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TWO NONTRIVIAL SOLUTIONS FOR A DISCRETE FOURTH ORDER PERIODIC BOUNDARY VALUE PROBLEM

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ABSTRACT. We study the discrete fourth order periodic boundary value problem with a parameter

$$\begin{cases} \Delta^4 u(t-2) - \Delta\big(p(t-1)\Delta u(t-1)\big) + q(t)u(t) = \lambda f(t,u(t)), \quad t \in [1,N]_{\mathbb{Z}}, \\ \Delta^i u(-1) = \Delta^i u(N-1), \quad i = 0,1,2,3. \end{cases}$$

By using variational methods and the mountain pass lemma, sufficient conditions are found under which the above problem has at least two nontrivial solutions. One example is included to illustrate the result.

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1. INTRODUCTION

Throughout this paper, for any integers c and d with $c \le d$, let $[c,d]_{\mathbb{Z}}$ denote the discrete interval $\{c, c+1, \dots, d\}$. Here, we study the discrete nonlinear fourth order periodic boundary value problem (BVP, for short) with a parameter

$$\begin{cases} \Delta^4 u(t-2) - \Delta\big(p(t-1)\Delta u(t-1)\big) + q(t)u(t) = \lambda f(t,u(t)), \ t \in [1,N]_{\mathbb{Z}}, \\ \Delta^i u(-1) = \Delta^i u(N-1), \quad i = 0,1,2,3, \end{cases} \tag{1.1}$$

where $N \ge 1$ is an integer, Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$, $\Delta^0 u(t) = u(t)$, $\Delta^i u(t) = \Delta^{i-1}(\Delta u(t))$ for $i \ge 1$, $p : [0,N]_{\mathbb{Z}} \to \mathbb{R}$ with $p(0) = p(N)$, $q : [1,N]_{\mathbb{Z}} \to \mathbb{R}$, $f : [1,N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is continuous in its second argument, and λ is a positive parameter. By a *solution* of BVP (1.1), we mean a function $u : [-1, N+2]_{\mathbb{Z}} \to \mathbb{R}$ such that u satisfies (1.1).

Difference equations appear naturally as discrete analogues and as numerical solutions of differential equations and delay differential equations which model various diverse phenomena in statistics, computing, electrical circuit analysis, dynamical systems, economics, and biology (see, for example, [1, 17, 18]). In recent years, many

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researchers have studied discrete fourth order BVPs with various boundary conditions. The reader may refer to [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 19, 20] for some recent work on this topic. Among those cited papers, [3, 4, 8, 9, 10, 11] are for discrete fourth order periodic problems. In particular, papers [8, 9, 10, 11] investigated BVP (1.1), and existence results for one, two, and more than two solutions are obtained by using variational arguments.

In this paper, we apply variational methods and the well-known mountain pass lemma of Ambrosetti and Rabinowitz to obtain some new conditions for the existence of two nontrivial solutions of BVP (1.1). The eigenvalues of a certain symmetric matrix associated with the problem are used in our discussion.

The rest of this paper is organized as follows. Section 2 contains some preliminary results; in particular, the variational structure of BVP (1.1) is established there. Section 3 contains the main result of this paper and its proof, and one illustrative example.

2. PRELIMINARY RESULTS

In this section, we collect some necessary preliminary results. The presentation of this section can be found in papers [8, 9, 10, 11].

We define a vector space X by

$$X = \left\{u : [-1, N+2]_{\mathbb{Z}} \to \mathbb{R} \mid \Delta^i u(-1) = \Delta^i u(N-1), \quad i = 0, 1, 2, 3\right\}, \tag{2.1}$$

and for any $u \in X$, let

$$\|u\| = \left(\sum_{t=1}^{N} |u(t)|^2\right)^{1/2}.$$

Remark 2.1. It is easy to see that, for any $u \in X$, we have

$$u(-1) = u(N-1), \quad u(0) = u(N), \quad u(1) = u(N+1), \quad \text{and} \quad u(2) = u(N+2). \tag{2.2}$$

Then, equipped with $\|\cdot\|$, X is an N dimensional reflexive and separable Banach space. In fact, X is isomorphic to $\mathbb{R}^N$. In this paper, when we write $u = (u(1), \dots, u(N)) \in \mathbb{R}^N$, we always imply that the vector u has been extended to a vector in X so that (2.2) holds, i.e., u has been extended to the vector

$$\big(u(N-1), u(N), u(1), \dots, u(N), u(1), u(2)\big) \in X,$$

and when we write $X = \mathbb{R}^N$, we mean that the elements in $\mathbb{R}^N$ have been extended in the above sense.

