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EXISTENCE OF A SOLUTION TO A CONJUGATE BOUNDARY VALUE PROBLEM APPLYING A COROLLARY OF THE OMITTED RAY FIXED POINT THEOREM

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In honor of Professor Johnny Henderson for his many years of service and contributions to the mathematical community and being an integral part of so many of our research projects.

ABSTRACT. This paper presents a corollary of the omitted ray fixed point theorem with an example that utilizes a non-standard existence of solutions argument, in conjunction with the mean value theorem, to prove the existence of a solution to a conjugate boundary value problem.

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1. INTRODUCTION

The Krasnosel'skii [13] fixed point theorem, as well as functional generalizations of Krasnosel'skii's fixed point theorem, rely on conditions of the form

$(K1)$ if $x \in P$ with $\alpha(x) = a$, then $\alpha(Tx) < a$;
$(K2)$ if $x \in P$ with $\beta(x) = b$, then $\beta(Tx) > b$.

In this paper, we show how to use a homogeneous function, in conjunction with the omitted ray fixed point theorem [8], to arrive at conditions of the form

$(D2)$ if $x \in P$ with $\beta(x) = b$, then $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$;
$(D4)$ if $x \in P$ with $\kappa(x) = c$ then $\psi(Tx - x_1) < \psi(x - x_1) + \psi(Tx - x)$.

These conditions have a natural triangle inequality flavor as compared to the corresponding conditions of the omitted ray fixed point theorem [8], which take the form

$(A2)$ if $x \in P$ with $\beta(x) = b$, then $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$;

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$(A6)$ if $x \in P$ with $\kappa(x) = c$, then $\psi(Tx - x_1) > \psi(x - x_1) + \psi(Tx - x)$.

As an application, we show how one can apply Corollary 2.6 to the operator

$$By(t) = f\left(\int_0^1 G(t,s)y(s)ds\right),$$

whose fixed points can be used to construct solutions of our conjugate boundary value problem. By using a function y_0 of the form

$$y_0 = f\left(\frac{bt(1-t)}{4}\right),$$

in conjunction with this alternative inversion technique involving the operator B, we introduce a new method that utilizes the mean value theorem to show that the conditions of Corollary 2.6 are satisfied; hence, our conjugate boundary value problem has a solution.

2. PRELIMINARIES

In this section we will state the definitions that are used in the remainder of the paper.

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* if for all $x \in P$ and $\lambda \geq 0$ it follows that $\lambda x \in P$; if both $x, -x \in P$, then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. An operator is called *completely continuous* if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. A map α is said to be a *nonnegative continuous concave functional* on a cone P of a real Banach space E if $\alpha : P \to [0,\infty)$ is continuous, and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0,1]$. Similarly we say the map β is a *nonnegative continuous convex functional* on a cone P of a real Banach space E if $\beta : P \to [0,\infty)$ is continuous, and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0,1]$. We say the map γ is a *continuous sub-homogeneous functional* on a real Banach space E if $\gamma : E \to \mathbb{R}$ is continuous, and

$$\gamma(tx) \leq t\gamma(x) \quad \text{for all} \quad x \in E, \quad t \in [0,1] \quad \text{and} \quad \gamma(0) = 0.$$

Similarly we say the map ρ is a *continuous super-homogeneous functional* on a real Banach space E if $\rho : E \to \mathbb{R}$ is continuous, and

$$\rho(tx) \geq t\rho(x) \quad \text{for all} \quad x \in E, \quad t \in [0,1] \quad \text{and} \quad \rho(0) = 0.$$

Let ψ and δ be nonnegative continuous functionals on P. Then, for positive real numbers a and b, we define the following sets:

$$P(\psi, b) := \{x \in P : \psi(x) < b\}$$

and

$$P(\delta, \psi, b, a) := P(\delta, b) - \overline{P(\psi, a)} = \{x \in P : a < \psi(x) \text{ and } \delta(x) < b\}.$$

The following theorem is the omitted ray fixed point theorem [8]. This theorem utilizes a functional version of Altman's condition [2], applying the techniques found in the Leggett-Williams fixed point theorem [14] and generalizations of the Leggett-Williams fixed point theorem [3, 4, 5].

