

## EXISTENCE OF POSITIVE SOLUTIONS TO A SECOND-ORDER DIFFERENTIAL EQUATION AT RESONANCE

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**ABSTRACT.** We establish sufficient conditions for the existence of positive solutions to the multipoint boundary value

$$\begin{aligned} -u'' &= f(t, u(t)), \quad t \in (0, 1), \\ u(0) &= u(1), \\ u'(0) &= u'(\eta). \end{aligned}$$

Since the associated homogeneous boundary value problem is not invertible the problem is said to be at resonance. The main tool employed is a variant of a fixed point index theorem due to Cremins for A-proper semilinear operators defined on cones.

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### 1. INTRODUCTION

We consider the existence of positive solutions to the second-order boundary value problem

$$-u''(t) = f(t, u(t)), \quad t \in (0, 1), \tag{1.1}$$

$$u(0) = u(1), \tag{1.2}$$

$$u'(0) = u'(\eta). \tag{1.3}$$

Throughout we assume that  $f : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$  is continuous. The boundary conditions (1.2) and (1.3) are such that the homogeneous boundary value problem  $-u'' = 0, u(0) = u(1), u'(0) = u'(\eta)$ , is not invertible. We say that the system is at resonance.

Due to their significance in real world applications many authors have studied the existence of positive solutions to boundary value problems; see [1, 2, 3, 4, 5, 9, 12, 13, 14, 15, 16, 17, 20, 26, 28, 29, 30, 31, 32] and references therein. Recently, several authors have also studied boundary value problems at resonance, see for example [2, 6, 7, 8, 10, 11, 18, 19, 21, 22, 23, 25, 26, 30].

Of particular interest are the papers by O'Regan and Zima [26], Bai and Fang [2] and Fang and Zhang [30]. The main result in [26] gives conditions in terms of norms for the existence of solutions to equations of the form  $Lx = Nx$ . The authors use their main result to show the existence of a positive solution to the periodic boundary value problem  $x'(t) = f(t, x(t)), x(0) = x(1)$ . Their results are extension of the work done by Santanilla [29]. In [2] the authors consider the three point boundary value problem

$$\begin{aligned}(p(t)x'(t))' &= f(t, x(t), x'(t)), \quad t \in (0, 1), \\ x'(0) &= 0, \quad x(1) = x(\eta).\end{aligned}$$

The boundary condition  $x(1) = x(\eta)$  ensures that the system is at resonance. Using a Cremins' type fixed point theorem the authors give sufficient conditions for the existence of at least one positive solution. In [30], the authors give sufficient conditions for the existence of at least one positive solution to the resonant multi-point boundary value problem

$$\begin{aligned}-x'' &= f(t, x), \quad t \in (0, 1), \\ x(0) &= \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \quad x(1) = \sum_{i=1}^{m-2} \beta_i x(\eta_i),\end{aligned}$$

when  $\sum_{i=1}^{m-2} \alpha_i = \sum_{i=1}^{m-2} \beta_i = 1$ . They state and prove an existence result based on the theorems due to Cremins [3]. We will use the existence theorem found in [30] to give sufficient conditions for the existence of positive solutions of the boundary value problem (1.1)–(1.3). All of these papers employ Mawhin's coincidence theory [24].

In Section 2 we give the necessary preliminary results from the theory of the fixed point index for A-proper semilinear operators on cones. Also, we state a fixed point theorem due to Wang and Zhang [30], which we employ to establish the existence to a positive solutions to (1.1)–(1.3). We state and prove our main result in Section 3.

## 2. PRELIMINARIES

This section begins with some basic definitions and notation associated with fixed point index theory for A-proper semilinear operators defined on a cone as outlined by Cremins, see [3]. We also present the necessary background from Mawhin's coincidence theory, see [24]. At the end of this section we state a variant of the fixed point theorem found in [30].

Let  $X$  and  $Z$  be Banach spaces and  $D$  a linear subspace of  $X$ . Let  $\{X_n\} \subset D$  and  $\{Z_n\} \subset Z$  be sequences of oriented finite dimensional subspaces such that  $Q_n z \rightarrow z$  in  $Z$  for every  $z$  and  $\text{dist}(x, X_n) \rightarrow 0$  for every  $x \in D$  where  $Q_n : Z \rightarrow Z_n$  and  $P_n : X \rightarrow X_n$  are sequences of continuous linear projections. The projection scheme  $\Gamma = \{X_n, Z_n, P_n, Q_n\}$  is then said to be *admissible* for maps from  $D \subset X$  to  $Z$ .

