t,u)$ *with respect to the second variable and satisfies*

$$H^{\Delta_u}(t,u) + H(t,u)\frac{\eta^{\Delta}(u)}{\delta^{\sigma}(u)} = -\frac{h(t,u)}{\delta^{\sigma}(u)}\sqrt{H(t,u)}, \tag{3.32}$$

and, for all sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$ and for every odd integer $m \in \{1, \dots, n-1\}$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,u)\,\delta(u)p(u) - \frac{[h_-(t,u)]^2 R_{m,1}^{1-\alpha}(u,T)}{4\gamma\delta(u)\tau^{\Delta}(u)R_{m,2}(u,T)A(u)B(u)} \right] \Delta u = \infty, \tag{3.33}$$

where A is defined by (3.24) and

$$B(t) := \begin{cases} c_1, c_1 \text{ is any positive constant}, & \text{when } \gamma > \alpha; \\ 1, & \text{when } \gamma = \alpha; \\ c_2 P^{(\alpha-\gamma)(\alpha-1)}(t), c_2 \text{ is any positive constant} & \text{when } \gamma < \alpha. \end{cases} \tag{3.34}$$

Then every solution of Eq. (1.1) is oscillatory.

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g_j(t)) > 0$, $j = 0, 1, 2, \dots, N$, on $[t_0, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 3.6, we obtain for $t \in [t_3, \infty)_{\mathbb{T}}$

$$\begin{aligned} w^{\Delta}(t) &\le -\delta(t)p(t) + \delta^{\Delta}(t)\left(\frac{w(t)}{\delta(t)}\right)^{\sigma} - \gamma\delta(t)\tau^{\Delta}(t)A(t)R_{m,2}(t,t_1)\left[\left(\frac{w(t)}{\delta(t)}\right)^{\sigma}\right]^{1+1/\alpha} \\ &= -\delta(t)p(t) + \delta^{\Delta}(t)\left(\frac{w(t)}{\delta(t)}\right)^{\sigma} \\ &\quad - \gamma\delta(t)\tau^{\Delta}(t)A(t)R_{m,2}(t,t_1)\frac{[w^{\sigma}(t)]^{1/\alpha-1}}{[\delta^{\sigma}(t)]^{1+1/\alpha}}[w^{\sigma}(t)]^2. \end{aligned} \tag{3.35}$$

In view of the definition of w and (3.12), we get

$$\begin{aligned} [w^{\sigma}(t)]^{1/\alpha-1} &= \left[\frac{\delta^{\sigma}(t)\left(x^{[n-1]}(t)\right)^{\sigma}}{x^{\gamma}(\tau^{\sigma}(t))}\right]^{\frac{1-\alpha}{\alpha}} \ge \left[\frac{\delta^{\sigma}(t)x^{[n-1]}(t)}{x^{\gamma}(\tau(t))}\right]^{\frac{1-\alpha}{\alpha}} \\ &\ge \left[\frac{\delta^{\sigma}(t)[x(\tau(t))]^{\alpha-\gamma}}{R_{m,1}^{\alpha}(t,T_1)}\right]^{\frac{1-\alpha}{\alpha}} \\ &= \frac{(\delta^{\sigma}(t))^{\frac{1-\alpha}{\alpha}}}{R_{m,1}^{1-\alpha}(t,T_1)}[x(\tau(t))]^{\frac{(\gamma-\alpha)(\alpha-1)}{\alpha}}. \end{aligned} \tag{3.36}$$

If $\gamma > \alpha$, then $x(\tau(t)) \ge x(\tau(t_3))$ for $t \ge t_3$, we have

$$[x(\tau(t))]^{\frac{(\gamma-\alpha)(\alpha-1)}{\alpha}} \ge [x(\tau(T_2))]^{\frac{(\gamma-\alpha)(\alpha-1)}{\alpha}} =: c_1 > 0.$$

If $\gamma = \alpha$, we have $[x(\tau(t))]^{\frac{(\gamma-\alpha)(\alpha-1)}{\alpha}} = 1$ for $t \ge t_3$; whereas if $\gamma < \alpha$, then there exist $b > 0$ and $t_4 \ge t_3$ such that $x^{[n-1]}(t) \le b$ for all $t \ge t_4$, and hence from (3.6), we have

$$b \ge x^{[n-1]}(t) \ge \phi_{\gamma}(x^{\sigma}(\tau(t)))\int_t^{\infty} p(s)\,\Delta s \ge \phi_{\gamma}(x(\tau(t)))\int_t^{\infty} p(s)\,\Delta s,$$

and so

$$[x(\tau(t))]^{\frac{(\gamma-\alpha)(\alpha-1)}{\alpha}} = [x(\tau(t))]^{\frac{(\gamma-\alpha)(\alpha-1)}{\alpha}} = [\phi_\gamma(x(\tau(t)))]^{\frac{(\gamma-\alpha)(\alpha-1)}{\alpha\gamma}}$$
$$\geq c_2 \left[\int_t^\infty p(s)\Delta s\right]^{\frac{(\alpha-\gamma)(\alpha-1)}{\alpha}} = c_2\ P^{(\alpha-\gamma)(\alpha-1)}(t),$$

where $c_2 := b^{\frac{(\gamma-\alpha)(\alpha-1)}{\alpha}} > 0$. Combining all these, we see that

$$[x(\tau^\sigma(t))]^{\frac{(\gamma-\alpha)(\alpha-1)}{\alpha}} \geq B(t), \qquad \text{for } t \geq t_4.$$

Substituting into (3.36), we get

$$[w^\sigma(t)]^{1/\alpha-1} \geq (\delta^\sigma(t))^{\frac{1-\alpha}{\alpha}} \frac{B(t)}{R_{m,1}^{1-\alpha}(t,T_1)}.$$

