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POSITIVE SOLUTIONS OF NONLOCAL MULTIPOINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. We investigate the existence of positive solutions of nonlocal multipoint boundary value problems for second order nonlinear differential equations. The nonlinearity depends on the derivative of the unknown function, and is allowed to change sign infinitely many times. We rely the method of lower and upper solutions to prove our main result. In fact, using the same technique, we obtain a multiplicity result without extra assumptions.

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1. INTRODUCTION

In this paper, we investigate the existence of positive solutions of second order differential equations with nonlocal multipoint boundary conditions. The nonlinearity is a continuous functions depending on the first derivative of the unknown function and may changes sign with respect to its second argument. We provide sufficient conditions that guarantee the existence of at least one positive solution. More specifically, we are concerned with the problem of the existence of positive solutions of the following boundary value problem

$$\begin{cases} y''(t) + f(t, y(t), y'(t)) = 0, \quad t \in (0,1), \\ y(0) - \sum_{i=1}^{n} a_i y(\xi_i) = 0, \\ y(1) - \sum_{i=1}^{m} b_i y(\eta_i) = 0, \end{cases} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $a_i, i = 1, 2, \ldots, n$ and $b_j, j = 1, 2, \ldots, m$ are nonnegative real parameters and their interior boundary points $\xi_i, \eta_j \in (0,1)$ are not necessarily ordered. Problems dealing with the existence of positive solutions of second order differential equations are very important in the applied sciences; for instance, they arise thermal conduction problems [4], semiconductor problems [10],

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hydrodynamic problems [5], where only positive solutions, i.e., solutions y satisfying $y(t) > 0$ for all $t \in (0,1)$ are meaningful. It is well known that Krasnoselskii's fixed point theorem in a cone has been instrumental in proving existence of positive solutions of two-point boundary value problems for second order differential equations. See for instance [1, 14] and the references therein. Several authors have investigated nonlocal problems similar to (1.1). For integral boundary conditions we refer to [2, 9, 15, 16] and for multipoint boundary conditions we refer to [6, 11, 13, 17]. In this paper, we consider a more general problem where the nonlinear term is a continuous function depending also on the first derivative of the unknown function and is allowed to change sign infinitely many times. We assume the existence of positive lower and upper solutions, and we exploit the fact that the nonlinearity changes sign to prove our main result. In fact, by the same technique, we obtain a multiplicity result as a byproduct of our main result with no extra assumptions and without relying on the five functionals fixed point theorem, see for instance [11]. We do not rely on cone preserving mappings, and the sign of the Green's function of the corresponding linear homogeneous problem plays no role in our study.

2. PRELIMINARIES

Let I denote the real interval $[0,1]$. $C^2(I)$ denotes the space of all continuous functions $u : I \to \mathbb{R}$, together with their derivatives up to order 2. For $u \in C^2(I)$ we define its norm by $\|u\| = \max(\|u\|_0, \|u'\|_0, \|u''\|_0)$, where $\|u\|_0 = \max\{|u(t)|; t \in I\}$. Equipped with this norm $C^2(I)$ is a Banach space.

Definition 2.1. $\alpha \in C^2(I)$ is called a lower solution of (1.1) if

$$\alpha''(t) + f(t, \alpha(t), \alpha'(t)) \geq 0, \quad t \in (0,1),$$

$$\alpha(0) - \sum_{i=1}^{n} a_i \alpha(\xi_i) \leq 0,$$

$$\alpha(1) - \sum_{j=1}^{m} b_j \alpha(\eta_j) \leq 0.$$

Similarly, we say that $\beta \in C^2(I)$ is an upper solution of (1.1) if the above inequalities are reversed when we substitute β for α.

The following result, known as the Leray-Schauder alternative (see [7, Theorem 2.4, page 4]), plays and important role in our study.