For $u \in X$, let the functionals Φ and Ψ be defined by

$$\Phi(u) = \frac{1}{2}\sum_{t=1}^{N}\left[|\Delta^2 u(t-2)|^2 + p(t-1)|\Delta u(t-1)|^2 + q(t)|u(t)|^2\right] \tag{2.3}$$

and

$$\Psi(u) = \sum_{t=1}^{N} F(t, u(t)), \tag{2.4}$$

where

$$F(t,x) = \int_0^x f(t,s)ds \quad \text{for } (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}. \tag{2.5}$$

Then, Φ and Ψ are continuously Gâteaux differentiable, and their Gâteaux derivatives at $u \in X$ are the functionals $\Phi'(u)$ and $\Psi'(u)$ given by

$$\Phi'(u)(v) = \sum_{t=1}^{N} \left[\Delta^2 u(t-2)\Delta^2 v(t-2) + p(t-1)\Delta u(t-1)\Delta v(t-1) + q(t)u(t)v(t)\right]$$

and

$$\Psi'(u)(v) = \sum_{t=1}^{N} f(t,u(t))v(t)$$

for any $v \in X$.

The following lemma follows from [10, Lemma 2.3] or [11, Lemma 2.3].

Lemma 2.2. *A function $u \in X$ is a critical point of the functional $\Phi - \lambda\Psi$ if and only if $u(t)$ is a solution of BVP* (1.1).

In the remainder of this section, we introduce an equivalent form of the functional Φ. Define the $N \times N$ matrices A, B, and C as follows: If $N \geq 5$, let

$$A = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 & -4 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 6 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & 6 & -4 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 & -4 \\ -4 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 6 \end{pmatrix}, \tag{2.6}$$

and if $N = 1, 2, 3, 4$, let A be respectively given by

$$(0), \quad \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix}, \quad \begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 6 & -4 & 2 & -4 \\ -4 & 6 & -4 & 2 \\ 2 & -4 & 6 & -4 \\ -4 & 2 & -4 & 6 \end{pmatrix}.$$

If $N \geq 3$, let

$$B = \begin{pmatrix} p(0)+p(1) & -p(1) & 0 & \cdots & -p(0) \\ -p(1) & p(1)+p(2) & -p(2) & \cdots & 0 \\ 0 & -p(2) & p(2)+p(3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -p(N-1) \\ -p(0) & 0 & 0 & \cdots & p(N-1)+p(0) \end{pmatrix}, \tag{2.7}$$

and if $N = 1, 2$, let B be respectively given by

$$(0) \quad \text{and} \quad \begin{pmatrix} p(0)+p(1) & -p(0)-p(1) \\ -p(0)-p(1) & p(0)+p(1) \end{pmatrix}.$$

Finally, for $N \geq 1$, let

$$C = \begin{pmatrix} q(1) & 0 & 0 & \cdots & 0 & 0 \\ 0 & q(2) & 0 & \cdots & 0 & 0 \\ 0 & 0 & q(3) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q(N-1) & 0 \\ 0 & 0 & 0 & \cdots & 0 & q(N) \end{pmatrix}. \tag{2.8}$$

Clearly, A, B, and C are symmetric. Let

$$u = (u(1), u(2), \ldots, u(N))^T \in \mathbb{R}^N.$$

Then, for $u \in X$, it is easy to verify that

$$\sum_{t=1}^{N} |\Delta^2 u(t-2)|^2 = u^T A u,$$

$$\sum_{t=1}^{N} p(t-1)|\Delta u(t-1)|^2 = u^T B u,$$

and

$$\sum_{t=1}^{N} q(t)|u(t)|^2 = u^T C u.$$

Thus, from (2.3), Φ can be rewritten as (see, for example, [8, (2.7) and Remark 2.3])

$$\Phi(u) = \frac{1}{2} u^T (A + B + C) u.$$

Remark 2.3. The matrices A, B, and C satisfy the following properties:

(a) A is positive semidefinite. In fact, it is clear that 0 is an eigenvalue of A with an eigenvector $(1, 1, \ldots, 1)^T$. Moreover, it can be shown that the $(N-1)$th leading principal submatrix of A is positive definite. Thus, A is positive semidefinite.