Then (3.35) becomes

$$w^\Delta(t) \leq -\delta(t)p(t) + \delta^\Delta(t)\left(\frac{w(t)}{\delta(t)}\right)^\sigma - \frac{\gamma\delta(t)\tau^\Delta(t)R_{m,2}(t,t_1)A(t)B(t)}{R_{m,1}^{1-\alpha}(t,t_1)}\left[\left(\frac{w(t)}{\delta(t)}\right)^\sigma\right]^2. \tag{3.37}$$

Multiplying both sides of (3.37), with t replaced by u, by $H(t,u)$, integrating with respect to u from t_4 to $t \in [t_4,\infty)_{\mathbb{T}}$, we have

$$\int_{t_4}^t H(t,u)\,\delta(u)p(u)\Delta u \leq -\int_{t_4}^t H(t,u)\,w^\Delta(u)\Delta u + \int_{t_4}^t H(t,u)\,\delta^\Delta(u)\left(\frac{w(u)}{\delta(u)}\right)^\sigma \Delta u$$
$$-\int_{t_3}^t \left[\frac{\gamma\delta(u)\tau^\Delta(u)R_{m,2}(u,t_1)A(u)B(u)H(t,u)}{R_{m,1}^{1-\alpha}(u,t_1)}\left[\left(\frac{w(u)}{\delta(u)}\right)^\sigma\right]^2\right]\Delta u.$$

Integrating by parts and using (3.21) and (3.32), we obtain

$$\int_{t_4}^t H(t,u)\,\delta(u)p(u)\Delta u \leq H(t,t_4)\,w(t_4) + \int_{t_4}^t H^{\Delta_u}(t,u)\,w^\sigma(u)\,\Delta u$$
$$+\int_{t_4}^t H(t,u)\,\delta^\Delta(u)\left(\frac{w(u)}{\delta(u)}\right)^\sigma \Delta u$$
$$-\int_{t_3}^t \left[\frac{\gamma\delta(u)\tau^\Delta(u)R_{m,2}(u,t_1)A(u)B(u)H(t,u)}{R_{m,1}^{1-\alpha}(u,t_1)}\left[\left(\frac{w(u)}{\delta(u)}\right)^\sigma\right]^2\right]\Delta u$$
$$\leq H(t,t_4)\,w(t_4) + \int_{t_4}^t \left[h_-(t,u)\sqrt{H(t,u)}\left(\frac{w(u)}{\delta(u)}\right)^\sigma\right.$$
$$\left.-\frac{\gamma\delta(u)\tau^\Delta(u)R_{m,2}(u,t_1)A(u)B(u)H(t,u)}{R_{m,1}^{1-\alpha}(u,t_1)}\left[\left(\frac{w(u)}{\delta(u)}\right)^\sigma\right]^2\right]\Delta u.$$

It follows that

$$\int_{t_4}^{t} H(t,u)\,\delta(u)p(u)\Delta u \le H(t,t_4)\,w(t_4)$$

$$- \int_{t_4}^{t} \left[\sqrt{\frac{\gamma\delta(u)\tau^{\Delta}(u)R_{m,2}(u,t_1)A(u)B(u)H(t,u)}{R_{m,1}^{1-\alpha}(u,t_1)}} \left(\frac{w(u)}{\delta(u)}\right)^{\sigma} \right.$$

$$\left. - \frac{h_-(t,u)}{2\sqrt{\frac{\gamma\delta(u)\tau^{\Delta}(u)R_{m,2}(u,t_1)A(u)B(u)}{R_{m,1}^{1-\alpha}(u,t_1)}}} \right]^2 \Delta u$$

$$+ \int_{t_4}^{t} \frac{[h_-(t,u)]^2 R_{m,1}^{1-\alpha}(u,t_1)}{4\gamma\delta(u)\tau^{\Delta}(u)R_{m,2}(u,t_1)A(u)B(u)}\Delta u$$

$$\le H(t,t_4)\,w(t_4) + \int_{t_4}^{t} \frac{[h_-(t,u)]^2 R_{m,1}^{1-\alpha}(t,t_1)}{4\gamma\delta(u)\tau^{\Delta}(u)R_{m,2}(u,t_1)A(u)B(u)}\Delta u.$$

Consequently,

$$\frac{1}{H(t,t_4)} \int_{t_4}^{t} \left[H(t,u)\,\delta(u)p(u) - \frac{[h_-(t,u)]^2 R_{m,1}^{1-\alpha}(u,t_4)}{4\gamma\delta(u)\tau^{\Delta}(u)R_{m,2}(u,t_4)A(u)B(u)} \right] \Delta u \le w(t_4),$$

which contradicts assumption (3.33). This completes that proof. □

Theorem 3.8. *Assume that* (2.4) *and* (2.1) *hold and there exists* $\delta \in C_{rd}^1([t_0,\infty)_{\mathbb{T}},(0,\infty))$ *such that for every sufficiently large* $T \in [t_0,\infty)_{\mathbb{T}}$

$$\limsup_{t\to\infty} \int_{T}^{t} \left[\delta(u)p(u) - \frac{\left(\delta^{\Delta}(u)\right)_+ C(u)}{R_{m,1}^{\alpha}(u,T)} \right] \Delta u = \infty, \tag{3.38}$$

where

$$C(t) := \begin{cases} c_1, c_1 \text{ is any positive constant}, & \text{when } \gamma > \alpha; \\ 1, & \text{when } \gamma = \alpha; \\ c_2 P^{\alpha(\gamma-\alpha)/\gamma}(t), c_2 \text{ is any positive constant} & \text{when } \gamma < \alpha. \end{cases} \tag{3.39}$$

