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EXISTENCE RESULTS FOR NONLINEAR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSE

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ABSTRACT. In this paper, we establish the existence and uniqueness of solutions for a class of initial value problems for implicit fractional differential equations with impulse and Caputo fractional derivative. The arguments are based upon the Banach contraction principle, and Schaefer's fixed point theorem. As applications, two examples are included to show the applicability of our results.

AMS (MOS) Subject Classifications: 26A33, 34A08, 34A37

1. Introduction

In this paper, we establish existence and uniqueness results to the following implicit fractional-order differential equation with impulse

$$^cD^{\alpha}_{t_k}y(t) = f(t, y, {}^cD^{\alpha}_{t_k}y(t)), \text{ for each } t \in (t_k, t_{k+1}],\ k = 0, \dots, m,\ 0 < \alpha \le 1, \quad (1.1)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (1.2)$$

$$y(0) = y_0, \quad (1.3)$$

where $^cD^{\alpha}_{t_k}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k : \mathbb{R} \to \mathbb{R}$, and $y_0 \in \mathbb{R}$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order (non-integer). Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes (see [1, 5, 6, 21, 24, 28, 30]). On the other hand, impulsive fractional differential equations are a very important class of fractional differential equations because many phenomena from physics, chemistry, engineering, biology, etc., can be represented by the impulsive fractional differential equations. The theory of impulsive

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differential equations describes the process subject to abrupt change in their states at times. Impulsive differential equations have received much attention, we refer the reader to books [4, 7, 19, 23, 26, 29], and the papers [2, 3, 9, 10, 12, 11, 16, 20, 31], the references therein.

In [12], Benchohra and Slimani considered the existence and uniqueness of solutions for the initial value problems with impulses,

$$\begin{gathered} {}^cD^\alpha y(t) = f(t, y(t)), \quad t \in J = [0,T],\ t \neq t_k,\ 0 < \alpha \leq 1, \\ \Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \\ y(0) = y_0, \end{gathered}$$

where $k = 1, \dots, m$, ${}^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k : \mathbb{R} \to \mathbb{R}$, and $y_0 \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

In [11], Benchohra and Seba, using Mönch's fixed point theorem combined with the technique of measures of noncompactness, considered the existence and uniqueness of solutions for the initial value problems with impulses,

$$\begin{gathered} {}^cD^\alpha y(t) = f(t, y(t)), \quad t \in J = [0,T],\ t \neq t_k,\ 0 < \alpha \leq 1, \\ \Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \\ y(0) = y_0, \end{gathered}$$

where $k = 1, \dots, m$, ${}^cD^\alpha$ is the Caputo fractional derivative, $f : J \times E \to E$ is a given function, $I_k : E \to E$, $y_0 \in E$, E is a Banach space, and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$.

In [2], Agarwal *et al.* studied the existence and uniqueness of solutions for the initial value problems, for fractional order differential equations with impulses

$$\begin{gathered} {}^cD^\alpha y(t) = f(t, y(t)), \quad t \in J = [0,T],\ t \neq t_k,\ k = 1, \dots, m,\ 1 < \alpha \leq 2, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ \Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y(0) = y_0, \quad y'(0) = y_1, \end{gathered}$$

where $k = 1, \dots, m$, ${}^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k : \mathbb{R} \to \mathbb{R}$, $y_0 \in \mathbb{R}$ and $y_1 \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$.

In [8], Benchohra *et al.* discussed the existence of solutions for the initial value problems, for fractional order differential inclusions,

$$ {}^cD^\alpha y(t) \in F(t, y(t)), \quad t \in J = [0,T],\ t \neq t_k,\ k = 1, \dots, m,\ 1 < \alpha \leq 2, $$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m,$$
$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m,$$
$$y(0) = y_0, \quad y'(0) = y_1,$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $F : J\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), I_k and $\overline{I}_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \dots, m$, and $y_0, y_1 \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$.