Theorem 2.4. *Suppose P is a cone in a real Banach space E, α and κ are nonnegative continuous concave functionals on P, β and θ are nonnegative continuous convex functionals on P, γ and δ are continuous sub-homogeneous functionals on E, ρ and ψ are continuous super-homogeneous functionals on E, and $T : P \to P$ is a completely continuous operator. Furthermore, suppose that there exist nonnegative numbers a, b, c and d, and functions $x_0, x_1 \in P$, such that*

($A1$) $x_0 \in \{x \in P : a \leq \alpha(x)$ *and* $\beta(x) < b\}$;
($A2$) *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(x) \geq a$, *then* $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$;
($A3$) *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(Tx) < a$, *then* $\delta(Tx - x_0) < \delta(x - x_0) + \delta(Tx - x)$;
($A4$) $x_1 \in \{x \in P : c < \kappa(x)$ *and* $\theta(x) \leq d\}$ *and* $P(\kappa, c) \neq \emptyset$;
($A5$) *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(x) \leq d$, *then* $\rho(Tx - x_1) > \rho(x - x_1) + \rho(Tx - x)$;
($A6$) *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(Tx) > d$, *then* $\psi(Tx - x_1) > \psi(x - x_1) + \psi(Tx - x)$.

If

($H1$) $\overline{P(\kappa, c)} \subsetneq P(\beta, b)$, *then* T *has a fixed point* $x \in P(\beta, \kappa, b, c)$,

whereas, if

($H2$) $\overline{P(\beta, b)} \subsetneq P(\kappa, c)$, *then* T *has a fixed point* $x \in P(\kappa, \beta, c, b)$.

Note that if ρ is a continuous homogeneous functional on E, then both ρ and $-\rho$ are continuous super-homogeneous and sub-homogeneous functionals on E. Using $-\rho$ in property ($A5$) of Theorem 2.4

$$-\rho(Tx - x_1) > -\rho(x - x_1) - \rho(Tx - x),$$

we have

$$\rho(Tx - x_1) < \rho(x - x_1) + \rho(Tx - x);$$

this is the justification of the following Corollary of the omitted ray fixed point theorem.

Corollary 2.5. *Suppose P is a cone in a real Banach space E, α and κ are nonnegative continuous concave functionals on P, β and θ are nonnegative continuous convex functionals on P, γ and δ are continuous sub-homogeneous functionals on E, ρ and ψ are continuous homogeneous functionals on E, and $T : P \to P$ is a completely continuous operator. Furthermore, suppose that there exist nonnegative numbers a, b, c and d, and functions $x_0, x_1 \in P$, such that*

$(A1)$ $x_0 \in \{x \in P : a \leq \alpha(x)$ *and* $\beta(x) < b\}$;
$(A2)$ *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(x) \geq a$, *then* $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$;
$(A3)$ *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(Tx) < a$, *then* $\delta(Tx - x_0) < \delta(x - x_0) + \delta(Tx - x)$;
$(A4)$ $x_1 \in \{x \in P : c < \kappa(x)$ *and* $\theta(x) \leq d\}$ *and* $P(\kappa, c) \neq \emptyset$;
$(A5)$ *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(x) \leq d$, *then* $\rho(Tx - x_1) < \rho(x - x_1) + \rho(Tx - x)$;
$(A6)$ *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(Tx) > d$, *then* $\psi(Tx - x_1) < \psi(x - x_1) + \psi(Tx - x)$.

If

$(H1)$ $\overline{P(\kappa, c)} \subsetneq P(\beta, b)$, *then* T *has a fixed point* $x \in P(\beta, \kappa, b, c)$,

whereas, if

$(H2)$ $\overline{P(\beta, b)} \subsetneq P(\kappa, c)$, *then* T *has a fixed point* $x \in P(\kappa, \beta, c, b)$.

In the event that $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$ for all $x \in P$ with $\beta(x) = b$, as is done in standard Krasnosel'skii-type arguments, one can combine conditions $(A2)$ and $(A3)$, as well as a similar argument for $(A5)$ and $(A6)$. This results in slight modifications of conditions $(A1)$ and $(A4)$ as stated in the following Corollary.

Corollary 2.6. *Suppose P is a cone in a real Banach space E, κ is a nonnegative continuous concave functional on P, β is a nonnegative continuous convex functional on P, γ is a continuous sub-homogeneous functional on E, ψ is a continuous homogeneous functional on E, and $T : P \to P$ is a completely continuous operator. Furthermore, suppose that there exist nonnegative numbers b and c, and functions $x_0, x_1 \in P$, such that*

$(D1)$ $x_0 \in \{x \in P : \beta(x) < b\}$;
$(D2)$ *if* $x \in P$ *with* $\beta(x) = b$ *then* $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$;
$(D3)$ $x_1 \in \{x \in P : c < \kappa(x)\}$ *and* $P(\kappa, c) \neq \emptyset$;
$(D4)$ *if* $x \in P$ *with* $\kappa(x) = c$ *then* $\psi(Tx - x_1) < \psi(x - x_1) + \psi(Tx - x)$.