**Definition 2.1.** A map  $T : D \subset X \rightarrow Y$  is called *approximation-proper* (abbreviated *A-proper*) at a point  $z \in Z$  with respect to  $\Gamma$  if  $T_n \equiv P_n T|_{D \cap X_n}$  is continuous for each  $n \in \mathbb{N}$  and, whenever  $\{x_{n_j} : x_{n_j} \in D \cap X_{n_j}\}$  is bounded with  $T_{n_j} x_{n_j} \rightarrow z$ , then there exists a subsequence  $\{x_{n_{j_k}}\}$  such that  $x_{n_{j_k}} \rightarrow x \in D$  and  $Tx = z$ .  $T$  is said to be A-proper on a set  $\Omega$  if it is A-proper at all points in  $\Omega$ .

Let  $C$  be a cone in the Banach space  $X$ . Let  $\Omega \subset X$  be an open and bounded set with  $\Omega \cap C = \Omega_C \neq \emptyset$ . Let  $T : \Omega_C \rightarrow C$  be a continuous operator such that  $Tx \neq x$  for all  $u \in \partial\Omega_C$ .

Our goal is to rewrite (1.1)-(1.3) in the form  $Lu = Nu$  where  $L$  is a Fredholm mapping. With this objective in mind we define what is meant by a Fredholm mapping on index zero and then state some needed properties.

**Definition 2.2.** A *Fredholm mapping* is a linear mapping  $L : \text{dom } L \subset X \rightarrow Z$  that satisfies the following two conditions:

- (i)  $\text{Ker } L$  has a finite dimension, and
- (ii)  $\text{Im } L$  is closed and has finite codimension.

If  $L$  is a Fredholm mapping, its (Fredholm) *index* is the integer  $\text{Ind } L = \dim \text{Ker } L - \text{codim } \text{Im } L$ .

Let  $L : \text{dom } L \subset X \rightarrow Z$  be a Fredholm map of index zero. Then there exist continuous projectors  $P : X \rightarrow X$  and  $Q : Z \rightarrow Z$  such that

$$\text{Im } P = \text{Ker } L, \text{Ker } Q = \text{Im } L, X = \text{Ker } L \oplus \text{Ker } P, Z = \text{Im } L \oplus \text{Im } Q \quad (2.1)$$

and the mapping  $L|_{\text{dom } L \cap \text{Ker } P} : \text{dom } L \cap \text{Ker } P \rightarrow \text{Im } L$  is invertible. The inverse of  $L_P := L|_{\text{dom } L \cap \text{Ker } P}$  is denoted by  $L^{-1} : \text{Im } L \rightarrow \text{dom } L \cap \text{Ker } P$ . Since  $\dim \text{Im } Q = \text{codim } \text{Im } L$  there exists an isomorphism  $J : \text{Im } Q \rightarrow \text{Ker } L$ . Let  $H = L + J^{-1}P$ . Then  $H : \text{dom } L \subset X \rightarrow Z$  is a linear bijection with bounded inverse. Hence  $C_1 = H(C \cap \text{dom } L)$  is a cone in the Banach space  $Z$ .

We assume that there is a continuous bilinear form  $[z, x]$  defined on  $Z \times X$  such that  $z \in \text{Im } L$  if and only if  $[z, x] = 0$  for all  $x \in \text{Ker } L$ .

The following variant of Cremins' fixed point theorem is found in Wang and Zhang (see [30]) and is used to establish our main results.

**Theorem 2.3.** Let  $L : \text{dom } L \rightarrow Z$  be a Fredholm operator of index zero,  $C \subset X$  be a cone,  $\Omega_1$  and  $\Omega_2$  be open bounded sets such that  $\theta \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ , and  $\Omega_2 \cap C \cap \text{dom } L \neq \emptyset$ , where  $\theta$  is the zero element in  $X$ . Suppose that  $L - \lambda N$  is A-proper for  $\lambda \in [0, 1]$  with  $N : \overline{\Omega_2 \cap C} \rightarrow Z$  bounded. Assume that

- (C1)  $(P + JQN)(C) \subset C$  and  $(P + JQN + L^{-1}(I - Q)N)(C) \subset C$ ;
- (C2)  $Lu \neq \lambda Nu$  for all  $u \in \partial\Omega_2 \cap C$  and  $\lambda \in (0, 1]$ ;