Theorem 2.2. *Let E be a normed space and let $T : E \to E$ be completely continuous operator (i.e. a map that restricted to any bounded set in E is compact). Let $\mathcal{S}(T) = \{x \in E; x = \lambda T(x)$ for some $0 < \lambda < 1\}$. Then either the set $\mathcal{S}(T)$ is unbounded, or T has at least one fixed point.*

3. MAIN RESULTS

Consider the nonlinear problem

$$\begin{cases} y''(t) + f(t, y(t), y'(t)) = 0 \quad t \in (0,1), \\ y(0) - \sum_{i=1}^{n} a_i y(\xi_i) = 0, \\ y(1) - \sum_{j=1}^{m} b_j y(\eta_j) = 0, \end{cases}$$

where the nonlinearity $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies

(H1) there exist a lower solution α, and an upper solution β such that $0 \leq \alpha(t) \leq \beta(t)$ for all $t \in [0,1]$ and$f(t, \beta(t), \beta'(t)) > 0 > f(t, \alpha(t), \alpha'(t))$;

(H2) there exists $\Psi : [0, +\infty) \to [1, +\infty)$ continuous and nondecreasing with $\int_0^{+\infty} \frac{u\,du}{\Psi(u)} = +\infty$, such that $|f(t, y, z)| \leq \Psi(|z|)$, $\forall t \in [0,1]$, $\alpha \leq y \leq \beta$, $z \in \mathbb{R}$.

Remark 3.1. $\alpha \leq y \leq \beta$ means $\alpha(t) \leq y(t) \leq \beta(t)$ for all $t \in I$.

Define $\triangle : C^2(I) \to [\alpha, \beta]$ by $\triangle(y) = \min\{\beta, \max\{y, \alpha\}\} = \max\{\alpha, \min\{y, \beta\}\}$. It is clear that $\triangle$ is a bounded operator and $\alpha(t) \leq \triangle(y(t)) \leq \beta(t)$ for each $t \in I$. Moreover, $\triangle(y(t)) = \beta(t)$ if $y(t) > \beta(t)$ and $\triangle(y(t)) = \alpha(t)$ if $y(t) < \alpha(t)$.

Theorem 3.2. *Assume* (*H*1) *and* (*H*2) *are satisfied. Then* (1.1) *has at least one positive solution,* $y \in [\alpha, \beta]$.

Proof. **Step 1.** A priori estimates on solutions.

For $\lambda \in [0,1]$ consider the following modified equation, for $t \in (0,1)$

$$y''(t) + \lambda f(t, \triangle(y(t)), y'(t)) = (1-\lambda)y(t) + \lambda[y(t) - \triangle(y(t))]\Psi(|y'(t)|), \tag{3.1}$$

and the boundary conditions

$$y(0) = \lambda \sum_{i=1}^{n} a_i \triangle(y(\xi_i)) + (1-\lambda)\triangle(y(0)), \tag{3.2}$$

$$y(1) = \lambda \sum_{i=1}^{n} b_i \triangle(y(\eta_i)) + (1-\lambda)\triangle(y(1)). \tag{3.3}$$

Claim 1. Every solution of (3.1), (3.2), (3.3) satisfies $\alpha(t) \leq y(t) \leq \beta(t)$ for all $t \in I$.

We show that $y(t) \leq \beta(t)$ for each $t \in I$. Suppose this is not true. Then there is $\tau \in I$ such that $y(\tau) > \beta(\tau)$. Let $z(t) = y(t) - \beta(t)$ for each $t \in I$ and $z(t_0) = \max_{t \in I}\{z(t)\}$. Then $z(\tau) > 0$ implies that $z(t_0) > 0$. If $t_0 \in (0,1)$ then $z'(t_0) = 0$ and $z''(t_0) \leq 0$. It follows from (3.1) and the definition of β, that

(i) for $\lambda = 0$ we have, from (3.1), that $y''(t_0) = y(t_0)$, so that

$$0 \geq y''(t_0) - \beta''(t_0) = y(t_0) - \beta''(t_0) \geq y(t_0) + f(t_0, \beta(t_0), \beta'(t_0))$$

$$> \beta(t_0) + f(t_0, \beta(t_0), \beta'(t_0)) > 0.$$

This is clearly a contradiction.