Then every solution of Eq. (1.1) *is oscillatory.*

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g_j(t)) > 0$, $j = 0,1,2,\ldots,N$, on $[t_0,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 3.6, we obtain for sufficiently large t_3

$$\begin{aligned} w^{\Delta}(t) &\le -\delta(t)p(t) + \delta^{\Delta}(t)\left(\frac{w(t)}{\delta(t)}\right)^{\sigma} - \gamma\delta(t)\tau^{\Delta}(t)A(t)R_{m,2}(t,t_1)\left[\left(\frac{w(t)}{\delta(t)}\right)^{\sigma}\right]^{1+1/\alpha} \\ &\le -\delta(t)p(t) + \left(\delta^{\Delta}(t)\right)_+ \frac{\left(x^{[n-1]}(t)\right)^{\sigma}}{\phi_\gamma\left[x\left(\tau^{\sigma}(t)\right)\right]} \\ &\le -\delta(t)p(t) + \left(\delta^{\Delta}(t)\right)_+ \frac{x^{[n-1]}(t)}{\phi_\gamma\left[x\left(\tau(t)\right)\right]} \qquad \text{for } t \in [t_3,\infty)_{\mathbb{T}}. \end{aligned}$$

In view of (3.12), we get

$$x(\tau(t)) \ge \phi_\alpha^{-1}\left(x^{[n-1]}(t)\right) R_{m,1}(t,t_2), \qquad \text{for } t \in [t_3,\infty)_{\mathbb{T}}.$$

Therefore

$$w^{\Delta}(t) \le -\delta(t)p(t) + \frac{\left(\delta^{\Delta}(t)\right)_+}{R_{m,1}^{\alpha}(t,t_2)}\left[x\left(\tau(t)\right)\right]^{\alpha-\gamma}, \qquad \text{for } t \in [t_3,\infty)_{\mathbb{T}}. \tag{3.40}$$

Let $\gamma > \alpha$. Since $x(t) \ge x(t_3) := c > 0$ for all $t \ge t_3$, we have $\left[x\left(\tau(t)\right)\right]^{\alpha-\gamma} \le c^{\alpha-\gamma} := c_1$ for all $t \ge t_3$. If $\gamma = \alpha$, then $\left[x\left(\tau(t)\right)\right]^{\alpha-\gamma} = 1$ for all $t \ge t_3$. If $\gamma < \alpha$, then there exist $b > 0$ and $t_4 \ge t_3$ such that $x^{[n-1]}(t) \le b$ for all $t \ge t_4$ and hence from (3.6), we have

$$b \ge x^{[n-1]}(t) \ge \phi_\gamma\left(x^{\sigma}\left(\tau\left(t\right)\right)\right)\int_t^{\infty} p\left(s\right)\Delta s \ge \phi_\gamma\left(x\left(\tau\left(t\right)\right)\right)\int_t^{\infty} p\left(s\right)\Delta s,$$

and so

$$\left[x\left(\tau(t)\right)\right]^{\alpha-\gamma} = \left[x^{\gamma}\left(\tau(t)\right)\right]^{\frac{\alpha-\gamma}{\gamma}} \le c_2\left[\int_t^{\infty} p\left(s\right)\Delta s\right]^{\frac{\gamma-\alpha}{\gamma}} = c_2\, P^{\alpha(\gamma-\alpha)/\gamma}(t),$$

where $c_2 := b^{\frac{\alpha-\gamma}{\alpha}} > 0$. Combining all these we see that

$$\left[x\left(\tau(t)\right)\right]^{\alpha-\gamma} \le C(t), \qquad \text{for } t \ge t_4.$$

From (3.40), we have

$$w^{\Delta}(t) \le -\delta(t)p(t) + \left(\delta^{\Delta}(t)\right)_+ \frac{C(t)}{R_{m,1}^{\alpha}(t,t_1)} \le -\delta(t)p(t) + \left(\delta^{\Delta}(t)\right)_+ \frac{C(t)}{R_{m,1}^{\alpha}(t,t_4)}.$$

Integrating this inequality from t_4 to t, we find

$$\int_{t_4}^{t}\left[\delta(s)p(s) - \left(\delta^{\Delta}(s)\right)_+ \frac{C(s)}{R_{m,1}^{\alpha}(s,t_4)}\right]\Delta s \le w(t_4).$$

Taking limit superior as $t \to \infty$, we obtain a contradiction to condition (3.38). This completes the proof. □

As a direct consequence of Theorems 3.1-3.8, we obtain oscillation criteria for Eq. (1.1) with $n = 2$; namely, for the equation

$$\left(r_1(t)\phi_{\alpha_1}\left(x^{\Delta}(t)\right)\right)^{\Delta} + \sum_{j=0}^{N} p_j\left(t\right)\phi_{\gamma_j}\left(x^{\sigma}\left(g_j\left(t\right)\right)\right) = 0. \tag{3.41}$$

Corollary 3.9. *Assume that* (3.1) *holds. Then every solution of Eq.* (3.41) *is oscillatory.*

Corollary 3.10. *Assume that* (2.4) *and* (2.1) *hold. Every solution of Eq.* (3.41) *is oscillatory provided one of the following conditions is satisfied for sufficiently large* $T \in [t_0, \infty)_{\mathbb{T}}$:

(a)

$$\int_{t_0}^{\infty} p(t)\, R_1^{\gamma}(t,T)\Delta t = \infty, \qquad \text{if} \quad \gamma < \alpha;$$

$$\limsup_{t\to\infty} P(t) R_1(t,T) > 1, \qquad \text{if} \quad \gamma = \alpha;$$

$$\text{either} \quad \int_{t_0}^{\infty} \tau^{\Delta}(t) P(t) R_2(t)\Delta t \qquad \text{if} \quad \gamma > \alpha;$$

$$\text{or} \quad \int_{T}^{\infty} \tau^{\Delta}(t) Q(t)\Delta t = \infty,$$

(b) there exist $\delta \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, (0,\infty))$ *and* $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv \{(t,u) : t \geq u \geq t_0\}$ *such that* (3.21), (3.22) *and*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\delta(u)p(u)H(t,u) - \frac{(\alpha_1/\gamma)^{\alpha_1}\, h_-^{\alpha_1+1}(t,u)}{(\alpha_1+1)^{\alpha_1+1}\left[\delta(u)\tau^{\Delta}(u)A(u)R_2(u)\right]^{\alpha_1}}\right]\Delta u = \infty,$$

where $\tau \circ \sigma = \sigma \circ \tau$ *on* $[t_0,\infty)_{\mathbb{T}}$*;*