- (C3)  $QNu \neq 0$  for  $u \in \partial\Omega_2 \cap C \cap \text{Ker } L$ ;  
 (C4)  $[QNu, u] \leq 0$  for all  $u \in \partial\Omega_2 \cap C \cap \text{Ker } L$ ;  
 (C5) there exists  $e \in C_1 \setminus \{\theta\}$  such that

$$Lu - Nu \neq \mu e \text{ for every } \mu \geq 0, u \in \partial\Omega_1 \cap C.$$

Then there exists an  $u \in \text{dom } L \cap C \cap (\bar{\Omega}_2 \setminus \Omega_1)$  such that  $Lu = Nu$ .

### 3. MAIN RESULT

In order to use Theorem 2.3 we must first write (1.1) in the form  $Lu = Nu$ . To this end, we use the Banach spaces  $X = Z = C[0, 1]$  with norm  $\|u\| = \max_{t \in [0, 1]} |u(t)|$ . The operator  $L : \text{dom } L \subset X \rightarrow Z$  is defined to be

$$Lu(t) = -u''(t)$$

with

$$\text{dom } L = \{u \in X : u'' \in C[0, 1], u(0) = u(1), u'(0) = u'(\eta)\}.$$

Define the mapping  $N : X \rightarrow Z$  by

$$Nu(t) = f(t, u(t))$$

and the cone  $C \subset X$  by

$$C = \{u \in X : u(t) \geq 0\}.$$

It is easy to verify that

$$\text{Ker } L = \{u \in \text{dom } L \subset X : u(t) \equiv c, c \in \mathbb{R}\}$$

$$\text{Im } L = \left\{ g \in Z : \int_0^\eta g(s) ds = 0 \right\}, \quad \text{and}$$

$$\dim \text{Ker } L = \text{codim Im } L = 1.$$

The projectors  $P : X \rightarrow X$  and  $Q : Z \rightarrow Z$  that arise naturally for this problem are

$$Qg(t) = \frac{1}{\eta} \int_0^\eta g(s) ds,$$

and

$$Pu(t) = \int_0^1 u(s) ds,$$

respectively. It is easy to check that the projectors  $P$  and  $Q$  are exact; that is, they satisfy (2.1). To simplify notation define the function  $h$  by

$$h(s) = \begin{cases} 0, & \eta < s \leq 1, \\ \frac{1}{\eta}, & 0 \leq s \leq \eta. \end{cases}$$

Then we can rewrite  $Q$  as

$$Qg(t) = \int_0^1 h(s)g(s) ds.$$

Our first Lemma is easy to verify. See [23] for a typical proof.

**Lemma 3.1.** *The mapping  $L : \text{dom } L \subset X \rightarrow Z$  is a Fredholm mapping of index zero.*

Define the isomorphism  $J := \text{Im } Q \rightarrow \text{Ker } L$  by  $Jg = g$ . The inverse mapping  $L^{-1} : \text{Im } L \rightarrow \text{dom } L \cap \text{Ker } P$  is given by

$$L^{-1}g(t) = \int_0^1 G(t, s)g(s) ds \tag{3.1}$$

where

$$G(t, s) = \frac{1}{2} \begin{cases} s(1 - 2t + s), & 0 \leq s < t, \\ (1 - s)(2t - s), & t \leq s \leq 1. \end{cases}$$

Also, define the function  $\tilde{G}(t, s)$  by

$$\tilde{G}(t, s) = h(s) + G(t, s) + \frac{6t^2 - 6t + 1}{6}h(s).$$

In order to employ Theorem 2.3,  $L - \lambda N$  must be A-proper for all  $\lambda \in [0, 1]$  and  $N : \overline{\Omega_2 \cap C} \rightarrow Z$  must be bounded.

**Lemma 3.2.** *Suppose that*

(H1) *There exist constants  $\alpha$  and  $\beta$  such that  $|f(t, u)| \leq \alpha + \beta|u|$  for all  $t \in [0, 1]$  and all  $u \geq 0$ .*

*Then the operator  $L^{-1} : \text{Im } L \rightarrow \text{dom } L \cap \text{Ker } P$  defined by (3.1) is compact,  $N : \overline{\Omega_2 \cap C} \rightarrow Z$  is bounded, where the set  $\Omega_2$  is defined in (3.2), and the operators  $L - \lambda N$  are A-proper for  $\lambda \in [0, 1]$ .*

*Proof.* An application of the Arzelà-Ascoli can be used to show that  $L^{-1} : \text{Im } L \rightarrow \text{dom } L \cap \text{Ker } P$  is compact. The boundedness of  $N$  follows trivially from (H1). Since  $L^{-1}$  is compact, then by Lemma 2(a) [27], the homotopy  $L - \lambda N, \lambda \in [0, 1]$  is A-proper. □

We are now ready to state our main result.