Motivated by the works mentioned above, we present, in this paper, two results for the problem (1.1)–(1.3). The first one is based on the Banach contraction principle, the second one on Schaefer's fixed point theorem. In Section 4 we indicate a generalization to problems (1.1)–(1.3). Finally, in the last Section, we give two examples to demonstrate our main results.

2. **Preliminaries**

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

Definition 2.1 ([22, 27])**.** The fractional (arbitrary) order integral of the function $h \in L^1([0,T],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds,$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \displaystyle\int_0^\infty t^{\alpha-1}e^{-t}dt$, $\alpha > 0$.

Definition 2.2 ([25, 27])**.** For a function h given on the interval $[0,T]$, the Caputo fractional-order derivative of order α of h, is defined by

$$({}^cD_0^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$. Here $[\alpha]$ denotes the integer part of α.

Lemma 2.3 ([25, 27])**.** *Let $\alpha \geq 0$ and $n = [\alpha] + 1$. Then*

$$I^\alpha({}^cD_0^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^k(0)}{k!} t^k.$$

We need the following auxiliary lemmas.

Lemma 2.4 ([32])**.** *Let $\alpha > 0$, then the differential equation*

$$^{c}D_0^{\alpha} k(t) = 0$$

has solutions $k(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, c_i \in \mathbb{R},\ i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 2.5 ([32])**.** *Let $\alpha > 0$, then*

$$I^{\alpha c} D_0^{\alpha} k(t) = k(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Theorem 2.6 (Banach's fixed point theorem [18])**.** *Let C be a non-empty closed subset of a Banach space X, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 2.7 (Schaefer's fixed point theorem [18])**.** *Let X be a Banach space, and $N : X \longrightarrow X$ a completely continuous operator. If the set*

$$\mathcal{E} = \{y \in X : y = \lambda N y,\ \textit{for some}\ \lambda \in (0, 1)\}$$

is bounded, then N has fixed points.

3. **Existence of Solutions**

Denote by $C(J, \mathbb{R})$ the Banach space of continuous functions $J \to \mathbb{R}$, with the usual supremum norm

$$\|y\|_{\infty} = \sup\{|y(t)|,\ t \in J\}.$$

Consider the set of functions

$$\begin{aligned} PC(J, \mathbb{R}) &= \{y : J \to \mathbb{R} : y \in C((t_k, t_{k+1}], \mathbb{R}), k = 0, \ldots, m \text{ and there exist } y(t_k^-) \\ &\quad \text{and } y(t_k^+),\ k = 1, \ldots, m \text{ with } y(t_k^-) = y(t_k)\}. \end{aligned}$$

$PC(J, \mathbb{R})$ is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t \in J} |y(t)|.$$

Let $J_0 = [t_0, t_1]$ and $J_k = (t_k, t_{k+1}]$ where $k = 1, \ldots, m$.

Definition 3.1. A function $y \in PC(J, \mathbb{R})$ whose α-derivative exists on J_k is said to be a solution of (1.1)–(1.3) if y satisfies the equation $^{c}D_{t_k}^{\alpha} y(t) = f(t, y(t), ^{c}D_{t_k}^{\alpha} y(t))$ on J_k, and satisfies the conditions

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 0, \ldots, m,$$

$$y(0) = y_0.$$

To prove the existence of solutions of (1.1)–(1.3), we need the following auxiliary lemmas.

Lemma 3.2 ([12]). *Let $0<\alpha\le 1$ and let $\sigma : J\to\mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t)=\begin{cases} y_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sigma(s)ds & \text{if } t\in[0,t_1],\\ y_0+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\sigma(s)ds & \\ +\frac{1}{\Gamma(\alpha)}\int_{t_k}^t(t-s)^{\alpha-1}\sigma(s)ds+\sum_{i=1}^k I_i(y(t_i^-)), & \text{if } t\in J_k:=(t_k,t_{k+1}], \end{cases} \tag{3.1}$$

where $k=1,\dots,m$, if and only if, y is a solution of the fractional IVP

$$^cD^\alpha_{t_k}y(t)=\sigma(t),\quad t\in J_k, \tag{3.2}$$

$$\Delta y|_{t=t_k}=I_k(y(t_k^-)),\quad k=1,\dots,m, \tag{3.3}$$

$$y(0)=y_0. \tag{3.4}$$

We are now in a position to state and prove our existence result for the problem (1.1)–(1.3) based on Banach's fixed point theorem.