(b) If $p(t) > 0$ for $t \in [0, N-1]_{\mathbb{Z}}$, then B is positive semidefinite. In fact, it is clear that 0 is an eigenvalue of B with an eigenvector $(1, 1, \ldots, 1)^T$. Moreover, it can be shown that the $(N-1)$th leading principal submatrix of B is positive definite. Thus, B is positive semidefinite.

(c) If $q(t) > 0$ for $t \in [1, N]_{\mathbb{Z}}$, then C is positive definite.

3. MAIN RESULT

In this section, we present our main result and its proof. We need the following conditions. Below, X and F are defined by (2.1) and (2.5), respectively.

(H1) $p(t) > 0$ for $t \in [0, N-1]_{\mathbb{Z}}$ and $q(t) > 0$ for $t \in [1, N]_{\mathbb{Z}}$;

(H2) $\lim_{|x|\to 0} \frac{|F(t,x)|}{|x|^2} = 0$ for $t \in [1, N]_{\mathbb{Z}}$;

(H3) $\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^2} \le 0$ for $t \in [1, N]_{\mathbb{Z}}$;

(H4) there exists $w \in X$ such that $\sum_{t=1}^{N} F(t, w(t)) > 0$.

Under condition (H1), in view of Remark 2.3, we see that $A+B+C$ is symmetric and positive definite, and so all of its eigenvalues are positive. Let ν_i, $i = 1, \ldots, N$, be the eigenvalues of $A + B + C$ satisfying

$$0 < \nu_1 \le \nu_2 \le \cdots \le \nu_N,$$

and let ξ_i be an eigenvector of $A + B + C$ associated with ν_i such that

$$\langle \xi_i, \xi_j \rangle = \begin{cases} 0, & i \ne j, \\ 1, & i = j, \end{cases}$$

that is, $\xi_i,\ i = 1, \ldots, N$, are an orthonormal basis.

Then, for any $u = (u(1), \ldots, u(N))^T \in \mathbb{R}^N$, it is easy to check that

$$\frac{1}{2}\nu_1 \|u\|^2 \le \Phi(u) = \frac{1}{2} u^T (A + B + C) u \le \frac{1}{2}\nu_N \|u\|^2. \tag{3.1}$$

We now state our main result in this paper.

Theorem 3.1. *Assume that (H1)–(H4) hold. Then, for each $\lambda > \underline{\lambda}$, BVP (1.1) has at least two nontrivial solutions, where $\underline{\lambda} = \inf_{u\in S} \underline{\lambda}(u)$ with*

$$\underline{\lambda}(u) = \frac{\nu_N \|u\|^2}{2\sum_{t=1}^{N} F(t, u(t))} \tag{3.2}$$

and

$$S = \left\{ u \in X \ :\ \sum_{t=1}^{N} F(t, u(t)) > 0 \right\}.$$

Remark 3.2. In view of (H4), $S \ne \emptyset$. Clearly, the conclusion of Theorem 3.1 still holds if $\lambda > \underline{\lambda}(w)$.

The following example illustrates the applicability of Theorem 3.1.

Example 3.3. In BVP (1.1), let $N = 6$, $p(0) = 1 + \sin 6$, $p(t) = 1 + \sin t$ and $q(t) = t^2 + 2$ for $t \in [1, 6]_{\mathbb{Z}}$, and

$$f(t,x) = \begin{cases} (\eta+1)|x|^{\eta-1}x, & |x| \geq 1, \\ (\zeta+1)|x|^{\zeta-1}x, & |x| < 1, \end{cases}$$

where $0 < \eta < 1 < \zeta < \infty$. Then, we claim that, for each $\lambda > 23.5590$, BVP (1.1) has at least two nontrivial solutions.

Clearly, (H1) holds, and for f defined above, we have

$$F(t,x) = \begin{cases} |x|^{\eta+1}, & |x| \geq 1, \\ |x|^{\zeta+1}, & |x| < 1, \end{cases}$$

so (H2) and (H3) hold as well. Moreover, (H4) also holds with $w(t) \equiv 1 \in X$.