**Theorem 3.3.** *In addition to (H1), assume there exists  $0 < a < b$  such that*

- (H2)  $f(t, u) \geq -\kappa u$  for all  $t \in [0, 1], u \geq 0$ , where  $\kappa \leq \min\{\eta, 1/\max_{(t,s) \in [0,1]} \tilde{G}(t, s)\}$ ,
- (H3)  $f(t, v) > 0$  for all  $(t, v) \in [0, 1] \times [0, a]$ ,
- (H4)  $\max_{t \in [0,1]} f(t, b) < 0$ .

*Then there exists at least one positive solution  $u$  of (1.1)–(1.3) such that  $a \leq \|u\| \leq b$ .*

*Proof.* We begin by defining the sets

$$\Omega_1 = \{u \in X : \|u\| < a\}$$

and

$$\Omega_2 = \{u \in X : \|u\| < b\}. \tag{3.2}$$

Note that both  $\Omega_1$  and  $\Omega_2$  are open and bounded in  $X$  and that

$$\theta \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2.$$

We first show that condition (C1) of Theorem 2.3,  $(P + JQN)(C) \subset C$  and  $(P + JQN + K_P(I - Q)N)(C) \subset C$ , is satisfied. From assumption (H2) we have

$$\begin{aligned} Pu + JQNu &= \int_0^1 u(s) ds + \frac{1}{\eta} \int_0^\eta f(s, u(s)) ds \\ &\geq \int_0^1 u(s) ds - \frac{\kappa}{\eta} \int_0^\eta u(s) ds \\ &= \int_\eta^1 u(s) ds + \left(1 - \frac{\kappa}{\eta}\right) \int_0^\eta u(s) ds \geq 0. \end{aligned}$$

Also by (H1),

$$\begin{aligned} Pu + JQNu + L^{-1}(I - Q)Nu &= \int_0^1 u(s) ds + \frac{1}{\eta} \int_0^\eta f(s, u(s)) ds \\ &\quad + \int_0^1 G(t, s) \left[ f(s, u(s)) - \frac{1}{\eta} \int_0^\eta f(\tau, u(\tau)) d\tau \right] ds \\ &= \int_0^1 u(s) ds + \frac{1}{\eta} \int_0^\eta f(s, u(s)) ds \\ &\quad + \int_0^1 G(t, s) f(s, u(s)) ds - \frac{1}{\eta} \int_0^\eta f(\tau, u(\tau)) d\tau \int_0^1 G(t, s) ds \\ &\geq \int_0^1 (1 - \kappa \tilde{G}(t, s)) u(s) ds \geq 0. \end{aligned}$$

Hence, condition (C1) of Theorem 2.3 is satisfied.

Now consider condition (C2),  $Lu \neq \lambda Nu$ , for all  $u \in \partial\Omega_2 \cap C$ ,  $\lambda \in (0, 1]$ . Suppose that the condition fails. Then there exists a  $u_0 \in \partial\Omega_2 \cap C$  and a  $\lambda_0 \in (0, 1]$  such that  $Lu_0 = \lambda Nu_0$ . Since  $u_0 \in \partial\Omega_2 \cap C$  then there exists a  $t_0 \in (0, 1)$  such that  $u(t_0) = b$ . Then  $u'(t_0) = 0$  and  $u''(t_0) < 0$ . We have  $Lu_0(t_0) = -u''(t_0) > 0$ . This produces the contradiction to (H4)

$$0 < Lu_0(t_0) = Nu_0(t_0) = f(t_0, b) < 0.$$

Hence condition (C2) is valid.

Next we show that condition (C3),  $QNu \neq 0$  for all  $u \in \partial\Omega_2 \cap C \cap \text{Ker } L$ , is valid. Note that

$$QNu = \frac{1}{\eta} \int_0^\eta f(s, u(s)) ds.$$

Let  $u_0 \in \partial\Omega_2 \cap C \cap \text{Ker } L$ . Then  $u_0 \equiv b$  and so again by (H4)

$$QNu = \frac{1}{\eta} \int_0^\eta f(s, b) ds < 0.$$

Thus (C3) is satisfied.