(ii) For $0 < \lambda \leq 1$, we have

$$\begin{aligned}
0 &\geq y''(t_0) - \beta''(t_0) \\
&\geq -\lambda f(t_0, \triangle(y(t_0)), y'(t_0)) + (1-\lambda)y(t_0) \\
&\quad + \lambda[y(t_0) - \triangle(y(t_0))]\Psi(|y'(t_0)|) + f(t_0, \beta(t_0)), \beta'(t_0)) \\
&= (1-\lambda)f(t_0, \beta(t_0)), \beta'(t_0)) + (1-\lambda)y(t_0) + \lambda z(t_0)\Psi(|y'(t_0)|) \\
&> (1-\lambda)f(t_0, \beta(t_0)), \beta'(t_0)) + (1-\lambda)\beta(t_0) + \lambda z(t_0)\Psi(|\beta'(t_0)|) > 0,
\end{aligned}$$

which gives a contradiction.

Now, if $t_0 = 0$ then $z(0) > 0, z'(0) \leq 0$, so that $y(0) > \beta(0)$ and $y'(0) \leq \beta'(0)$. We have the following contradiction.

$$\begin{aligned}
\beta(0) < y(0) &= \lambda \sum_{i=1}^{n} a_i \triangle(y(\xi_i)) + (1-\lambda)\triangle(y(0)) \\
&\leq \lambda \sum_{i=1}^{n} a_i \beta(\xi_i) + (1-\lambda)\beta(0) \\
&\leq \lambda\beta(0) + (1-\lambda)\beta(0) = \beta(0).
\end{aligned}$$

Also, if $t_0 = 1$, again we arrive at a contradiction. In this case we have $y(1) > \beta(1)$ and $y'(1) \geq \beta'(1)$, and so

$$\begin{aligned}
\beta(1) < y(1) &= \lambda \sum_{i=1}^{n} b_i \triangle(y(\eta_i)) + (1-\lambda)\triangle(y(1)) \\
&\leq \lambda \sum_{i=1}^{n} b_i \beta(\eta_i) + (1-\lambda)\beta(1) \\
&\leq \lambda\beta(1) + (1-\lambda)\beta(1) = \beta(1).
\end{aligned}$$

Therefore, we have proved that $y(t) \leq \beta(t)$ for each $t \in I$. Similarly, we can show that $y(t) \geq \alpha(t)$ for each $t \in I$.

Claim 2. Let $R_1 := \|\beta - \alpha\|_0$. Then there exists $R_2 > 0$, independent of λ, such that every solution $y \in [\alpha, \beta]$ of (3.1), (3.2), (3.3) satisfies $|y'(t)| \leq R_2$ for every $t \in I$.

Let $\Phi(z) := (1+3\|\beta\|_0)\Psi(|z|)$. Then Φ has the same properties as Ψ (see assumption (H2)). Choose R_2 so that $\int_0^{R_2} \frac{u du}{\Phi(u)} \geq R_1$. We want to show that $|y'(t)| \leq R_2$ for every $t \in I$.

Suppose, on the contrary that there exists $\tau_1 \in I$ such that $|y'(\tau_1)| > R_2$. Then, by the continuity of y' on I, there exists an interval $[\mu, \xi] \subset [0,1]$ such that the following situations occur:

(i) $y'(\mu) = 0$, $y'(\xi) = R_2$, $0 < y'(t) < R_2$, for all $t \in (\mu, \xi)$,

(ii) $y'(\mu) = R_2$, $y'(\xi) = 0$, $0 < y'(t) < R_2$, for all $t \in (\mu, \xi)$,

(iii) $y'(\mu) = 0$, $y'(\xi) = -R_2$, $-R_2 < y'(t) < 0$, for all $t \in (\mu, \xi)$,

(iv) $y'(\mu) = -R_2$, $y'(\xi) = 0$, $-R_2 < y'(t) < R_2$, for all $t \in (\mu, \xi)$.

We study the first case. The other cases can be handled in a similar way.