Theorem 3.3. *Assume*

(H1) *The function $f : J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous.*

(H2) *There exist constants $K>0$ and $0<L<1$ such that*

$$|f(t,u,v)-f(t,\bar u,\bar v)|\le K|u-\bar u|+L|v-\bar v|$$

for any $u,v,\bar u,\bar v\in\mathbb{R}$ and $t\in J$.

(H3) *There exists a constant $\ell>0$ such that*

$$|I_k(u)-I_k(\overline{u})|\le \ell|u-\overline{u}|,$$

for each $u,\overline{u}\in\mathbb{R}$ and $k=1,\dots,m$.

If

$$\frac{KT^\alpha(m+1)}{(1-L)\Gamma(\alpha+1)}+m\ell<1, \tag{3.5}$$

then there exists a unique solution for IVP (1.1)–(1.3) on J.

Proof. Transform the problem (1.1)–(1.3) into a fixed point problem. Consider the operator $N : PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ defined by

$$N(y)(t)=y_0+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}g(s)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^t(t-s)^{\alpha-1}g(s)ds+\sum_{0<t_k<t}I_k(y(t_k^-)), \tag{3.6}$$

where $g\in C(J,\mathbb{R})$ is such that

$$g(t)=f(t,y(t),g(t)).$$

Clearly, the fixed points of operator N are solutions of problem (1.1)–(1.3).

Let $u, w \in PC(J,\mathbb{R})$. Then for $t \in J$, we have

$$|N(u)(t) - N(w)(t)| \leq \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g(s) - h(s)| ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |g(s) - h(s)| ds + \sum_{0<t_k<t} |I_k(u(t_k^-)) - I_k(w(t_k^-))|,$$

where $g, h \in C(J,\mathbb{R})$ are such that

$$g(t) = f(t, u(t), g(t)),$$

and

$$h(t) = f(t, w(t), h(t)).$$

By $(H2)$ we have

$$\begin{aligned} |g(t) - h(t)| &= |f(t, u(t), g(t)) - f(t, w(t), h(t))| \\ &\leq K|u(t) - w(t)| + L|g(t) - h(t)|. \end{aligned}$$

Thus

$$|g(t) - h(t)| \leq \frac{K}{1-L} |u(t) - w(t)|.$$

Then, for $t \in J$

$$\begin{aligned} |N(u)(t) - N(w)(t)| &\leq \frac{K}{(1-L)\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |u(s) - w(s)| ds \\ &+ \frac{K}{(1-L)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |u(s) - w(s)| ds \\ &+ \sum_{k=1}^{m} \ell |u(t_k^-) - w(t_k^-)|. \\ &\leq \frac{mKT^\alpha}{(1-L)\Gamma(\alpha+1)} \|u - w\|_{PC} + \frac{T^\alpha K}{(1-L)\Gamma(\alpha+1)} \|u - w\|_{PC} \\ &+ m\ell \|u - w\|_{PC}. \end{aligned}$$

Thus

$$\|N(u) - N(w)\|_{PC} \leq \left[\frac{KT^\alpha (m+1)}{(1-L)\Gamma(\alpha+1)} + m\ell \right] \|u - w\|_{PC}.$$

By (3.5), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point which is a unique solution of the problem (1.1)–(1.3).

Our second result is based on Schaefer's fixed point theorem.

