Communications in Applied Analysis **19** (2015), 497–504

A FRACTIONAL BOUNDARY VALUE PROBLEM WITH DIRICHLET BOUNDARY CONDITION

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ABSTRACT. In this paper, the authors study a nonlinear fractional boundary value problem consisting of the equation

$$-D^{\alpha}_{0+}u + aD^{\beta}_{0+}u = w(t)f(u), \quad 1 < \alpha < 2,\ 0 \le \beta < 1,$$

and the Dirichlet boundary condition. The associated Green's function is derived in terms of the generalized Mittag-Leffler function, and the existence of solutions is established based on it.

AMS (MOS) Subject Classification. Primary 34B15; secondary 34B27.

1. INTRODUCTION

In this paper, we consider the boundary value problem (BVP) consisting of the fractional differential equation

$$-D^{\alpha}_{0+}u + aD^{\beta}_{0+}u = w(t)f(u), \quad 0 < t < 1, \tag{1.1}$$

and the Dirichlet boundary condition (BC)

$$u(0) = u(1) = 0, \tag{1.2}$$

Received December 15, 2014

where $0 \le \beta < 1 < \alpha < 2$, $a \in \mathbb{R}$, $w \in L[0,1]$ such that $w(t) \ge 0$ a.e. on $[0,1]$, and $f \in C(\mathbb{R}, \mathbb{R})$. Here, $D^{\gamma}_{0+}h$ is the γ-th left Riemann-Liouville fractional derivative of $h : [0,1] \to \mathbb{R}$ defined by

$$(D^{\gamma}_{0+}h)(t) = \frac{1}{\Gamma(l-\gamma)} \frac{d^l}{dt^l} \int_0^t (t-s)^{l-\gamma-1} h(s)ds, \quad l = \lfloor \gamma \rfloor + 1, \tag{1.3}$$

whenever the right-hand side exists with $\Gamma(\cdot)$ being the Gamma function.

Fractional differential equations have extensive applications in various fields of science and engineering and have been a focus of research for decades; see [15, 17, 20] and the references therein. The existence of solutions or positive solutions of nonlinear fractional BVPs with various BCs has been investigated by many researchers; see [3, 5, 11, 14] for some recent development. Due to certain special properties of fractional calculus, critical point theory can only be applied to study equations involving both the left and right Riemann-Liouville fractional derivatives; see for example [3]. To the best of our knowledge, if only the left (or right) Riemann-Liouville fractional derivatives are involved, the only feasible approach is to convert the BVP to an integral equation and use various techniques to find the fixed points. This idea has been widely used in recent works; see, for example, [1, 2, 4, 5, 7, 8, 10, 9, 11, 12, 14, 16, 19, 21, 23]. Many of those operators are constructed based on the establishment of the associated Green's functions.

However, due to the unusual feature of the fractional calculus, the Green's functions for fractional BVPs have not been well developed. In most existing literature, the Green's functions were known only for the BVPs consisting of the equation

$$-D^{\alpha}_{0+}u = f(t,u), \quad 0 < t < 1, \tag{1.4}$$

and certain BCs, see for example [2, 4, 5, 7, 8, 14]. When a more general equation such as

$$-D^{\alpha}_{0+}u + a(t)u = f(t,u), \quad 0 < t < 1, \tag{1.5}$$

is involved, the method employed in those papers fails to work due to the complexity caused by the extra term $a(t)u$.

Recently, the present authors [10, 11] studied the BVPs consisting of Eq. (1.5) and two types of BCs. The associated Green's functions are constructed as series of functions based on the spectral theory. We refer the reader to [10, Theorem 2.1] and [11, Theorem 2.1] for the details. It is notable that the approach used in [11] can be easily extended to BVPs consisting of Eq. (1.5) and some other BCs. However, this approach cannot be directly applied to BVP (1.1), (1.2) due to the appearance of the term $D^{\beta}_{0+}u$. Furthermore, it is natural to raise the following question: Is it possible to find Green's functions in closed forms or in terms of special functions?

Motivated by these concerns, in this paper, we first derive an associated Green's function for BVP (1.1), (1.2) and express it in terms of the generalized Mittag-Leffler function $E_{\gamma,\delta}: \mathbb{C} \to \mathbb{C}$ defined by

$$E_{\gamma,\delta}[z] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\gamma n + \delta)}. \tag{1.6}$$

It is known that the Mittag-Leffler function is an entire function and hence is convergent for all $z \in \mathbb{C}$. Then, we establish the existence of solutions using the Green's function.