To check that condition(C4),  $[QNu, U] \leq 0$  for all  $u \in \partial\Omega_2 \cap C \cap \text{Ker } L$ , we define the bilinear form  $[\cdot, \cdot] : Z \times X \rightarrow \mathbb{R}$  by

$$[g, u] = \int_0^\eta g(s)u(s) ds.$$

The bilinear form is continuous and if  $u \in \text{Ker } L$ , i.e.  $u \equiv c$  for some  $c \in \mathbb{R}$ , then

$$c \int_0^\eta g(s) ds = 0 \Leftrightarrow g \in \text{Im } L.$$

Let  $u \in \partial\Omega_2 \cap C \cap \text{Ker } L$ . By condition (H4) we have

$$\begin{aligned} [QNu, u] &= \int_0^\eta \frac{1}{\eta} \int_0^\eta f(s, b) ds b d\tau \\ &= \frac{b}{\eta} \int_0^\eta d\tau \int_0^\eta f(s, b) ds \\ &= b \int_0^\eta f(s, b) ds < 0. \end{aligned}$$

Thus (C4) is satisfied.

Finally we show that (C5), there exists  $e \in C_1 \setminus \{\theta\}$  such that  $Lu - Nu \neq \mu e$  for every  $\mu \geq 0, u \in \partial\Omega_1 \cap C$ , is true. We may assume that  $Lu \neq Nu$  for all  $u \in \partial\Omega_1 \cap C \cap \text{dom } L$  otherwise the proof is complete. Let  $e = 1 \in C_1 \setminus \{\theta\}$ . Suppose that  $u_1 \in \partial\Omega_1 \cap C \cap \text{dom } L$  and  $\mu_1 > 0$  are such that

$$Lu_1 - Nu_1 = \mu_1 e = \mu_1. \tag{3.3}$$

Since  $Qg = 0$  for all  $g \in \text{Im } L$ , then  $QLu_1 = 0$ , and so after applying  $Q$  to both sides of (3.3) we have

$$QNu_1 + Q\mu_1 = 0. \tag{3.4}$$

We have from (3.4) that

$$\int_0^\eta f(s, u(s)) + \mu_1 ds = 0.$$

However, from condition (H3),

$$\int_0^\eta f(s, u(s)) ds > 0$$

which is a contradiction since  $\mu_1 > 0$  and  $\eta > 0$ . Thus (C5) is valid.

Since all conditions of Theorem 2.3 are satisfied then there exists a solution  $u^* \in \text{dom } L \cap (\overline{\Omega}_2 \setminus \Omega_1)$  such that  $Lu^* = Nu^*$ . That is,  $u^*$  is a solution of (1.1)–(1.3) such that  $u^*(t) \geq 0$  and  $a \leq \|u^*\| \leq b$ . The proof is complete.