Theorem 3.4. *Assume (H1), (H2) and*

(H4) *There exist* $p, q, r \in C(J, \mathbb{R}_+)$ *with* $r^* = \sup_{t\in J} r(t) < 1$ *such that*

$$|f(t, u, w)| \le p(t) + q(t)|u| + r(t)|w| \text{ for } t \in J \text{ and } u, w \in \mathbb{R}.$$

(H5) *The functions* $I_k : \mathbb{R} \to \mathbb{R}$ *are continuous and there exist constants* $M^*,\ N^* > 0$ *such that*

$$|I_k(u)| \le M^*|u| + N^* \text{ for each } u \in \mathbb{R}, k = 1, \dots, m.$$

If

$$mM^* + \frac{(m+1)T^\alpha q^*}{(1 - r^*)\Gamma(\alpha + 1)} < 1,$$

then the IVP (1.1)–(1.3) *has at least one solution on* J.

Proof. Consider the operator N defined in (3.6). We shall use Schaefer's fixed point theorem to prove that N has a fixed point. The proof will be given in several steps.

Step 1: *N is continuous.* Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC(J, \mathbb{R})$. Then for each $t \in J$,

$$\begin{aligned}|N(u_n)(t) - N(u)(t)| \le{}& \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g_n(s) - g(s)| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |g_n(s) - g(s)| ds \\ &+ \sum_{0<t_k<t} |I_k(u_n(t_k^-)) - I_k(u(t_k^-))|,\end{aligned}$$

where $g_n, g \in C(J, \mathbb{R})$ are such that

$$g_n(t) = f(t, u_n(t), g_n(t)),$$

and

$$g(t) = f(t, u(t), g(t)).$$

By $(H2)$ we have

$$\begin{aligned}|g_n(t) - g(t)| &= |f(t, u_n(t), g_n(t)) - f(t, u(t), g(t))| \\ &\le K|u_n(t) - u(t)| + L|g_n(t) - g(t)|.\end{aligned}$$

Then

$$|g_n(t) - g(t)| \le \frac{K}{1 - L} |u_n(t) - u(t)|.$$

Since $u_n \to u$, then we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. And let $\eta > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \le \eta$ and $|g(t)| \le \eta$. Then, we have

$$\begin{aligned}(t - s)^{\alpha-1} |g_n(s) - g(s)| &\le (t - s)^{\alpha-1} [|g_n(s)| + |g(s)|] \\ &\le 2\eta(t - s)^{\alpha-1},\end{aligned}$$

and

$$(t_k - s)^{\alpha-1} |g_n(s) - g(s)| \le (t_k - s)^{\alpha-1} [|g_n(s)| + |g(s)|]$$

$$\leq \quad 2\eta(t_k - s)^{\alpha-1}.$$

For each $t \in J$, the functions $s \to 2\eta(t-s)^{\alpha-1}$ and $s \to 2\eta(t_k - s)^{\alpha-1}$ are integrable on $[0,t]$; then the Lebesgue Dominated Convergence Theorem and (3.4) imply that

$$|N(u_n)(t) - N(u)(t)| \to 0 \text{ as } n \to \infty,$$

and hence

$$\|N(u_n) - N(u)\|_{PC} \to 0 \text{ as } n \to \infty.$$

Consequently, N is continuous.

Step 2: *F maps bounded sets into bounded sets in $PC(J,\mathbb{R})$.* Indeed, it is enough to show that for any $\eta^* > 0$, there exists a positive constant ℓ_1 such that for each $u \in B_{\eta^*} = \{u \in PC(J,\mathbb{R}) : \|u\|_{PC} \leq \eta^*\}$, we have $\|N(u)\|_{PC} \leq \ell_1$. We have for each $t \in J$,

$$\begin{aligned} N(u)(t) = y_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)ds + \sum_{0<t_k<t} I_k(u(t_k^-)), \end{aligned} \tag{3.7}$$