This paper is organized as follows: after this introduction, the main results are stated in Section 2. One example is also given therein. All the proofs are given in Section 3.

2. MAIN RESULTS

We first consider the Green's function for the associated linear BVP consisting of the equation

$$-D_{0+}^{\alpha} u + a D_{0+}^{\beta} u = 0, \quad 0 < t < 1, \tag{2.1}$$

and BC (1.2).

Let $G: [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$G(t,s) = \begin{cases} \tilde{G}_1(t,s), & 0 \le t \le s \le 1, \\ \tilde{G}_1(t,s) - \tilde{G}_2(t,s), & 0 \le s \le t \le 1, \end{cases} \tag{2.2}$$

where

$$\tilde{G}_1(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1} E_{\alpha-\beta,\alpha}[a t^{\alpha-\beta}] E_{\alpha-\beta,\alpha}[a(1-s)^{\alpha-\beta}]}{E_{\alpha-\beta,\alpha}[a]}, \tag{2.3}$$

$$\tilde{G}_2(t,s) = (t-s)^{\alpha-1} E_{\alpha-\beta,\alpha}[a(t-s)^{\alpha-\beta}], \tag{2.4}$$

and the Mittag-Leffler function $E_{\alpha-\beta,\alpha}$ is defined by (1.6). It is easy to see that G is continuous on $[0,1] \times [0,1]$.

Theorem 2.1. *Assume* $|a| < \Gamma(\alpha - \beta + 1)$. *Then* G *defined by* (2.2) *is the Green's function for BVP* (2.1), (1.2). *Furthermore,* $G(0,\cdot) = G(1,\cdot) = G(\cdot,0) = G(\cdot,1) = 0$.

Remark 2.2. Note that $|a| < \Gamma(\alpha-\beta+1)$ is an important condition in Theorem 2.1 to guarantee the existence of the Green's function for BVP (2.1), (1.2). We would like to point out that if $\beta = 0$, the existence of the Green's function has been proved in [11, Theorem 2.1] under the wider condition $|a| < 4^{\alpha-1}\Gamma(\alpha)$ using a different approach that is not applicable to BVP (2.1), (1.2) with $\beta \neq 0$.

With the Green's function G obtained in Theorem 2.1, we are ready to study the nonlinear BVP (1.1), (1.2). Let

$$\overline{G}(s) = 2(1-s)^{\alpha-1}E_{\alpha-\beta,\alpha}[a(1-s)^{\alpha-\beta}] \tag{2.5}$$

and

$$U = \int_0^1 \overline{G}(s)w(s)ds. \tag{2.6}$$

For any $u \in C[0,1]$, define $\|u\| = \max_{t\in[0,1]} |u(t)|$.

Theorem 2.3. *Let $|a| < \Gamma(\alpha-\beta+1)$. Assume there exists $r > 0$ such that*

$$|f(x)| \le r/U, \quad x \in [0,r]. \tag{2.7}$$

Then BVP (1.1), (1.2) has at least one solution u with $\|u\| \le r$.

To illustrate the application of our result, we consider the following example.

Example 2.4. Consider the BVP

$$\begin{cases} -D^{\alpha}_{0+}u + aD^{\beta}_{0+}u = e^t(\sin(u)+b), \\ u(0) = u(1) = 0, \end{cases} \tag{2.8}$$

where $0 \le \beta < 1 < \alpha < 2$ and $|a| < \Gamma(\alpha-\beta+1)$. We claim that for any $b \in \mathbb{R}$, BVP (2.8) has at least one solution.

In fact, let $w(t) = e^t$ and $f(x) = \sin(x) + b$. It is easy to see that (2.7) holds when r is large enough. Then by Theorem 2.3, BVP (2.8) has at least one solution.

3. PROOFS

The following lemma on spectral theory in Banach spaces is used to derive the associated Green's function; see [22, page 795, items 57b and 57d] for details.