If

$(H1)$ $\overline{P(\kappa, c)} \subsetneq P(\beta, b)$, *then* T *has a fixed point* $x \in P(\beta, \kappa, b, c)$,

whereas, if

(H2) $\overline{P(\beta,b)} \subsetneq P(\kappa,c)$*, then* T *has a fixed point* $x \in P(\kappa,\beta,c,b)$.

3. APPLICATION

In this section, as an application of Corollary 2.6, we are concerned with the existence of at least one positive solution for the second-order boundary value problem given by the nonlinear equation

$$x'' + f(x) = 0, \qquad 0 \le t \le 1, \tag{3.1}$$

where $f : \mathbb{R} \to [0,\infty)$ is continuous, together with the conjugate boundary conditions

$$x(0) = 0 = x(1). \tag{3.2}$$

We will also assume that f is increasing and concave ($f''(x) < 0$ for all $x \ge 0$) to illustrate a new technique for existence of solutions arguments that utilizes the omitted ray fixed point theorem. We look for solutions $x \in C^{(2)}[0, 1]$ which are both nonnegative and concave on $[0, 1]$. We will impose growth conditions on f which ensure the existence of at least one nonnegative, symmetric solution. See Henderson et al in [9, 12] for a more complete consideration of symmetric arguments that arise in the study of conjugate boundary value problems, and see [1, 16] for other types of problems with similar properties. The Green's function for

$$-x'' = 0 \tag{3.3}$$

satisfying the boundary conditions is given by

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{3.4}$$

Let the Banach space $E = C[0,1]$ be endowed with the maximum norm given by

$$\|x\| = \max_{0\le t\le 1} |x(t)|\,,$$

and define the cone $P \subset E$ by

$$P = \{x \in E \,|\, x \text{ is nonnegative, concave and symmetric on } [0,1]\}\,.$$

Define the completely continuous operator A by

$$Ax(t) = \int_0^1 G(t,s)f(x(s))ds.$$

We have that P is a cone in the Banach space E with the sup norm, and

$$A : P \to P$$

since $G(t,s)$ is nonnegative on its domain and $f(x) \ge 0$ on $[0,\infty)$. In the literature concerning the existence of positive solutions of various boundary value problems, the common procedure is to apply fixed point theorems to operators analogous to the operator A above. Rather, in this paper we will apply the omitted ray fixed point theorem to the operator B, with

$$B : P \to P$$

defined by

$$By(t) = f\left(\int_0^1 G(t,s)y(s)ds\right)$$

for $t \in [0,1]$. We now show that if $y \in P$ is a fixed point of B, then

$$x(t) := \int_0^1 G(t,s)y(s)ds \tag{3.5}$$

is a fixed point of A (and hence is a solution of the BVP (3.1), (3.2)). To see this, assume $y \in P$ is a fixed point of B, and define x by (3.5). Then, for $t \in [0,1]$,

$$\begin{aligned} Ax(t) &= \int_0^1 G(t,s)f(x(s))ds \\ &= \int_0^1 G(t,s)f\left(\int_0^1 G(s,r)y(r)dr\right)ds \\ &= \int_0^1 G(t,s)B(y(s))ds \\ &= \int_0^1 G(t,s)y(s)ds \\ &= x(t). \end{aligned}$$

Conversely, if we assume x is a fixed point of A, then

$$y(t) := f(x(t)) \tag{3.6}$$

is a fixed point of B (in many applications $f(0) = 0$). Suppose $x \in P$ is a fixed point of A, and define y by (3.6). Then, for $t \in [0,1]$, we see that

$$\begin{aligned} By(t) &= f\left(\int_0^1 G(t,s)y(s)ds\right) \\ &= f\left(\int_0^1 G(t,s)f(x(s))ds\right) \\ &= f(x(t)) \\ &= y(t). \end{aligned}$$

This alternative inversion technique can be traced back to Avery-Peterson and Burton-Zhang in the late nineties. For more details, see [6, 7, 10, 11], as well as the more recent work by Muresan-Nica [15]. In the following application we demonstrate how one can use the nonstandard operator B with the mean value theorem to show the existence of solutions to our boundary value problem using Corollary 2.6 of the omitted ray fixed point theorem.