where $g \in C(J,\mathbb{R})$ is such that

$$g(t) = f(t, u(t), g(t)).$$

By (H4) we have for each $t \in J$,

$$\begin{aligned} |g(t)| &= |f(t,u(t),g(t))| \\ &\leq p(t) + q(t)|u(t)| + r(t)|g(t)| \\ &\leq p(t) + q(t)\eta^* + r(t)|g(t)| \\ &\leq p^* + q^*\eta^* + r^*|g(t)|, \end{aligned}$$

where $p^* = \sup_{t\in J} p(t)$, and $q^* = \sup_{t \in J} q(t)$. Then

$$|g(t)| \leq \frac{p^* + q^*\eta^*}{1 - r^*} := M.$$

Thus (3.7) implies

$$\begin{aligned} |N(u)(t)| &\leq |y_0| + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} + m(M^*|u| + N^*) \\ &\leq |y_0| + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} + m(M^*\eta^* + N^*). \end{aligned}$$

Then

$$\|N(u)\|_{PC} \leq |y_0| + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)} + m(M^*\eta^* + N^*) := \ell_1.$$

Step 3: *F maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R})$.*

Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, B_{η^*} be a bounded set of $PC(J,\mathbb{R})$ as in Step 2, and let $u \in B_{\eta^*}$. Then

$$\begin{aligned}
&|N(u)(\tau_2) - N(u)(\tau_1)| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} |(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}||g(s)|ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{\alpha-1}||g(s)|ds + \sum_{0<t_k<\tau_2-\tau_1} |I_k(u(t_k^-))| \\
&\le \frac{M}{\Gamma(\alpha+1)}[2(\tau_2-\tau_1)^\alpha + (\tau_2^\alpha - \tau_1^\alpha)] + (\tau_2-\tau_1)(M^*|u| + N^*) \\
&\le \frac{M}{\Gamma(\alpha+1)}[2(\tau_2-\tau_1)^\alpha + \tau_2^\alpha - \tau_1^\alpha] + (\tau_2-\tau_1)(M^*\eta^* + N^*).
\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is completely continuous.

Step 4: *A priori bounds.* Now it remains to show that the set

$$E = \{u \in PC(J,\mathbb{R}) : u = \lambda N(u) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let $u \in E$, then $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$\begin{aligned}
u(t) = \lambda y_0 &+ \frac{\lambda}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s)ds \\
&+ \frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)ds + \lambda \sum_{0<t_k<t} I_k(u(t_k^-)).
\end{aligned}$$

And, by (H3) we have for each $t \in J$,

$$\begin{aligned}
|g(t)| &= |f(t, u(t), g(t))| \\
&\le p(t) + q(t)|u(t)| + r(t)|g(t)| \\
&\le p^* + q^*|u(t)| + r^*|g(t)|.
\end{aligned}$$

Thus

$$|g(t)| \le \frac{1}{1-r^*}(p^* + q^*|u(t)|).$$

This implies, by (H4) and (H5) (as in Step 2), that for each $t \in J$ we have

$$\begin{aligned}
|u(t)| \le |y_0| &+ \frac{m\left(\frac{1}{1-r^*}(p^* + q^*|u(t)|)\right) T^\alpha}{\Gamma(\alpha+1)} \\
&+ \frac{\left(\frac{1}{1-r^*}(p^* + q^*|u(t)|)\right) T^\alpha}{\Gamma(\alpha+1)} + m(M^*|u(t)| + N^*) \\
\le |y_0| &+ \frac{m\left(\frac{1}{1-r^*}(p^* + q^*\|u\|_{PC})\right) T^\alpha}{\Gamma(\alpha+1)}
\end{aligned}$$

$$+\frac{\left(\frac{1}{1-r^*}(p^*+q^*\|u\|_{PC})\right)T^\alpha}{\Gamma(\alpha+1)}+m(M^*\|u\|_{PC}+N^*).$$