(c) there exist $\delta \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, (0,\infty))$ *and* $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv \{(t,u) : t \geq u \geq t_0\}$ *such that* (3.21), (3.32) *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,u)\,\delta(u)p(u) - \frac{\left[h_-(t,u)\right]^2 R_1^{1-\alpha}(u,T)}{4\gamma\delta(u)\tau^{\Delta}(u)R_2(u)A(u)B(u)}\right]\Delta u = \infty,$$

for $\alpha \geq 1$ *and where* $\tau \circ \sigma = \sigma \circ \tau$ *on* $[t_0,\infty)_{\mathbb{T}}$*;*

(d) there exist $\delta \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, (0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_T^t \left[\delta(u)p(u) - \frac{\left(\delta^{\Delta}(u)\right)_+ C(u)}{R_1^{\alpha}(u,T)}\right]\Delta u = \infty,$$

where A, B *and* C *are defined by* (3.24), (3.34) *and* (3.39) *respectively and where*

$$R_1(t,T) := R_{1,1}(t,T) = \int_T^{\tau(t)} r_1^{-1/\alpha_1}(s)\ \Delta s, \ \ R_2(t) := R_{1,2}(t,T) = r_1^{-1/\alpha_1}\left(\tau(t)\right),$$

and

$$Q(t) := Q_1(t,T) = \left[\int_t^{\infty} p(u)\Delta u \Big/ r_1(\tau(t))\right]^{1/\alpha_1}.$$

Corollary 3.11. *Assume that* (2.4) *and* (2.1) *hold and for sufficiently large* $T \in [t_0,\infty)_{\mathbb{T}}$

$$\limsup_{t\to\infty} P(t)R_1(t,T) = \infty.$$

Then every bounded solution of Eq. (3.41) *is oscillatory.*

For Eq. (1.1) with an even $n \geq 4$, we have further criteria for oscillation as shown below. We denote

$$\bar{P}_i(t) := \begin{cases} \left[\int_t^\infty \bar{P}_{i-1}(s)\,\Delta s / r_{n-i}(t)\right]^{1/\alpha_{n-i}} & i = 1, \ldots, n-1, \\ \sum_{j=0}^N p_j(t), & i = 0. \end{cases}$$

Theorem 3.12. *Assume that* (2.4) *and* (2.1) *hold and*

$$\text{either} \quad \int_{t_0}^\infty \bar{P}_1(t)\Delta t = \infty \quad \text{or} \quad \int_{t_0}^\infty \bar{P}_2(t)\Delta t = \infty, \tag{3.42}$$

and for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$,

$$\begin{aligned} &\textstyle\int_{t_0}^\infty p(t)\, R_{n-1,1}^\gamma(t,T)\Delta t = \infty, && \text{if} \quad \gamma < \alpha; \\ &\limsup_{t\to\infty} P(t)R_{n-1,1}(t,T) > 1, && \text{if} \quad \gamma = \alpha; \\ &\textstyle\int_{t_0}^\infty \tau^\Delta(t)P(t)R_{n-1,2}(t,T)\Delta t = \infty, && \text{if} \quad \gamma > \alpha. \end{aligned} \tag{3.43}$$

Then every solution of Eq. (1.1) *is oscillatory.*

Theorem 3.13. *The conclusions of Theorem* 3.12 *hold if the third condition in* (3.43) *is replaced by*

$$\int_T^\infty \tau^\Delta(t)Q_{n-1}(t,T)\Delta t = \infty, \tag{3.44}$$

for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$ *and where*

$$Q_{n-1}(t,T) := \phi_{\alpha[1,n-1]}^{-1}\left[\int_s^\infty p(u)\Delta u\right]\hat{R}_{n-2}(\tau(t),T).$$

Theorem 3.14. *Assume that* (2.4), (2.1) *and* (3.42) *hold and for sufficiently large* $T \in [t_0, \infty)_{\mathbb{T}}$

$$\limsup_{t\to\infty} P(t)R_{n-1,1}(t,T) = \infty. \tag{3.45}$$

Then every bounded solution of Eq. (1.1) *is oscillatory.*

Theorem 3.15. *Assume* (2.4), (2.1) *and* (3.42) *hold and* $\tau \circ \sigma = \sigma \circ \tau$ *on* $[t_0, \infty)_{\mathbb{T}}$. *Furthermore, suppose that there exist functions* $\delta \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ *and* $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv \{(t,u) : t \geq u \geq t_0\}$ *such that* (3.21), (3.22) *and*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\delta(u)p(u)H(t,u) - \frac{(\alpha/\gamma)^\alpha [h_-(t,u)]^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left[\delta(u)\tau^\Delta(u)A(u)R_{n-1,2}(u,T)\right]^\alpha}\right]\Delta u = \infty, \tag{3.46}$$

where A *is defined by* (3.24). *Then every solution of Eq.* (1.1) *is oscillatory.*

Theorem 3.16. *Assume* (2.4), (2.1) *and* (3.42) *hold,* $\alpha \geq 1$ *and* $\tau \circ \sigma = \sigma \circ \tau$ *on* $[t_0, \infty)_{\mathbb{T}}$. *Furthermore, suppose that there exist functions* $\delta \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, (0, \infty))$

and $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv \{(t,u) : t \geq u \geq t_0\}$ *such that* (3.21), (3.32) *and for all sufficiently large* $T \in [t_0, \infty)_{\mathbb{T}}$

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,u)\,\delta(u)p(u) - \frac{[h_-(t,u)]^2 R_{n-1,1}^{1-\alpha}(u,T)}{4\gamma\delta(u)\tau^{\Delta}(u)R_{n-1,2}(u,T)A(u)B(u)} \right] \Delta u$$
$$= \infty, \tag{3.47}$$

where A *and* B *are defined by* (3.24) *and* (3.34). *Then every solution of Eq.* (1.1) *is oscillatory.*