Since $\lambda \in [0, 1]$, it follows from (3.1) that for $t \in (\mu, \xi)$

$$\begin{aligned} y''(t) &= -\lambda f(t, \triangle(y(t)), y'(t)) = (1-\lambda)y(t) + \lambda[y(t) - \triangle(y(t))]\Psi(|y'(t)|) \\ &\leq |f(t, \triangle(y(t)), y'(t))| + |y(t)| + |y(t) - \triangle(y(t))|\Psi(y'(t)) \\ &\leq \Psi(y'(t)) + \|\beta_0\| + [|y(t)| + |\triangle(y(t))|]\Psi(y'(t)) \\ &\leq \Psi(y'(t)) + \|\beta_0\| + 2\|\beta\|_0 \Psi(y'(t)). \end{aligned}$$

Since $\Psi(y'(t)) \geq 1$ it follows that

$$y''(t) \leq (1 + 3\|\beta\_|0)\Psi(y'(t)) = \Phi(y'(t)).$$

Therefore

$$\frac{y''(t)y'(t)}{\Phi(y'(t))} \leq y'(t), \quad \text{for all } t \in (\mu, \xi).$$

This inequality implies

$$\int_\mu^t \frac{y''(s)y'(s)}{\Phi(y'(s))}\, ds \leq \int_\mu^t y'(s)\, ds = y(t) - y(\mu) \leq \beta(t) - \alpha(\mu) \leq R_1.$$

Hence

$$\int_0^{y'(t)} \frac{udu}{\Phi(u)} \leq R_1 \leq \int_0^{R_2} \frac{udu}{\Phi(u)},$$

so that

$$\int_{y'(t)}^{R_2} \frac{udu}{\Phi(u)} \geq 0.$$

Since $\frac{u}{\Phi(u)} > 0$ it follows that $y'(t) \leq R_2$ for all $t \in (\mu, \xi)$. Taking into account all the four cases we see that $|y'(t)| \leq R_2$ for all $t \in I$.

It follows from the continuity of f and Ψ and the boundedness of the operator $\triangle$ that there exists $R_3 > 0$ such that $|y''(t)| \leq R_3$ for all $t \in I$.

Consequently, we have shown that all possible solutions y, of (3.1), (3.2), (3.3) satisfy the a priori bound $\|y\| \leq R$, where $R = \max(\|\beta\|_0, R_2, R_3)$.

Step 2. Existence of solutions of (3.1), (3.2), (3.3).

It is clear the problem (3.1), (3.2), (3.3) is equivalent to the abstract equation

$$Ly = \Gamma(\lambda, y),$$

where $L : C^2(I) \to C(I) \times \mathbb{R}^2$ is defined by

$$Ly(t) = (y''(t), y(0), y(1)),$$

and

$$\Gamma(\lambda, y) = (F(\lambda, y), A(\lambda, y), B(\lambda, y)),$$

with

$$F(\lambda, y)(t) = -\lambda f(t, \triangle(y(t)), y'(t)) + (1-\lambda)y(t) + \lambda[y(t) - \triangle(y(t))]\Psi(|y'(t)|),$$

$$A(\lambda, y) = \lambda \sum_{i=1}^{n} a_i \triangle(y(\xi_i)) + (1-\lambda)\triangle(y(0)),$$

$$B(\lambda, y) = \lambda \sum_{i=1}^{n} b_i \triangle(y(\eta_i)) + (1-\lambda)\triangle(y(1)).$$

Since L^{-1} exists, is compact and $\Gamma(\lambda, \cdot)$ is continuous, the operator $L^{-1}\Gamma(\lambda, \cdot)$ is completely continuous. Also, it follows from the previous step that the set of solutions of $y = L^{-1}\Gamma(\lambda, y)$ for $0 < \lambda < 1$, is bounded. Then by Theorem 2.2 the operator $L^{-1}\Gamma(1, \cdot)$ has a fixed point z_0. It is clear from Claim 1 that $z_0 \in [\alpha, \beta]$ and, therefore it is a solution of problem (1.1). □

4. ITERATIVE METHOD

In this section we shall develop an iterative method, which is not necessary monotone, to construct a sequence of functions which converges uniformly to a solution of (1.1). For this purpose we shall assume that, $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies, in addition to (H1) and (H2),

(H3) $(f(t, u_2, v) - f(t, u_1, v))(u_2 - u_1) > 0$ whenever $u_2 \geq u_1$ for all $v \in \mathbb{R}$.