Then we have

$$\begin{aligned}\|u\|_{PC}\le |y_0|&+\frac{m\left(\frac{1}{1-r^*}(p^*+q^*\|u\|_{PC})\right)T^\alpha}{\Gamma(\alpha+1)}\\&+\frac{\left(\frac{1}{1-r^*}(p^*+q^*\|u\|_{PC})\right)T^\alpha}{\Gamma(\alpha+1)}+m(M^*\|u\|_{PC}+N^*).\end{aligned}$$

Thus

$$\|u\|_{PC}\le\frac{|y_0|+mN^*+\frac{(m+1)p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}}{1-mM^*-\frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}}:=R.$$

This shows that the set E is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point which is a solution of the problem (1.1)–(1.3). □

4. Nonlocal Impulsive Differential Equations

This section is concerned with a generalization of the results presented in the previous section to implicit nonlocal fractional differential equations with impulses. More precisely, we shall present a result of existence and uniqueness for the following implicit nonlocal problem

$$^cD^\alpha_{t_k}y(t)=f(t,y,{}^cD^\alpha_{t_k}y(t)),\ \text{for each } t\in(t_k,t_{k+1}],k=0,\ldots,m,0<\alpha\le 1, \tag{4.1}$$

$$\Delta y|_{t=t_k}=I_k(y(t_k^-)),\quad k=1,\ldots,m, \tag{4.2}$$

$$y(0)+\varphi(y)=y_0, \tag{4.3}$$

where f, y_0, I_k, are as in Section 3 and $\varphi : C(J,\mathbb{R})\to\mathbb{R}$ is a continuous function.

Nonlocal conditions were initiated by Byszewski [15] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [13, 14], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, in [17], the author used

$$\varphi(y)=\sum_{i=1}^{p}c_iy(\tau_i) \tag{4.4}$$

where c_i, $i=1,\ldots,p$, are given constants and $0<\tau_1<\cdots<\tau_p\le T$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, (4.4) allows the additional measurements at τ_i, $i=1,\ldots,p$.

Theorem 4.1. *Assume (H1)–(H3) and the following hypothesis holds:*

(H6) *There exists a constant* $\gamma > 0$ *such that*

$$|\varphi(u) - \varphi(\overline{u})| \leq \gamma |u - \overline{u}| \text{ for each } u, \overline{u} \in PC(J, \mathbb{R}).$$

If

$$\left[\frac{KT^{\alpha}(m+1)}{(1-L)\Gamma(\alpha+1)} + m\ell + \gamma\right] < 1, \tag{4.5}$$

then the nonlocal problem (4.1)–(4.3) *has a unique solution on* J.

Proof. We transform the problem (4.1)–(4.3) into a fixed point problem. Consider the operator $\widetilde{N} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ defined by

$$\widetilde{N}(y)(t) = y_0 - \varphi(y) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s) ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s) ds + \sum_{0<t_k<t} I_k(y(t_k^-)),$$

where $g \in C(J, \mathbb{R})$ be such that

$$g(t) = f(t, y(t), g(t)).$$

Clearly, the fixed points of the operator $\widetilde{N}$ are solution of the problem (4.1)-(4.3). We can easily show the $\widetilde{N}$ is a contraction. □

5. **Examples**

Example 1. Consider the following impulsive Cauchy problem

$$^cD_{t_k}^{\frac{1}{2}} y(t) = \frac{1}{99e^{t+2}(1 + |y(t)| + |^cD^{\frac{1}{2}} y(t)|)}, \text{ for each } t \in J_0 \cup J_1, \tag{5.1}$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{|y(\frac{1}{2}^-)|}{55 + |y(\frac{1}{2}^-)|}, \tag{5.2}$$

$$y(0) = 1, \tag{5.3}$$

where $J_0 = \left[0, \frac{1}{2}\right]$, $J_1 = \left(\frac{1}{2}, 1\right]$, $t_0 = 0$, and $t_1 = \frac{1}{2}$.