Theorem 3.17. *Assume* (2.4), (2.1) *and* (3.42) *hold and there exists* $\delta \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, (0,\infty))$ *such that for every sufficiently large* $T \in [t_0,\infty)_{\mathbb{T}}$

$$\limsup_{t\to\infty} \int_T^t \left[\delta(u)p(u) - \frac{\left(\delta^{\Delta}(u)\right)_+ C(u)}{R_{n-1,1}^{\alpha}(u,T)} \right] \Delta u = \infty, \tag{3.48}$$

where C *is defined by* (3.39). *Then every solution of Eq.* (1.1) *is oscillatory.*

Proofs of Theorems 3.12–3.17. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g_j(t)) > 0$, $j = 0, 1, 2, \ldots, N$, on $[t_0,\infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an odd integer $m \in \{1, \ldots, n-1\}$ such that (2.2) and (2.3) hold for $t \geq t_1 \in [t_0,\infty)_{\mathbb{T}}$. This implies $x(t)$ is strictly increasing on $[t_1,\infty)_{\mathbb{T}}$. We claim that (3.42) implies that $m = n-1$. In fact, if $1 \leq m \leq n-3$, then for $t \geq t_1$

$$x^{[n]}(t) < 0,\ x^{[n-1]}(t) > 0,\ x^{[n-2]}(t) < 0,\ x^{[n-3]}(t) > 0. \tag{3.49}$$

As seen in the proof of Theorem 3.1, there exists $t_2 \in [t_1,\infty)_{\mathbb{T}}$ and $L > 0$ such that

$$-\left(x^{[n-1]}(t)\right)^{\Delta} = \sum_{j=0}^{N} p_j(t)\,\phi_{\gamma_j}\left(x^{\sigma}(g_j(t))\right) \geq L \sum_{j=0}^{N} p_j(t) = L\ \bar{P}_0(t) \quad \text{for } t \in [t_2,\infty)_{\mathbb{T}}. \tag{3.50}$$

Integrating (3.50) from t to $v \in [t,\infty)_{\mathbb{T}}$ and using (3.49), we get that

$$x^{[n-1]}(t) \geq -x^{[n-1]}(v) + x^{[n-1]}(t) \geq L \int_t^v \bar{P}_0(s)\,\Delta s.$$

By taking limits as $v \to \infty$, we have

$$x^{[n-1]}(t) \geq L \int_t^{\infty} \bar{P}_0(s)\,\Delta s.$$

It is known from Theorem 3.1 that $\int_t^{\infty} \bar{P}_0(s)\,\Delta s < \infty$. Thus

$$\left(x^{[n-2]}(t)\right)^{\Delta} \geq L^{1/\alpha_{n-1}} \left[\frac{1}{r_{n-1}(t)} \int_t^{\infty} \bar{P}_0(s)\,\Delta s \right]^{1/\alpha_{n-1}} = L^{1/\alpha_{n-1}} \bar{P}_1(t). \tag{3.51}$$

Assume $\int_{t_0}^{\infty} \bar{P}_1(t)\Delta t = \infty$. Integrating above inequality from t_2 to $t \in [t_2, \infty)_{\mathbb{T}}$ and noting that $x^{[n-2]} < 0$ eventually, we get

$$x^{[n-2]}(t) - x^{[n-2]}(t_2) \geq L^{1/\alpha_{n-1}} \int_{t_2}^{t} \bar{P}_1(s)\Delta s.$$

Then by (3.42), we have $\lim_{t\to\infty} x^{[n-2]}(t) = \infty$, which contradicts the fact that $x^{[n-2]}(t) < 0$ on $[t_2, \infty)_{\mathbb{T}}$. Assume $\int_{t_0}^{\infty} \bar{P}_2(t)\Delta t = \infty$. By integrating the inequality (3.51) from t to $v \in [t, \infty)_{\mathbb{T}}$ and then taking limits as $v \to \infty$ and using the fact $x^{[n-2]} < 0$ eventually, we get

$$-x^{[n-2]}(t) > L^{1/\alpha_{n-1}} \int_{t}^{\infty} \bar{P}_1(s)\Delta s,$$

which implies

$$-\left(x^{[n-3]}(t)\right)^{\Delta} > L^{1/\alpha[n-2,n-1]} \left[\frac{1}{r_{n-2}(t)} \int_{t}^{\infty} \bar{P}_1(s)\Delta s\right]^{1/\alpha_{n-2}} = L^{1/\alpha[n-2,n-1]} \bar{P}_2(t).$$

Again, integrating above inequality from t_2 to $t \in [t_2, \infty)_{\mathbb{T}}$ and noting that $x^{[n-3]} > 0$ eventually, we get

$$x^{[n-3]}(t_2) - x^{[n-3]}(t) \geq L^{1/\alpha[n-2,n-1]} \int_{t_2}^{t} \bar{P}_2(s)\Delta s.$$

As a result, $\lim_{t\to\infty} x^{[n-3]}(t) = -\infty$, which contradicts the fact that $x^{[n-3]} > 0$ on $[t_2, \infty)_{\mathbb{T}}$. This shows that if (3.42) holds, then $m = n - 1$. The rest of the proof of Theorems 3.12–3.17 are similar to the proof of Theorems 3.2–3.8 with $m = n - 1$ respectively and hence can be omitted. ☐

Remark 3.18. The conclusions of Theorem 3.12–3.17 remain intact if assumption (3.42) is replaced by one of the following conditions holds:

$$\text{either} \quad \int_{t_0}^{\infty} P_1(t)\Delta t = \infty \quad \text{or} \quad \int_{t_0}^{\infty} P_2(t)\Delta t = \infty. \tag{3.52}$$

4. OSCILLATION CRITERIA FOR ODD ORDER EQUATIONS

In this section, we establish the oscillation criteria for Eq. (1.1) when n is odd. It follows from Lemma 2.1 that there exists an even integer $m \in \{0, \dots, n-1\}$ such that (2.2) and (2.3) hold eventually.

Theorem 4.1. *Assume that* (2.1) *and* (3.1) *hold. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero eventually.*

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g_j(t)) > 0$, $j = 0, 1, 2, \dots, N$, on $[t_0, \infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an even integer $m \in \{0, \dots, n-1\}$ such that (2.2) and (2.3) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$.