Lemma 3.1. *Let X be a Banach space and $\mathcal{A} : X \to X$ be a linear operator with the operator norm $\|\mathcal{A}\|$ and the spectral radius $r(\mathcal{A})$ of $\mathcal{A}$. Then*

(a) $r(\mathcal{A}) \le \|\mathcal{A}\|$;
(b) *if $r(\mathcal{A}) < 1$, then $(\mathcal{I}-\mathcal{A})^{-1}$ exists and $(\mathcal{I}-\mathcal{A})^{-1} = \sum_{n=0}^{\infty} \mathcal{A}^n$, where $\mathcal{I}$ stands for the identity operator.*

Let $D^{\gamma}_{0+}h$ be defined by (1.3) and let $I^{\gamma}_{0+}h$ be the γ-th Riemann-Liouville fractional integral of h defined by

$$(I^{\gamma}_{0+}h)(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}h(s)ds. \tag{3.1}$$

The following properties of fractional calculus are taken from [17, Section 2.1].

Lemma 3.2. *Assume $\gamma > 0$, $\delta > 0$, and $h \in C[0,1]$. Then:*

(a) $I^{\gamma}_{0+}t^{\delta} = \frac{\Gamma(\delta+1)t^{\gamma+\delta}}{\Gamma(\gamma+\delta+1)}$;

(b) $I^{\gamma}_{0+}I^{\delta}_{0+}h = I^{\gamma+\delta}_{0+}h$;

(c) $(I^{\gamma}_{0+}D^{\gamma}_{0+}h)(t) = h(t) + \sum_{i=1}^{n} c_i t^{\gamma-i}$, *where* $n = \lfloor \gamma \rfloor + 1$, c_i, $i = 1, \dots, n$, *are constants depending on* h.

In the sequel, we let $X = C[0,1]$ be the Banach space with the standard maximum norm.

Proof of Theorem 2.1. For any $h \in X$, assume u is a solution of the BVP consisting of the equation

$$-D^{\alpha}_{0+}u + aD^{\beta}_{0+}u = h(t), \tag{3.2}$$

and (1.2). By Lemma 3.2,

$$I^{\alpha}_{0+}D^{\beta}_{0+}u = I^{\alpha-\beta}_{0+}(I^{\beta}_{0+}D^{\beta}_{0+}u) = I^{\alpha-\beta}_{0+}(u + c_0 t^{\beta-1}) = I^{\alpha-\beta}_{0+}u + \tilde{c}_0 t^{\alpha-1}, \tag{3.3}$$

where c_0, $\tilde{c}_0$ are constants.

Now, we take the α-th integral on both sides of (3.2). By Lemma 3.2 (a), (c), and (3.3), we have

$$u(t) - a(I^{\alpha-\beta}_{0+}u)(t) = -(I^{\alpha}_{0+}h)(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \tag{3.4}$$

where c_1, c_2 are constants to be determined. Note that $u(0) = 0$ implies $c_2 = 0$. Hence, (3.4) becomes

$$u(t) - a(I^{\alpha-\beta}_{0+}u)(t) = -(I^{\alpha}_{0+}h)(t) + c_1 t^{\alpha-1}. \tag{3.5}$$

Define $\mathcal{A} : X \to X$ and $\mathcal{B} : X \to X$ by

$$(\mathcal{A}u)(t) = a(I^{\alpha-\beta}_{0+}u)(t) \quad \text{and} \quad (\mathcal{B}h)(t) = -(I^{\alpha}_{0+}h)(t) + c_1 t^{\alpha-1}.$$

Then

$$(\mathcal{I} - \mathcal{A})u = \mathcal{B}h. \tag{3.6}$$

By (3.1) and Lemma 3.2 (a), when $|a| < \Gamma(\alpha - \beta + 1)$,

$$\begin{aligned} \|A\| = \sup_{\|u\|=1} \|\mathcal{A}u\| \le \sup_{t\in[0,1]} \left| a\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} ds \right| = \sup_{t\in[0,1]} \left| (aI^{\alpha-\beta}_{0+}1)(t) \right| \\ = \frac{|a|}{\Gamma(\alpha-\beta+1)} < 1. \end{aligned}$$

Then by Lemma 3.1,

$$u = \sum_{n=0}^{\infty} \mathcal{A}^n \mathcal{B}h.$$

By Lemma 3.2 (a) and (b), we can show that

$$\begin{aligned}(\mathcal{A}^n\mathcal{B}h)(t) &= (\mathcal{A}^n(-I_{0+}^{\alpha}h))(t) + c_1\mathcal{A}^n(t^{\alpha-1})\\ &= -\frac{a^n}{\Gamma((n+1)\alpha-n\beta)}\int_0^t (t-s)^{(n+1)\alpha-n\beta-1}h(s)ds\\ &\quad+\frac{c_1a^n\Gamma(\alpha)}{\Gamma((n+1)\alpha-n\beta)}t^{(n+1)\alpha-n\beta-1}, \quad n=0,1,\ldots. \end{aligned}\tag{3.7}$$