With the above N, p, and q, let the matrices A, B, C be defined by (2.6)–(2.8). Then, using *MATLAB*, we find that the eigenvalues of $A + B + C$ are given by

$$\nu_1 \approx 6.5981, \quad \nu_2 \approx 13.7406, \quad \nu_3 \approx 21.1240,$$

$$\nu_4 \approx 27.7973, \quad \nu_5 \approx 34.4156, \quad \nu_6 \approx 47.1180.$$

In view of (3.2), we have $\underline{\lambda}(w) = \nu_6/2 \approx 23.5590$. The claim now follows from Theorem 3.1 and Remark 3.2.

Remark 3.4. In Example 3.3, it is interesting to observe that

$$\lim_{u \in S,\ \|u\| \to 0^+} \underline{\lambda}(u) = \infty \quad \text{and} \quad \lim_{u \in S,\ \|u\| \to \infty} \underline{\lambda}(u) = \infty.$$

In the remainder of this section, we prove Theorem 3.1. First, recall that a functional $I \in C^1(X, \mathbb{R})$ is said to satisfy the Palais-Smale (PS) condition if every sequence $\{u_n\} \subset X$, such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence. Here, the sequence $\{u_n\}$ is called a PS sequence of I.

In our proof, we need the following classic mountain pass lemma of Ambrosetti and Rabinowitz (see, for example, [15, Theorem 7.1]). Below, we denote by $B_r(u)$ the open ball centered at $u \in X$ with radius $r > 0$, $\overline{B}_r(u)$ its closure, and $\partial B_r(u)$ its boundary.

Lemma 3.5. *Let $(X, \|\cdot\|)$ be a real Banach space and $I \in C^1(X, \mathbb{R})$. Assume that I satisfies the PS condition and there exist $u_0, u_1 \in X$ and $\rho > 0$ such that*

(A1) $u_1 \notin \overline{B}_\rho(u_0)$;
(A2) $\max\{I(u_0), I(u_1)\} < \inf_{u \in \partial B_\rho(u_0)} I(u)$.

Then, I possesses a critical value which can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s)) \geq \inf_{u \in \partial B_\rho(u_0)} I(u),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\}.$$

In the sequel, let $I_\lambda = \Phi - \lambda\Psi$, where Φ and Ψ are defined by (2.3) and (2.4), respectively.

Lemma 3.6. *Assume that (H3) holds. Then, for any $\lambda > 0$, the functional I_λ is coercive and satisfies the PS condition.*

Proof. Let $\lambda > 0$ be fixed. We first show that I_λ is coercive, i.e.,

$$\lim_{\|u\|\to\infty} I_\lambda(u) = \infty \quad \text{for any } u \in X. \tag{3.3}$$

By (H3), there exists $K > 0$ such that

$$F(t,x) \le \epsilon|x|^2 \quad \text{for } (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}^N \text{ with } |x| > K, \tag{3.4}$$

where

$$0 < \epsilon < \frac{\nu_1}{2\lambda}. \tag{3.5}$$

On the other hand, by the continuity of f, there exists $c : [1,N]_{\mathbb{Z}} \to \mathbb{R}^+$ such that

$$|F(t,x)| \le c(t) \quad \text{for } (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}^N \text{ with } |x| \le K. \tag{3.6}$$

For any $u \in X$, let $S_1 = \{t \in [1,N]_{\mathbb{Z}} : |u(t)| \le K\}$ and $S_2 = \{t \in [1,N]_{\mathbb{Z}} : |u(t)| > K\}$. Then, from (3.1), (3.4), and (3.6), we have

$$\begin{aligned} I_\lambda(u) &\ge \frac{1}{2}\nu_1\|u\|^2 - \lambda\sum_{t\in S_1} F(t,u(t)) - \lambda\sum_{t\in S_2} F(t,u(t)) \\ &\ge \frac{1}{2}\nu_1\|u\|^2 - \lambda\sum_{t=1}^{N} c(t) - \lambda\epsilon\sum_{t=1}^{N} |u(t)|^2 \\ &= \left(\frac{1}{2}\nu_1 - \lambda\epsilon\right)\|u\|^2 - \lambda\sum_{t=1}^{N} c(t). \end{aligned}$$

Then, from (3.5), we see that $I_\lambda(u) \to \infty$ as $\|u\| \to \infty$, i.e., (3.3) holds.