Theorem 3.1. *If b and c are positive real numbers with $b < c$, and $f \in C^2[0,\infty)$ is a non-negative, increasing and concave function such that*

(a) $f\left(\frac{b}{16}\right) < \frac{b}{2}$,
(b) $\min_{y\in\left[\frac{c}{12},\frac{3c}{16}\right]} f(y) \geq \frac{19c}{16}$,
(c) $\max_{y\in\left[\frac{c}{12},\frac{3c}{16}\right]} f'(y) < 2$, *and*
(d) $\max_{y\in\left[\frac{b}{16},\frac{b}{8}\right]} f'(y) < 8$,

then the conjugate problem (3.1), (3.2) *has at least one positive solution* x^*.

Proof. For $y \in P$ let

$$\beta(y) = \kappa(y) = y\left(\frac{1}{2}\right),$$

and for $y \in E$ let

$$\gamma(y) = \psi(y) = \left|y\left(\frac{1}{2}\right)\right|.$$

By the properties of G and f, for any $y \in P$ we have

$$(By)(t) = f\left(\int_0^1 G(t,s)y(s)ds\right) \geq 0,$$

$$(By)'(t) = f'\left(\int_0^1 G(t,s)y(s)ds\right)\left(\int_t^1 y(s)ds - \int_0^1 sy(s)ds\right),$$

$$\begin{aligned}(By)''(t) &= f''\left(\int_0^1 G(t,s)y(s)ds\right)\left(\int_t^1 y(s)ds - \int_0^1 sy(s)ds\right)^2 \\ &+ f'\left(\int_0^1 G(t,s)y(s)ds\right)(-y(t)) \leq 0.\end{aligned}$$

Since y is symmetric, for any $t \in [0,1]$ we have

$$\begin{aligned}(By)(1-t) &= f\left(\int_0^1 G(1-t,s)y(s)ds\right) = f\left(\int_0^1 G(1-t,s)y(1-s)ds\right) \\ &= f\left(-\int_1^0 G(1-t,1-u)y(u)du\right) = f\left(-\int_1^0 G(t,u)y(u)du\right) \\ &= f\left(\int_0^1 G(t,s)y(s)ds\right) = (By)(t).\end{aligned}$$

Thus, $B : P \to P$. By the Arzela-Ascoli Theorem it is a standard exercise to show that B is a completely continuous operator using the properties of G and f. Clearly $P(\kappa, c)$ is a bounded subset of the cone P, since if $x \in P(\kappa, c)$, then

$$\|x\| = x\left(\frac{1}{2}\right) < c,$$

and if $x \in \overline{P(\beta, b)}$, then

$$c > b \geq \beta(x) = \kappa(x).$$

Consequently we have $\kappa(x) < c$, that is,

$$\overline{P(\beta, b)} \subset P(\kappa, c).$$

Moreover, $b + c \in P(\kappa, c) - \overline{P(\beta, b)}$, so that $\overline{P(\beta, b)} \subsetneq P(\kappa, c)$.

Let y_0 and y_1 be defined by

$$y_0(t) = f\left(\int_0^1 G(t,s)\frac{b}{2}ds\right) = f\left(\frac{bt(1-t)}{4}\right)$$

and

$$y_1(t) = f\left(\int_0^1 G(t,s)\frac{3c}{2}ds\right) = f\left(\frac{3ct(1-t)}{4}\right),$$

respectively. Then by (a) we have that

$$y_0 \in \{y \in P : \beta(y) < b\},$$

and by (b) we have that

$$y_1 \in \{y \in P : c < \kappa(y)\}.$$

Also, $0 \in P(\kappa, c)$. Thus, $P(\kappa, c) \neq \emptyset$, and we have verified $(D1)$ and $(D3)$ of Corollary 2.6.

Claim 1: If $\beta(y) = b$, then $\gamma(By - y_0) < \gamma(y - y_0) + \gamma(By - y)$.