Theorem 4.1. *Assume that (H1),(H2),(H3) hold. Then* (1.1) *has at least one solution* $u \in [\alpha, \beta]$.

To prove the theorem, we shall construct a sequence of functions satisfying some properties that will be specified later, and which converges uniformly on I to the desired solution.

The proof of the theorem shall be based on several lemmas. The first lemma is of independent interest.

Lemma 4.2. *Let* $\phi : I \times \mathbb{R} \to \mathbb{R}$ *be continuous, bounded and satisfy the following condition:*

(H_ϕ) $(\phi(t, u_2) - \phi(t, u_1))(u_2 - u_1) > 0$ *whenever* $u_2 \geq u_1$ *for all* $u_1, u_2 \in \mathbb{R}$.

Then for any real numbers δ, ρ *the boundary value problem*

$$\begin{cases} -u''(t) = \phi(t, u(t)), & t \in I, \\ u(0) = \delta, \\ u(1) = \rho, \end{cases} \tag{4.1}$$

has a unique solution u.

Proof. **Uniqueness.** Suppose that problem (4.1) has two solutions x and u. Put $z = x - u$. Then $z(0) = z(1) = 0$. Now, we have a function z continuous on I with $z(0) = z(1) = 0$. If $z(t) = 0$ for all $t \in I$, we are done. Suppose that there exists $s \in I$ such that $z(s) \neq 0$. We assume that $z(s) > 0$. The other case can be handled in a similar way. Then there exists $\tau \in I$ such that

$$z(\tau) = \max_{t\in I} z(t) > 0, \quad z'(\tau) = 0 \text{ and } z''(\tau) \leq 0.$$

Then

$$\begin{aligned} 0 \geq z(\tau)z''(\tau) &= -(x(\tau) - u(\tau))(\phi(\tau, u(\tau)) - \phi(\tau, x(\tau))) \\ &= (\phi(\tau, x(\tau)) - \phi(\tau, u(\tau)))(x(\tau) - u(\tau)). \end{aligned}$$

But condition (H_ϕ) implies that

$$(\phi(\tau, x(\tau)) - \phi(\tau, u(\tau)))(x(\tau) - u(\tau)) > 0.$$

This clear contradiction implies that $z(\tau) = 0$. It follows that $z(t) = 0$ for all $t \in I$, i.e. $x(t) = u(t)$ for all $t \in I$, which shows uniqueness of the solution.

Existence. For $\lambda \in [0, 1]$ consider the family of problems

$$\begin{cases} -u''(t) = \lambda\phi(t, u(t)), & t \in I, \\ u(0) = \lambda\delta, \\ u(1) = \lambda\rho, \end{cases} \tag{4.2}$$

For $\lambda = 0$, problem (4.2) has only the trivial solution. So, we consider the case $\lambda \in (0, 1]$.

(i) u is a solution of (4.2) if and only if it satisfies, for $t \in I$,

$$u(t) = \lambda\left(\rho t + \delta(1-t) + \int_0^1 (1-t)\phi(s, u(s))ds - \int_0^t (t-s)\phi(s, u(s))ds\right). \tag{4.3}$$

Indeed, it is clear that the differential equation in (4.2) implies

$$u(t) = u(0) + u'(0)t - \lambda \int_0^t \int_0^s \phi(\tau, u(\tau))d\tau\, ds.$$

Then

$$u(t) = u(0) + u'(0)t - \lambda \int_0^t (t-s)\phi(s, u(s))\, ds.$$

Multiplying the differential equation in (4.2) by $(1-t)$ and integrating the resulting equation from 0 to 1, we obtain

$$u'(0) = u(1) - u(0) + \lambda \int_0^1 (1-s)\phi(s, u(s))\, ds \tag{4.4}$$

Substituting the values of $u(1)$ and $u(0)$ gives (4.3).