□

## REFERENCES

- [1] D. Anderson, R. Avery, J. Henderson, and X. Liu, Existence of positive solutions of a second order right focal boundary value problem, *Comm. Appl. Nonlinear Anal.* **18** (2011), no. 3, 41–52.
- [2] C. Bai and J. Fang, Existence of positive solutions for three-point boundary value problems at resonance, *J. Math. Anal. Appl.* **291** (2004), 538–549.
- [3] C. T. Cremins, A fixed-point index and existence theorems for semilinear equations in cones, *Nonlinear Anal.* **46** (2001), 789–806.
- [4] C. J. Chyan and J. Henderson, A multiplicity result for positive solutions of right focal boundary value problems, *Commun. Appl. Anal.* **5** (2001), no. 4, 485–495.
- [5] J. Davis and J. Henderson, Triple positive solutions for  $(k, n - k)$  conjugate boundary value problems, *Math. Slovaca* **51** (2001), no. 3, 313–320.
- [6] Z. Du, Solvability of functional differential equations with multi-point boundary value problems at resonance, *Comput. Math. Appl.* **55** (2008), no. 11, 2653–2661.
- [7] Z. Du, G. Cai, and W. Ge, A class of third order multi-point boundary value problem, *Taiwanese J. Math.* **9** (2005), no. 1, 81–94.
- [8] Z. Du, X. Lin, and W. Ge, Nonlocal boundary value problem of higher order ordinary differential equations at resonance, *Rocky Mountain J. Math.* **36** (2006), no. 5, 1471–1486.
- [9] P. W. Eloe, J. Henderson and N. Kosmatov, Countable positive solutions of a conjugate boundary value problem, *Comm. Appl. Nonlinear Anal.* **7** (2000), no. 2, 47–55.
- [10] R.E. Gaines, J. Santanilla, A coincidence theorem in convex sets with applications to periodic solutions of ordinary differential equations, *Rocky Mountain J. Math.* **12** (1982), 669–678.
- [11] C. P. Gupta, Solvability of a multi-point boundary value problem at resonance, *Results Math.* **28** (1995), no. 3–4, 270–276.
- [12] J. Graef, J. Henderson and B. Yang, Existence and nonexistence of positive solutions of an  $N$ -th order nonlocal boundary value problem, *Dynamic Systems and Applications.* **5** (2008), 186–191.
- [13] X. Han, Positive solutions for a three-point boundary value problem at resonance, *J. Math. Anal. Appl.* **336** (2007), no. 1, 556–568.
- [14] J. Henderson and R. Luca, Positive solutions for singular systems of higher-order multi-point boundary value problems, *Math. Model. Anal.* **18** (2013), no. 3, 309–324.
- [15] J. Henderson and R. Luca, Positive solutions for systems of second-order integral boundary value problems, *Electron. J. Qual. Theory Differ. Equ.* 2013, no 70, 21 pp.
- [16] J. Henderson and W. Yin, Positive solutions and nonlinear eigenvalue problems for functional differential equations, *Appl. Math. Lett.* **12** (1999), no. 2, 63–68.
- [17] G. Infante and M. Zima, Positive solutions of multi-point boundary value problems at resonance, *Nonlinear Anal.* **69** (2008), 2458–2465.
- [18] W. H. Jiang, Y. P. Guo, and J. Qiu, Solvability of  $2n$ -order  $m$ -point boundary value problem at resonance, *Appl. Math. Mech. (English Ed.)* **28** (2007), no. 9, 1219–1226.
- [19] E. R. Kaufmann, Existence of solutions for a nonlinear second-order equation with periodic boundary conditions at resonance, *Comm. Appl. Anal.* **18** (2014), 163–174.
- [20] E. R. Kaufmann, A Kolmogorov predator-prey system on a time scale, *Dynamic Systems and Applications* **23** (2014), 561–574.
- [21] G. L. Karakostas and P. Ch. Tsamatos, On a nonlocal boundary value problem at resonance, *J. Math. Anal. Appl.* **259** (2001), no. 1, 209–218.



- [22] N. Kosmatov, A multipoint boundary value problem with two critical conditions, *Nonlinear Anal.* **65** (2006), 622–633.
- [23] N. Kosmatov, Multi-point boundary value problems on an unbounded domain at resonance, *Nonlinear Anal.* **68** (2008), 2158–2171.
- [24] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, in NSF-CBMS Regional Conference Series in Mathematics, **40**. Amer. Math. Soc., Providence RI 1979.
- [25] X. Ni and W. Ge, Multi-point boundary-value problems for the  $p$ -Laplacian at resonance, *Electron. J. Differential Equations* **2003** (2003), no. 112, 1–7.
- [26] D. O'Regan and M. Zima, Leggett-Williams norm-type theorems for coincidences, *Arch. Math.* **87** (2006), 233–244.
- [27] W. V. Petryshyn, Using degree theory for densely defined A-proper maps in the solvability of semilinear equations with unbounded and noninvertible linear part, *Nonlinear Anal.* **4** (1980), no. 2, 259–281.
- [28] W. V. Petryshyn, Existence of fixed points of positive  $k$ -set-contractive maps as consequences of suitable boundary conditions, *J. London Math. Soc.* **38** (1988), no. 2, 503–512.
- [29] J. Santanilla, Some coincidence theorems in wedges, cones, and convex sets, *J. Math. Anal. Appl.* **105** (1985), 357–371.
- [30] F. Wang and F. Zhang, Existence of  $n$  positive solutions to second-order multi-point boundary value problem at resonance, *Bull. Korean Math. Soc.* **49** (2012), no. 4, 815–827.
- [31] J. R. L. Webb and M. Zima, Multiple positive solutions of resonant and non-resonant nonlocal boundary value problems, *Nonlinear Anal.* **71** (2009), no. 3-4, 1369–1378.
- [32] H. Zhang and J. Sun, Positive solutions of third-order nonlocal boundary value problems at resonance, *Boundary Value Problems* **2012** 2012:102.