(I) We show that if $m=0$, then $\lim_{t\to\infty} x(t)=0$. In this case

$$(-1)^k x^{[k]} > 0 \quad \text{for} \quad k=0,1,\dots,n.$$

This implies that $x(t)$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. Then $\lim_{t\to\infty} x(t)=l\ge 0$. Assume $l>0$. Then for sufficiently large $t_2\in[t_1,\infty)_{\mathbb{T}}$, we have $x^\sigma(g_j(t))\ge l$ for $t\ge t_2$. It follows that

$$\phi_{\gamma_j}\left(x^\sigma\left(g_j\left(t\right)\right)\right)\ge l^{\gamma_j}\ge L \ \text{ for } t\in[t_2,\infty)_{\mathbb{T}},$$

where $L:=\inf_{0\le j\le N}\left\{l^{\gamma_j}\right\}>0$. Then from (1.1), we obtain

$$-\left(x^{[n-1]}\left(t\right)\right)^{\Delta}=\sum_{j=0}^{N} p_j\left(t\right)\phi_{\gamma_j}\left(x^\sigma\left(g_j\left(t\right)\right)\right)\ge L\sum_{j=0}^{N} p_j\left(t\right) \quad \text{for } t\in[t_2,\infty)_{\mathbb{T}}.$$

Replacing t by s in above inequality and integrating from t_2 to $t\in[t_2,\infty)_{\mathbb{T}}$, we obtain

$$-x^{[n-1]}\left(t\right)+x^{[n-1]}\left(t_2\right)\ge L\ \sum_{j=0}^{N}\int_{t_2}^{t} p_j\left(s\right)\Delta s.$$

Hence by (3.1), we have $\lim_{t\to\infty} x^{[n-1]}\left(t\right)=-\infty$, which contradicts the fact that $x^{[n-1]}\left(t\right)>0$ eventually.

(II) Assume $m\ge 2$. Then the same argument as in the proof of Theorem 3.1 leads to a contradiction to assumption (3.1). This completes the proof. □

Theorem 4.2. *Assume that* (2.1) *holds and*

$$\int_{t_0}^{\infty}\bar{P}_{n-1}(t)\Delta t=\infty. \tag{4.1}$$

If

$$\sum_{j=0}^{N}\int_{t_2}^{\infty} p_j\left(s\right)\int_{t_1}^{g_i(s)} r_1^{-1/\alpha_1}(\zeta)\Delta\zeta\Delta s=\infty, \tag{4.2}$$

then every solution of Eq. (1.1) *is either oscillatory or tends to zero eventually.*

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t)>0$ and $x\left(g_j\left(t\right)\right)>0$, $j=0,1,2,\dots,N$, on $[t_0,\infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an even integer $m\in\{0,\dots,n-1\}$ such that (2.2) and (2.3) hold for $t\ge t_1\in[t_0,\infty)_{\mathbb{T}}$.

(I) We show that if $m=0$, then $\lim_{t\to\infty} x(t)=0$. In this case

$$(-1)^k x^{[k]} > 0 \quad \text{for} \quad k=0,1,\dots,n. \tag{4.3}$$

This implies that $x(t)$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. Then $\lim_{t\to\infty} x(t)=l\ge 0$. Assume $l>0$. Then for sufficiently large $t_2\in[t_1,\infty)_{\mathbb{T}}$, we have $x^\sigma(g_j(t))\ge l$ for $t\ge t_2$. It follows that

$$\phi_{\gamma_j}\left(x^\sigma\left(g_j\left(t\right)\right)\right)\ge l^{\gamma_j}\ge L \text{ for } t\in[t_2,\infty)_{\mathbb{T}},$$

where $L := \inf_{0\leq j\leq N}\{l^{\gamma_j}\} > 0$. Then from (1.1), we obtain

$$-\left(x^{[n-1]}(t)\right)^{\Delta} = \sum_{j=0}^{N} p_j(t)\,\phi_{\gamma_j}\left(x^{\sigma}(g_j(t))\right) \geq L\sum_{j=0}^{N} p_j(t) = L\ \bar{P}_0(t).$$

Integrating above inequality from t to $v \in [t,\infty)_{\mathbb{T}}$, we get

$$-x^{[n-1]}(v) + x^{[n-1]}(t) \geq L\int_t^v \bar{P}_0(s)\,\Delta s,$$

and by (4.3) we see that $x^{[n-1]}(v) > 0$. Hence by taking limits as $v \to \infty$, we have

$$x^{[n-1]}(t) \geq L\int_t^\infty \bar{P}_0(s)\,\Delta s,$$

which implies

$$\left(x^{[n-2]}(t)\right)^{\Delta} \geq L^{1/\alpha_{n-1}}\ \left[\int_t^\infty \bar{P}_0(s)\,\Delta s \Big/ r_{n-1}(t)\right]^{1/\alpha_{n-1}} = L^{1/\alpha_{n-1}}\ \bar{P}_1(t).$$

Integrating from t to $v \in [t,\infty)_{\mathbb{T}}$ and letting $v \to \infty$ and using (4.3), we get

$$-x^{[n-2]}(t) \geq L^{1/\alpha_{n-1}}\ \int_t^\infty \bar{P}_1(s)\Delta s.$$

Continuing this process, we get

$$-x^{[1]}(t) \geq L^{1/\alpha[2,n-1]}\ \int_t^\infty \bar{P}_{n-2}(s)\ \Delta s,$$

which implies

$$-x^{\Delta}(t) \geq L^{1/\alpha[1,n-1]}\ \left[\int_t^\infty \bar{P}_{n-2}(s)\,\Delta s \Big/ r_1(t)\right]^{1/\alpha_1} = L^{1/\alpha[1,n-1]}\bar{P}_{n-1}(t).$$

Again, integrating the above inequality from t_2 to $t \in [t_2,\infty)_{\mathbb{T}}$, we get

$$-x(t) + x(t_2) \geq L^{1/\alpha[1,n-1]}\int_{t_2}^t \bar{P}_{n-1}(s)\ \Delta s$$

Hence by (4.1), we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that $x > 0$ eventually. This shows that if $m = 0$, then $\lim_{t\to\infty} x(t) = 0$.