(ii) We show that there exists a positive constant L_0, independent of λ, such that any possible solution u of (4.2) satisfies

$$\|u\| \leq L_0. \tag{4.5}$$

The boundedness of ϕ implies that there exists $M_\phi > 0$ such that $|\phi(t, u(t))| \leq M_\phi$ for all $t \in I$, so that $\|u''\|_0 \leq M_\phi$. Since $u'(t) = u'(0) - \lambda \int_0^t \phi(\tau, u(\tau))\, d\tau$, we have

$$|u'(t)| \leq |u'(0)| + M_\phi,$$

It follows from (4.4) that

$$|u'(t)| \leq |\rho| + |\delta| + M_\phi.$$

From (4.3), we infer that

$$|u(t)| \leq |\rho| + |\delta| + 2M_\phi.$$

Let $L_0 = |\rho| + |\delta| + 2M_\phi$. Then any possible solution of u of (4.2) satisfies (4.5).

(iii) Define and operator $\Upsilon : C^2(I) \to C^2(I)$ by $(\Upsilon u)(t) =$ the right hand side of (4.3). Let $\Omega := \{u \in C^2(I); \|u\| \leq L_0\}$. Then, it is easily seen that $(\Upsilon(\Omega))$ is uniformly bounded and equicontinuous. The Arzelà-Ascoli theorem implies that the operator Υ is compact. Moreover, the set of all solutions u of the equation $u = \lambda \Upsilon u$ is bounded (see (4.5)). It follows from Theorem 2.2 that $u = \Upsilon u$ has at least one solution. Thus, (4.2) has at least one solution for $\lambda = 1$, which is, in fact, unique from the previous step. Thus, u is a solution of (4.2). This completes the proof of Lemma 4.2. □

Let R_2 be the constant from Claim 2. Set $K = \max(R_2, \|\alpha'\|_0, \|\beta'\|_0)$ and consider the modified functions

$$F(t,u,v) = \begin{cases} f(t,u,K) & v > K \\ f(t,u,v), & -K \leq v \leq K. \\ f(t,u,-K), & v < -K \end{cases} \tag{4.6}$$

We construct a sequence of modified problems in the following way. Let $y_0 = \alpha$ and for $k = 1, 2, \ldots$ we consider the problems

$$\begin{cases} y_k''(t) + F(t, \triangle(y_k(t)), y_{k-1}'(t)) = 0, \quad t \in (0,1) \\ y_k(0) = \displaystyle\sum_{i=1}^{n} a_i y_{k-1}(\xi_i) \\ y_k(1) = \displaystyle\sum_{j=1}^{m} b_j y_{k-1}(\eta_j) \end{cases} \tag{4.7}$$

where $\triangle(y(t)) = \min\{\beta, \max(y(t), \alpha)\}$ for every $t \in I$. We show that every problem (4.7) has a unique solution which is uniformly bounded together with its first and second order derivatives.

Lemma 4.3. *The sequence* $\{y_k\}_{k\geq 1}$ *is well-defined and satisfies*

(i) $\alpha \le y_k \le \beta, k = 0, 1, \dots$

(ii) *If* $|y_l'(t)| \le K$ *for* $l = 0, 1, \dots, k-1$ *and for all* $t \in I$, *then there exists* K_1, *independent of* k, *such that* $|y_k'(t)| \le K_1$ *for all* $t \in I$.

(iii) $u = \lim_{k\to\infty} y_k$ *is a solution of* (1.1).

Proof. Since $y_0 = \alpha$ then y_0 is well-defined and satisfies (i) and (ii). Suppose that $y_1, \dots, y_k$ are well defined and satisfy (i) and (ii). We know that y_k is also well-defined and satisfies (i) and (ii). For this, let

$$\phi(t, y_k) = F(t, \triangle(y_k(t)), y_{k-1}'(t)), \delta = \sum_{i=1}^{n} a_i y_{k-1}(\xi_i)$$