Now, assume that $\{u_n\} \subset X$ is a PS sequence of I_λ. From (3.3), $\{u_n\}$ is bounded in X. Since the dimension of X is finite, $\{u_n\}$ has a convergent subsequence, i.e, I_λ satisfies the PS condition. This completes the proof of the lemma. □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. We first show that, for each $\lambda > 0$, 0 is a strict local minimizer of I_λ. Obviously,

$$I_\lambda(0) = \Phi(0) - \lambda\Psi(0) = 0.$$

For ϵ satisfying (3.5), by (H2), there exists $\kappa > 0$ such that

$$|F(t,x)| \le \epsilon|x|^2 \quad \text{for } (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}^N \text{ with } |x| \le \kappa.$$

Note that $u(t) \le \|u\|$ for $t \in [1, N]_{\mathbb{Z}}$. Then, for any $u \in B_\kappa(0) \setminus \{0\}$, from (3.1) and (3.5), we have

$$I_\lambda(u) \ge \frac{1}{2}\nu_1\|u\|^2 - \lambda\epsilon \sum_{t=1}^{N} |u(t)|^2 = \left(\frac{1}{2}\nu_1 - \lambda\epsilon\right)\|u\|^2 > 0.$$

Thus, for each $\lambda > 0$, 0 is a strict local minimizer of I_λ.

For $\lambda > \underline{\lambda}$, by the definition of $\underline{\lambda}$ and (3.2), there exists $v \in X$ such that

$$\lambda > \frac{\nu_N\|v\|^2}{2\sum_{t=1}^{N} F(t, v(t))}.$$

Thus, from (3.1), it follows that

$$I_\lambda(v) \le \frac{1}{2}\nu_N\|v\|^2 - \lambda \sum_{t=1}^{N} F(t, v(t)) < 0 \quad \text{if } \lambda > \underline{\lambda}.$$

Thus, 0 is not a global minimizer of I_λ if $\lambda > \underline{\lambda}$.

Below, for any λ satisfying $\lambda > \underline{\lambda}$, we show that I_λ has a global minimizer. Choose $\xi \in \mathbb{R}$ such that

$$I_\lambda(w) < \xi < 0.$$

Let

$$Y = \{u \in X : I_\lambda(u) \le \xi\}.$$

Then, $Y \ne \emptyset$ and is bounded since I_λ is coercive by Lemma 3.6. We claim that I_λ is bounded below on Y. Assume, to the contrary, that there exists a sequence $\{u_n\} \subset Y$ such that

$$\lim_{n\to\infty} I_\lambda(u_n) = -\infty. \tag{3.7}$$

Note that $\{u_n\}$ is bounded. Then, passing to a subsequence, if necessary, we may assume that $u_n \to u$ in X. Since I_λ is continuous in X, we have

$$\lim_{n\to\infty} I_\lambda(u_n) = \lim_{n\to\infty} (\Phi(u_n) - \lambda\Psi(u_n)) = \Phi(u) - \lambda\Psi(u).$$

This contradicts (3.7). Thus, we have

$$0 > \eta := \inf_{u\in Y} I_\lambda(u) = \inf_{u\in X} I_\lambda(u) > -\infty.$$

Let $\{u_n\} \subset Y$ be a sequence such that

$$\lim_{n\to\infty} I_\lambda(u_n) = \eta.$$

Arguing as above, we see that there exits $u_1 \in X$ such that, up to a subsequence, $u_n \to u_1$ in X. Hence, we have

$$I_\lambda(u_1) = \eta < 0, \tag{3.8}$$

and so $u_1 \not\equiv 0$. Clearly, u_1 is a critical point of I_λ. Then, by Lemma 2.2, u_1 is a nontrivial solution of BVP (1.1).

In the following, we apply Lemma 3.5 to find a second critical point of I_λ when $\lambda > \underline{\lambda}$. By Lemma 3.6, I_λ satisfies the PS condition. Since 0 is a strict local minimizer of I_λ, there exists $0 < \rho < \|u_1\|$ such that

$$r := \inf_{u \in \partial B_\rho(u_0)} I_\lambda(u) > 0.$$

Then, in view of the fact that $I_\lambda(0) = 0$ and (3.8) holds, we see that all the conditions of Lemma 3.5 are satisfied with $u_0 = 0$ and the above u_1. Thus, by Lemma 3.5, there exists a critical point u_2 of I_λ such that

$$I_\lambda(u_2) \geq r > 0. \tag{3.9}$$

By (3.8) and (3.9), we see that $u_1 \neq u_2$ and $u_2 \not\equiv 0$. Hence, Lemma 2.2 implies that u_2 is a second nontrivial solution of BVP (1.1). This completes the proof of the theorem. □

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