Set

$$f(t, u, v) = \frac{1}{99e^{t+2}(1 + |u| + |v|)}, \quad t \in [0, 1], \ u, v \in \mathbb{R}.$$

Clearly, the function f is jointly continuous.

For each $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0, 1]$:

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq \frac{1}{99e^2}(|u - \bar{u}| + |v - \bar{v}|).$$

Hence condition (H2) is satisfied with $K = L = \frac{1}{99e^2}$. And let

$$I_1(u) = \frac{u}{55 + u}, \quad u \in [0, \infty).$$

Let $u, v \in [0,\infty)$. Then we have

$$|I_1(u)-I_1(v)| = \Big|\frac{u}{55+u}-\frac{v}{55+v}\Big| = \frac{55|u-v|}{(55+u)(55+v)} \le \frac{1}{55}|u-v|.$$

Thus condition

$$\frac{KT^{\alpha}(m+1)}{(1-L)\Gamma(\alpha+1)}+m\ell = \frac{2}{(99e^2-1)\Gamma(\frac{3}{2})}+\frac{1}{55} = \frac{4}{(99e^2-1)\sqrt{\pi}}+\frac{1}{55} < 1,$$

is satisfied with $T=1$, $m=1$ and $\ell=\frac{1}{55}$. It follows from Theorem 3.3 that the problem (5.1)–(5.3) has a unique solution on $J=[0,1]$.

Example 2. Consider the following impulsive Cauchy problem

$$^cD^{\frac{1}{2}}_{t_k}y(t) = \frac{(2+|y(t)|+|^cD^{\frac{1}{2}}y(t)|)}{102e^{t+3}(1+|y(t)|+|^cD^{\frac{1}{2}}y(t)|)}, \text{ for each } t\in J_0\cup J_1, \tag{5.4}$$

$$\Delta y|_{t=\frac{1}{3}} = \frac{|y(\frac{1}{3}^-)|}{77+|y(\frac{1}{3}^-)|}, \tag{5.5}$$

$$y(0)=1, \tag{5.6}$$

where $J_0=\left[0,\frac{1}{3}\right]$, $J_1=\left(\frac{1}{3},1\right]$, $t_0=0$, and $t_1=\frac{1}{3}$. Set

$$f(t,u,v) = \frac{(2+|u|+|v|)}{102e^{t+3}(1+|u|+|v|)}, \quad t\in[0,1],\ u,v\in\mathbb{R}.$$

Clearly, the function f is jointly continuous.

For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t\in[0,1]$:

$$|f(t,u,v)-f(t,\bar{u},\bar{v})| \le \frac{1}{102e^3}(|u-\bar{u}|+|v-\bar{v}|).$$

Hence condition (H2) is satisfied with $K=L=\frac{1}{102e^3}$. We have, for each $t\in[0,1]$,

$$|f(t,u,v)| \le \frac{1}{102e^{t+3}}(2+|u|+|v|).$$

Thus condition (H4) is satisfied with $p(t)=\frac{1}{51e^{t+3}}$ and $q(t)=r(t)=\frac{1}{102e^{t+3}}$. And let

$$I_1(u) = \frac{u}{77+u}, \quad u\in[0,\infty).$$

We have, for each $u\in[0,\infty)$,

$$|I_1(u)| \le \frac{1}{77}u+1$$

Thus condition (H5) is satisfied with $M^*=\frac{1}{77}$ and $N^*=1$. Thus condition

$$mM^*+\frac{(m+1)T^{\alpha}q^*}{(1-r^*)\Gamma(\alpha+1)} = \frac{1}{77}+\frac{2}{(102e^3-1)\Gamma(\frac{3}{2})} = \frac{1}{77}+\frac{4}{(102e^3-1)\sqrt{\pi}} < 1,$$

is satisfied with $T=1$, $m=1$ and $q^*(t)=r^*(t)=\frac{1}{102e^3}$. It follows from Theorem 3.4 that the problem (5.4)–(5.6) has at least one solution on $J=[0,1]$.

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