and $\rho = \sum_{i=1}^{n} b_i y_{k-1}(\eta_i)$. If follows from Lemma 4.2 that (4.7) has a unique solution y_k. This shows that y_k is well-defined. We can proceed as in Claim 1 of Theorem 3.2 to show that $\alpha \le y_k \le \beta$. Since $\alpha \le y_k \le \beta$ and $|y_{k-1}'(t)| \le K$, it follows that $F(t, \triangle(y_k(t)), y_{k-1}'(t)) = f(t, y_k(t), y_{k-1}'(t))$ for all $t \in I$. Then,

$$y_k'(t) = y_k'(0) - \int_0^t f(s, y_k(s), y_{k-1}'(s)) ds,$$

with

$$\begin{aligned} y_k'(0) &= y_k(1) - y_k(0) + \int_0^1 (1-s) f(s, y_k(s), y_{k-1}'(s))\, ds \\ &= \sum_{i=1}^{n} b_i y_{k-1}(\eta_i) - \sum_{i=1}^{n} a_i y_{k-1}(\xi_i) + \int_0^1 (1-s) f(s, y_k(s), y_{k-1}'(s))\, ds. \end{aligned}$$

Hence

$$y_k'(t) = \sum_{j=1}^{m} b_j y_{k-1}(\eta_j) - \sum_{i=1}^{n} a_i y_{k-1}(\xi_i) + \int_0^1 (1-s) f(s, y_k(s), y_{k-1}'(s)) ds \tag{4.8}$$

$$- \int_0^t f(s, y_k(s), y_{k-1}'(s))\, ds. \tag{4.9}$$

Let $M_f := \max\{|f(t, y, z)| : t \in I, y \in [\alpha, \beta], |z| \le K\}$. Then (4.8) implies that there exists $K_1 = \max \left(\sum_{j=1}^{m} b_j + \sum_{i=1}^{n} a_i \right) \|\beta\|_0 + 2M_f, K)$ such that

$$|y_k'(t)| \le K_1 \text{ for all } t \in I.$$

Also, we have $|y_k''(t)| \le M_f$. We have shown that the sequences $\{y_k\}_{k\ge1}$, $\{y_k'\}_{k\ge1}$, $\{y_k''\}_{k\ge1}$, $\{y_{k-1}\}_{k\ge1}$, $\{y_{k-1}'\}_{k\ge1}$ are uniformly bounded.

The Bolanzo-Weierstrass Theorem implies that there are subsequences, which we label the same, which are uniformly convergent on the interval I. Let $u = \lim_{k\to\infty} y_k$, $v = \lim_{k\to\infty} y_k'$ and $w = \lim_{k\to\infty} y_k''$. Moreover, using a diagonalization process, if necessary, we assume that $u = \lim_{k\to\infty} y_{k-1}$ and $v = \lim_{k\to\infty} y_{k-1}'$. If follows from (4.8) and the continuity of the nonlinearity f that

$$w(t) + f(t, u(t), v(t)) = 0, \quad t \in (0, 1),$$

$$v(t) = v(0) - \int_0^t f(s, u(s), v(s))\, ds,$$

which gives $v'(t) = w(t), t \in (0,1)$. Next, the boundary conditions lead to

$$u(0) = \sum_{i=1}^{n} a_i u(\xi_i), u(1) = \sum_{j=1}^{m} b_j u(\eta_j).$$

Integrating both sides of (4.8) from 0 to t we get $u'(t) = v(t)$, so that $u''(t) = w(t)$. Finally, summarizing the above discussion, we see that u is a solution of (1.1). □

5. MULTIPLICITY OF SOLUTIONS

In this section we use the previous results to get multiplicity of solutions of problem (1.1), under the following assumption.

(H4) there exists sequences $\{\alpha_j\}, \{\beta_j\}$ in $C^2(I)$, of lower and upper solutions of (1.1), such that for all $j = 1, 2, \ldots$,
  (i) $0 < \alpha_j \le \beta_j \le \alpha_{j+1}$
  (ii) $f(t, \beta_j(t), \beta_j'(t)) > 0 > f(t, \alpha_j(t), \alpha_j')), t \in I$
  (iii) the condition (H2) holds on $[0,1] \times [\alpha_j, \beta_j] \times \mathbb{R}$.