Let $y \in P$ with $\beta(y) = y\left(\frac{1}{2}\right) = b$. Since y is concave and symmetric with $y(0) = y(1) \geq 0$ and $y\left(\frac{1}{2}\right) = b$, we have

$$y(t) \geq \begin{cases} 2bt & : 0 \leq t \leq \frac{1}{2}, \\ 2b(1-t) & : \frac{1}{2} \leq t \leq 1. \end{cases}$$

In particular we have that

$$\begin{aligned} \int_0^1 G\left(\frac{1}{2}, s\right) y(s)ds &\geq \int_0^{\frac{1}{2}} G(1/2, s)2bsds + \int_{\frac{1}{2}}^1 G(1/2, s)2b(1-s)ds \\ &= \int_0^{\frac{1}{2}} bs^2ds + \int_{\frac{1}{2}}^1 b(1-s)^2ds = \frac{b}{12} \end{aligned}$$

and

$$\int_0^1 G\left(\frac{1}{2}, s\right) y(s)ds \leq \int_0^1 G(1/2, s)bds = \frac{b}{8},$$

respectively. We also have that

$$\int_0^1 G\left(\frac{1}{2}, s\right) \frac{b}{2}ds = \frac{b}{16}.$$

Therefore, by the mean value theorem there exists a $w \in \left[\frac{b}{16}, \frac{b}{8}\right]$ such that

$$\begin{aligned}\gamma(By - y_0) &= \left| f\left(\int_0^1 G(1/2,s)y(s)ds\right) - f\left(\int_0^1 G(1/2,s)\frac{b}{2}ds\right)\right| \\ &= f'(w)\left|\left(\int_0^1 G(1/2,s)y(s)ds\right) - \left(\int_0^1 G(1/2,s)\frac{b}{2}ds\right)\right| \\ &< 8\left(\max_{t\in[0,1]}\left|y(t) - \frac{b}{2}\right|\right)\left(\int_0^1 G(1/2,s)ds\right) \\ &= \frac{b}{2} \le y\left(\frac{1}{2}\right) - f\left(\frac{b}{16}\right) = \gamma(y - y_0),\end{aligned}$$

since by (a) we have that

$$f\left(\frac{b}{16}\right) < \frac{b}{2}.$$

Claim 2: If $y \in P$ with $\kappa(y) = c$, then $\psi(By - y_1) < \psi(y - y_1) + \psi(By - y)$.

Let $y \in P$ with $\kappa(y) = y\left(\frac{1}{2}\right) = c$. Since y is concave and symmetric with $y(0) = y(1) \ge 0$ and $y\left(\frac{1}{2}\right) = c$, just like in Claim 1, we have

$$y(t) \ge \begin{cases} 2ct & 0 \le t \le \frac{1}{2}, \\ 2c(1-t) & \frac{1}{2} \le t \le 1. \end{cases}$$

In particular we have that

$$\frac{c}{8} \ge \int_0^1 G\left(\frac{1}{2}, s\right) y(s)ds \ge \frac{c}{12};$$

also, we have that

$$\int_0^1 G\left(\frac{1}{2}, s\right)\left(\frac{3c}{2}\right) ds = \frac{3c}{16}.$$

Therefore, by the mean value theorem there exists a $z \in \left[\frac{c}{12}, \frac{3c}{16}\right]$ such that

$$\begin{aligned}\psi(By - y_1) &= \left| f\left(\int_0^1 G(1/2,s)y(s)ds\right) - f\left(\int_0^1 G(1/2,s)\left(\frac{3c}{2}\right)ds\right)\right| \\ &= f'(z)\left|\left(\int_0^1 G(1/2,s)y(s)ds\right) - \left(\int_0^1 G(1/2,s)\left(\frac{3c}{2}\right)ds\right)\right| \\ &< 2\left(\max_{t\in[0,1]}\left|y(t) - \frac{3c}{2}\right|\right)\left(\int_0^1 G(1/2,s)ds\right) \\ &= 2\left(\frac{3c}{2}\right)\left(\frac{1}{8}\right) \le \left|c - f\left(\frac{3c}{16}\right)\right| + \left|f\left(\int_0^1 G(1/2,s)y(s)ds\right) - c\right| \\ &= \gamma(y - y_1) + \gamma(By - y),\end{aligned}$$

since by (b) we have that

$$\min_{y\in\left[\frac{c}{12}, \frac{3c}{16}\right]} f(y) \ge \frac{19c}{16}.$$

Therefore, the conditions of Theorem 2.6 are satisfied and the operator B has at least one fixed point y^* with

$$b < y^*\left(\frac{1}{2}\right) < c.$$

Thus, the operator A has at least one fixed point x^* which is a solution of the conjugate problem (3.1), (3.2) with

$$x^* = \int_0^1 G(t,s)y^*(s)ds.$$

□

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