(II) Assume $m \geq 2$. This implies

$$x^{[1]}(t) > 0, \quad x^{[2]}(t) > 0 \quad \text{and} \quad x^{[n-1]}(t) > 0 \quad \text{for } t \geq t_1.$$

Since $x^{[2]}(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$, we have

$$x^{[1]}(t) \geq x^{[1]}(t_1) =: c > 0.$$

Thus for $t \geq t_1$,

$$x(t) \geq x(t) - x(t_1) \geq c^{1/\alpha_1}\int_{t_1}^t r_1^{-1/\alpha_1}(s)\Delta s.$$

Choose $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that for $t \in [t_2, \infty)_{\mathbb{T}}$

$$\phi_{\gamma_j}\left(x^{\sigma}\left(g_j\left(t\right)\right)\right) \geq \phi_{\gamma_j}\left(x\left(g_j\left(t\right)\right)\right) \geq c^{\gamma_j/\alpha_1}\int_{t_1}^{g_i(t)} r_1^{-1/\alpha_1}(s)\Delta s \geq C\int_{t_1}^{g_i(t)} r_1^{-1/\alpha_1}(s)\Delta s, \tag{4.4}$$

where $C := \inf_{0\leq j\leq N}\left\{c^{\gamma_j/\alpha_1}\right\} > 0$. It follows from (1.1) and (4.4) that

$$-\left(x^{[n-1]}\left(t\right)\right)^{\Delta} \geq C\sum_{j=0}^{N} p_j\left(t\right)\int_{t_1}^{g_i(t)} r_1^{-1/\alpha_1}(s)\Delta s \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}.$$

Integrating both sides of the last inequality from t_2 to t, we have

$$-x^{[n-1]}\left(t\right) + x^{[n-1]}\left(t_2\right) \geq C\sum_{j=0}^{N}\int_{t_2}^{t} p_j\left(s\right)\int_{t_1}^{g_i(s)} r_1^{-1/\alpha_1}(\zeta)\Delta\zeta\Delta s.$$

Hence by (4.2), we have $\lim_{t\to\infty} x^{[n-1]}\left(t\right) = -\infty$, which contradicts the fact that $x^{[n-1]}\left(t\right) > 0$ eventually. This completes the proof. □

Theorem 4.3. *Assume that* (2.4), (2.1), (3.3) *and* (4.1) *hold for sufficiently large* $T \in [t_0, \infty)_{\mathbb{T}}$ *and for every even integer* $m \in \{2, \ldots, n-1\}$. *Then every solution of Eq.* (1.1) *is oscillatory or tends to zero eventually.*

Theorem 4.4. *The conclusions of Theorem* 4.3 *hold if the third condition in* (3.3) *is replaced by* (3.14) *for sufficiently large* $T \in [t_0, \infty)_{\mathbb{T}}$ *and for every even integer* $m \in \{2, \ldots, n-1\}$.

Theorem 4.5. *Assume that* (2.4), (2.1), (3.17) *and* (4.1) *hold for sufficiently large* $T \in [t_0, \infty)_{\mathbb{T}}$ *and for every odd integer* $m \in \{2, \ldots, n-1\}$. *Then every bounded solution of Eq.* (1.1) *is oscillatory or tends to zero eventually.*

Theorem 4.6. *Assume* (2.4) *and* (2.1) *hold and* $\tau \circ \sigma = \sigma \circ \tau$ *on* $[t_0, \infty)_{\mathbb{T}}$. *Furthermore, suppose that there exist functions* $\delta \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ *and* $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv \{(t, u) : t \geq u \geq t_0\}$ *such that* (3.21), (3.22)*and* (3.23) *and* (4.1) *hold for all sufficiently large* T *and for every even integer* $m \in \{2, \ldots, n-1\}$. *Then every solution of Eq.* (1.1) *is oscillatory or tends to zero eventually.*

Theorem 4.7. *Assume* (2.4) *and* (2.1) *hold,* $\alpha \geq 1$ *and* $\tau \circ \sigma = \sigma \circ \tau$ *on* $[t_0, \infty)_{\mathbb{T}}$. *Furthermore, suppose that there exist functions* $\delta \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ *and* $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv \{(t, u) : t \geq u \geq t_0\}$ *such that* (3.21), (3.32) *and* (3.33) *and* (4.1) *hold for all sufficiently large* T *and for every even integer* $m \in \{2, \ldots, n-1\}$. *Then every solution of Eq.* (1.1) *is oscillatory or tends to zero eventually.*

Theorem 4.8. *Assume* (2.4) *and* (2.1) *hold and that there exists* $\delta \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ *such that* (3.38) *and* (4.1) *hold for all sufficiently large* T *and for every even integer* $m \in \{2, \ldots, n-1\}$. *Then every solution of Eq.* (1.1) *is oscillatory or tends to zero eventually.*

Proofs of Theorems 4.3–4.8. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g_j(t)) > 0$, $j = 0, 1, 2, \ldots, N$, on $[t_0, \infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an even integer $m \in \{0, \ldots, n-1\}$ such that (2.2) and (2.3) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$.

(I) Assume $m = 0$. The same argument as in the proof of Theorem 4.2 and hence is omitted.

(II) Assume $m \geq 2$. The same argument as in the proof of Theorems 3.2–3.8 respectively and hence is omitted. This completes the proof. □

Remark 4.9. (1) If the assumption (4.1) is not satisfied, we have some sufficient conditions which ensure that every solution $x(t)$ of (1.1) oscillates or $\lim_{t\to\infty} x(t)$ exists (finite).