By the convergence of the Mittag-Leffler function defined in (1.6), we see that

$$\sum_{n=0}^{\infty}\frac{a^n(t-s)^{n(\alpha-\beta)}}{\Gamma(n(\alpha-\beta)+\alpha)}$$

is uniformly convergent on $[0,1]\times[0,1]$. Therefore, by (3.7),

$$\begin{aligned}u(t) &= \left(\sum_{n=0}^{\infty}\mathcal{A}^n\mathcal{B}h\right)(t)\\ &= \sum_{n=0}^{\infty}\frac{c_1a^n\Gamma(\alpha)t^{(n+1)\alpha-n\beta-1}}{\Gamma((n+1)\alpha-n\beta)} - \sum_{n=0}^{\infty}\int_0^t\frac{a^n(t-s)^{(n+1)\alpha-n\beta-1}}{\Gamma((n+1)\alpha-n\beta)}h(s)ds\\ &= c_1\Gamma(\alpha)t^{\alpha-1}\sum_{n=0}^{\infty}\frac{a^nt^{n(\alpha-\beta)}}{\Gamma(n(\alpha-\beta)+\alpha)} - \int_0^t(t-s)^{\alpha-1}\sum_{n=0}^{\infty}\frac{a^n(t-s)^{n(\alpha-\beta)}}{\Gamma(n(\alpha-\beta)+\alpha)}h(s)ds\\ &= c_1\Gamma(\alpha)t^{\alpha-1}E_{\alpha-\beta,\alpha}[at^{\alpha-\beta}] - \int_0^t(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}[a(t-s)^{\alpha-\beta}]h(s)ds,\end{aligned}\tag{3.8}$$

with $E_{\alpha-\beta,\alpha}$ being the generalized Mittag-Leffler function defined by (1.6). Note that $u(1)=0$ implies

$$c_1 = \int_0^1\frac{(1-s)^{\alpha-1}E_{\alpha-\beta,\alpha}[a(1-s)^{\alpha-\beta}]}{\Gamma(\alpha)E_{\alpha-\beta,\alpha}[a]}h(s)ds.$$

Hence by (3.8),

$$u(t) = \int_0^1 G(t,s)h(s)ds,\tag{3.9}$$

where G is defined by (2.2).

On the other hand, let u be defined by (3.9). By reversing the above process, we can show that u is a solution of Eq. (3.2). It is also easy to verify that BC (1.2) holds.

Therefore, G is the Green's function for BVP (2.1), (1.2).

Finally, by (1.6), (2.3), and (2.4), it is easy to verify that

$$G(0,\cdot)=G(1,\cdot)=G(\cdot,0)=G(\cdot,1)=0.$$

□

In the sequel, let $K \subset X$ be a set defined by

$$K = \{u \in X \mid \|u\| \leq r\}$$

where r is given in Theorem 2.3.

Define $T: X \to X$ by

$$(Tu)(t) = \int_0^1 G(t,s)w(s)f(u(s))ds, \tag{3.10}$$

where G is given in (2.2).

It is clear that u is a solution of BVP (1.1), (1.2) if and only if $u \in X$ is a fixed point of T. By a standard argument, we can show that T is completely continuous.

Proof of Theorem 2.3. By (2.2)–(2.5), it is easy to see that

$$|G(t,s)| \leq |\tilde{G}_1(t,s)| + |\tilde{G}_2(t,s)| \leq \overline{G}(s) \text{ on } [0,1] \times [0,1].$$

For any $u \in K$, we have $u(t) \leq r$ on $[0,1]$. By (2.6), (2.7), and (3.10),

$$\begin{aligned} |(Tu)(t)| = \left| \int_0^1 G(t,s)w(s)f(u(s))ds \right| &\leq \int_0^1 |G(t,s)|w(s)|f(u(s))|ds \\ &\leq \int_0^1 \frac{\overline{G}(s)w(s)r}{U} ds = \frac{r}{U} U = r, \quad t \in [0,1]. \end{aligned}$$

Hence, $\|Tu\| \leq r$. Therefore, $TK \subset K$.

By Schauder fixed point theorem, T has a fixed point u in K. Hence, BVP (1.1), (1.2) has at least one solution $u(t)$ with $\|u\| \leq r$. □

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