Theorem 5.1. *Assume that Condition (H4) holds. Then Problem (1.1) has infinitely many positive solutions y_j such that $\alpha_j \le y_j \le \beta_j$.*

Example 5.2. The problem

$$\begin{cases} y''(t) = (1 + \cos y'(t)) \sin y(t) \quad 0 < t < 1, \\ y(0) = \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} y(\xi_i), \\ y(1) = \dfrac{1}{m} \displaystyle\sum_{\ell=1}^{m} y(\eta_\ell), \end{cases} \tag{5.1}$$

has an infinite number of positive solutions, namely $y_j = 2(j+1)\pi, j = 0, 1, 2, \ldots$.

The results of the previously published works do not apply. However, our Theorem 5.1 does apply. In fact, the function f, defined by $f(t, y, z) = (1 + \cos z) \sin y$ changes sign infinitely many times. For, we have

$$f(t, \alpha_j, 0) < 0 \text{ for } \alpha_j = \left(\frac{3}{2} + 2j\right) 2\pi, \quad j = 0, 1, 2 \ldots$$

and

$$f(t, \beta_j, 0) > 0 \text{ for } \beta_j = \left(\frac{5}{2} + 2j\right) \pi, \quad j = 0, 1, 2, \ldots$$

It is clear that $y_j \in [\alpha_j, \beta_j]$ for $j = 0, 1, 2 \ldots$

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REFERENCES

[1] R. Agarwal, D. O'Regan and P. Wong, *Positive Solutions of Differential Difference and Integral Equations*, Kluwer Acad., Boston, 1999.

[2] A. Boucherif, Positive solutions of second order differential equations with integral boundary conditions, *Discrete Contin. Dyn. Syst.* (2007), Suppl., 155–159.

[3] A. Boucherif and J. Henderson, Positive solutions of second order boundary value problems with sign changing Caratheodory nonlinearities, *Electron. J. Qualitative Theory Diff. Equ.* **2006** (2006), no. 7, 1–14.

[4] J. Cannon, The solution of the heat equation subject to the specification of energy, *Quarterly Applied Math.* **2** 21 (1963), 155–160.

[5] R. Chegis, R. Chegis, Numerical solution of a heat conduction problem with in integral boundary condition, *Litovsk. Math. Sb.* **24** (1984), 209–215.

[6] P. W. Eloe and B. Ahmad, Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions, *Applied Math. Letters* **18** (2005), 521–527.

[7] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer Verlag, Berlin, 2003.

[8] M. Grossinho and F. Minhos, Existence results for some third order separated boundary value problems, *Nonlinear Anal.* **47** (2001), 2407–2418.

[9] G. Infante, Eigenvalues and postive solutions of ODEs involving integral boundary conditions, *Discrete Contin. Dyn. Syst.* **2005** (2005), Suppl., 436–442.

[10] N Ionkin, Solutions of a boundary value problem in heat conduction theory with nonlocal boundary conditions, *Differential Equations* **13** (1977), 294–304.

[11] W. Jiang and Y. Guo, Multiple positive solutions for second order m-point bounary value problems, *J. Math. Anal. Appl.* **327** (2007), 415–424.

[12] R. Khan, The generalized method of quasilinearization and nonlinear boundary value problems with integral boundary conditions, *Electron. J. Qualitative Theory Diff. Equ.* **2003** (2003), no. 10, 1–15.

[13] L. Kong and Q. Kong, Multipoint boundary value problems of second order differential equations, *Nonlinear Anal.* **58** (2004), 909–931.

[14] M. Krasnoselskii, *Positive Solutions of Operator Equations*, Noordhoff, Holland, 1964.

[15] J. Webb, Remarks on positive solutions of some three point boundary value problems, *Discrete Contin. Dyn. Syst.* **2003** (2003), Suppl., 905–915.

[16] Z. Yang, Positive solutions fo a second order integral boundary value problem, *J. Math. Anal. Appl.* **321** (2006), 751–765.

[17] X. Zhang, H. Zou and L. Liu, Positive solutions fo a second order m-point boundary value problems with changing sign singular nonlinearity, *Appplied Math. Letters* **20** (2007), 629–636.