(2) The conclusions of Theorem 4.2–4.8 remain intact if assumption (4.1)is replaced by one of the following conditions

$$\int_{t_0}^{\infty} \bar{P}_0(t)\Delta t = \infty, \int_{t_0}^{\infty} \bar{P}_1(t)\Delta t = \infty, \ldots, \int_{t_0}^{\infty} \bar{P}_{n-2}(t)\Delta t = \infty.$$

(3) The conclusions of Theorem 4.3–4.8 remain intact if assumption (4.1)is replaced by one of the following conditions either

$$\int_{t_0}^{\infty} P_0(t)\Delta t = \infty, \int_{t_0}^{\infty} P_1(t)\Delta t = \infty, \ldots, \int_{t_0}^{\infty} P_{n-1}(t)\Delta t = \infty.$$

In the following theorems we assume whether (3.42) or (3.52) holds.

Theorem 4.10. *Assume that* (2.4), (2.1), (3.42) *and* (3.43) *hold for sufficiently large* $T \in [t_0, \infty)_{\mathbb{T}}$. *Then every solution of Eq.* (1.1) *is oscillatory or tends to zero eventually.*

Theorem 4.11. *The conclusions of Theorem 4.10 hold if the third condition in* (3.43) *is replaced by* (3.44) *for sufficiently large* $T \in [t_0, \infty)_{\mathbb{T}}$.

Theorem 4.12. *Assume that* (2.4), (2.1), (3.42) *and* (3.45) *hold for sufficiently large* $T \in [t_0, \infty)_{\mathbb{T}}$. *Then every bounded solution of Eq.* (1.1) *is oscillatory or tends to zero eventually.*

Theorem 4.13. *Assume* (2.4) *and* (2.1) *hold and* $\tau \circ \sigma = \sigma \circ \tau$ *on* $[t_0, \infty)_{\mathbb{T}}$. *Furthermore, suppose that there exist functions* $\delta \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ *and* $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv \{(t, u) : t \geq u \geq t_0\}$ *such that* (3.21), (3.22)*and* (3.42) *and* (3.46) *hold for all sufficiently large* T *and for every even integer* $m \in \{2, \ldots, n-1\}$. *Then every solution of Eq.* (1.1) *is oscillatory or tends to zero eventually.*

Theorem 4.14. *Assume* (2.4) *and* (2.1) *hold,* $\alpha \geq 1$ *and* $\tau \circ \sigma = \sigma \circ \tau$ *on* $[t_0, \infty)_{\mathbb{T}}$. *Furthermore, suppose that there exist functions* $\delta \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ *and* $H, h \in$

$C_{rd}(\mathbb{D},\mathbb{R})$, where $\mathbb{D} \equiv \{(t,u) : t \geq u \geq t_0\}$ such that (3.21), (3.32) *and* (3.42) *and* (3.47) *hold for all sufficiently large T. Then every solution of Eq.* (1.1) *is oscillatory or tends to zero eventually.*

Theorem 4.15. *Assume* (2.4) *and* (2.1) *hold and that there exists $\delta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ such that* (3.42) *and* (3.48)*hold for all sufficiently large T and for every even integer $m \in \{2,\dots,n-1\}$. Then every solution of Eq.* (1.1) *is oscillatory or tends to zero eventually.*

Proofs of Theorems 4.10–4.15. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g_j(t)) > 0$, $j = 0,1,2,\dots,N$, on $[t_0,\infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an even integer $m \in \{0,\dots,n-1\}$ such that (2.2) and (2.3) hold for $t \geq t_1 \in [t_0,\infty)_{\mathbb{T}}$.

(I) Assume $m = 0$. In this case

$$(-1)^k x^{[k]} > 0 \quad \text{for} \quad k = 0,1,\dots,n.$$

This implies that $x(t)$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. Then $\lim_{t\to\infty} x(t) = l \geq 0$. Assume $l > 0$. Then for sufficiently large $t_2 \in [t_1,\infty)_{\mathbb{T}}$, we have $x^\sigma(g_j(t)) \geq l$ for $t \geq t_2$. It follows that

$$\phi_{\gamma_j}(x^\sigma(g_j(t))) \geq l^{\gamma_j} \geq L \ \text{ for } t \in [t_2,\infty)_{\mathbb{T}},$$

where $L := \inf_{0\leq j\leq N}\{l^{\gamma_j}\} > 0$. Then from (1.1), we obtain

$$-\left(x^{[n-1]}(t)\right)^\Delta = \sum_{j=0}^{N} p_j(t)\,\phi_{\gamma_j}(x^\sigma(g_j(t))) \geq L\sum_{j=0}^{N} p_j(t) = L\ \bar{P}_0(t).$$

The rest of the proof is similar to the proof of Theorems 3.12–3.17, which leads to a contradiction to the assumption (3.42). This shows that if $m = 0$, then $\lim_{t\to\infty} x(t) = 0$. The same argument as in the proof of Theorem 4.2 can be used and hence is omitted.

(II) Assume $m \geq 2$. The same argument as in the proof of Theorems 3.12–3.17 and hence is omitted. This completes the proof. ☐

Remark 4.16. (1) The results are more general than any of the results in the references since, by different choices for γ_i, we can get that all terms are sublinear, or all terms are superlinear, or a combination of sublinear and superlinear terms.

(2) The results in this paper are including the both cases and also we do not need to assume $g_j(t) \geq t$ or $g_j(t) \leq t$, for all sufficiently large t.

(3) The results in this paper are in a form with a high degree of generality, thus with an appropriate choice of the functions $\delta(t)$and $H(t,s)$, we can get several sufficient conditions for oscillation of equation (1.1). For instance, if we choose $H(t,s) = (t-s)^n$, $n \geq 2$, or $H(t,s) = (t-s)^{(n)}$, where $t^{(n)} = t(t-1)\cdots(t-n+1)$, $t^{(0)} = 1$,

or $H(t,s) = (A(t) - A(s))^n$, where $A(t) = \int_{t_0}^{t} \frac{\Delta s}{r(s)}$, for $t \geq s \geq t_0$; we may choose $\delta(t)$ by 1, or t, etc.

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