Communications in Applied Analysis **19** (2015), 515–526

EXISTENCE AND UNIQUENESS OF SOLUTIONS TO FRACTIONAL ORDER MULTI-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. We study sufficient conditions for existence and uniqueness of solutions to boundary value problems for fractional order differential equations of the form

$$-{}^cD^q u(t) = f(t, u(t)); \quad t \in J = [0,1], 1 < q \leq 2,$$
$$u(0) = g(u), \quad u(1) - \sum_{i=1}^{m-2} \lambda_i u(\eta_i) = h(u),$$

where $\lambda_i, \eta_i \in (0,1)$ with $\sum_{i=1}^{m-2} \lambda_i \eta_i < 1$, g, $h \in C(J, \mathbb{R})$ are boundary functions and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. We use a fixed point theorem for condensing maps to establish sufficient conditions for existence as well as uniqueness of solutions to the boundary value problem. We provide an example to verify the applicability of our results.

AMS (MOS) Subject Classification. 26A33, 33C45.

1. Introduction

The theory of fractional order differential equations is growing rapidly and a wide range of applications can be found in various scientific and engineering disciplines such as physics, mechanics, chemistry, biology, viscoelasticity, control theory, signal processing, economics, optimization theory etc, we refer to [20, 24, 25, 17, 21]. The theory of existence and uniqueness of solutions to boundary value problems for fractional order differential equations has also attracted some attentions; we refer the readers to [7, 2, 4, 3, 5, 23, 6] and the references therein for the recent development of the theory for boundary value problems. Existence and uniqueness results of solutions or positive solutions to multi-point boundary value problems via classical fixed point theorems such as the Banach contraction principle or the Schauder fixed point theorem are studied in [13, 26, 30, 14, 11, 19, 15, 29].

The use of coincidence degree theory approach to study existence of solutions of fractional order differential equations is quite recent and only few results can be

Received December 15, 2014 1083-2564 $15.00 ©Dynamic Publishers, Inc.

found in the literature dealing with boundary value problems (BVPs) [16, 28, 10, 27]. Here we remark that Johnny Henderson has a significant contribution in the field of fractional order differential equations. We refer to [8, 1] for the case of compact operators, and to [9] for the case of noncompactness.

In [28], Wang et. al. studied existence and uniqueness of solutions to a class of nonlocal Cauchy problems of the form

$$\begin{aligned} {}^cD^q u(t) &= f(t, u(t)); \quad t \in J = [0, T], \\ u(0) + g(u) &= u_0, \end{aligned}$$

where ${}^cD^q$ is the Caputo fractional derivative of order $q \in (0, 1)$, the function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and $u_0 \in \mathbb{R}$. Chen et al [10] studied sufficient conditions for existence results for the following two point boundary value problem

$$\begin{aligned} D^{\alpha}_{0+}\phi_p(D^{\beta}_{0+}u(t)) &= f(t, u(t), D^{\beta}_{0+}u(t)) \\ D^{\beta}_{0+}u(0) &= D^{\beta}_{0+}u(1) = 0, \end{aligned}$$

where D^{α}_{0+} and D^{β}_{0+} are Caputo fractional derivatives, $0 < \alpha$, $\beta \leq 1$, $1 < \alpha + \beta \leq 2$. Tang et al [27] studied the following two point boundary value problem for fractional differential equations

$$\begin{aligned} D^{\alpha}_{0+}\phi_p(D^{\beta}_{0+}u(t)) &= f(t, u(t), D^{\beta}_{0+}u(t)) \\ u(0) = 0, \quad D^{\beta}_{0+}u(0) &= D^{\beta}_{0+}u(1), \end{aligned}$$

where D^{α}_{0+} and D^{β}_{0+} are Caputo fractional derivatives, $0 < \alpha$, $\beta \leq 1$, $1 < \alpha + \beta \leq 2$.

Motivated by the above results, we use coincidence degree theory approach for condensing maps to study sufficient conditions for existence and uniqueness of solutions to some nonlinear multi-point boundary value problem with nonlinear boundary conditions of the form

$$\begin{aligned} -{}^cD^q u(t) &= f(t, u(t)); \quad t \in J, 1 < q \leq 2, \\ u(0) = g(u), \quad u(1) &- \sum_{i=1}^{m-2} \lambda_i u(\eta_i) = h(u), \end{aligned} \tag{1.1}$$

where $\lambda_i, \eta_i \in (0, 1)$ with $\sum\limits_{i=1}^{m-2} \lambda_i \eta_i < 1$, g, $h : C(J, \mathbb{R})$ and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

2. Basic results

We provide some basic definitions and results and refer the reader to [24, 25, 17, 21, 22, 12, 18] for more insight to the theory. The Banach space of all continuous functions from $J \to \mathbb{R}$ with the norm $\|u\| = \sup\{|u(t)| : t \in J\}$ is denoted by $X = C(J, \mathbb{R})$ and $\mathbb{B} \in P(X)$ denotes the family of all its bounded sets.

Definition 2.1. The fractional integral of order $q \in \mathbb{R}_+$ of a function $y \in L^1([a,b],\mathbb{R})$ is defined as

$$I^q_{a+}y(t) = \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}y(s)\,ds. \tag{2.1}$$

Definition 2.2. The Caputo fractional order derivative of a function y on the interval $[a,b]$ is defined by

$${}^cD^q_{a+}y(t) = \frac{1}{\Gamma(n-q)}\int_a^t (t-s)^{n-q-1}y^{(n)}(s)\,ds,$$

where $n = [q]+1$ and $[q]$ represents the integer part of q.

Lemma 2.3. *The fractional differential equation of order $q > 0$*

$${}^cD^q y(t) = 0, \quad n-1 < q \le n,$$

has a solution of the form $y(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$, where $c_i \in R$, $i = 0,1,2,\ldots,n-1$.

Lemma 2.4. *The following result holds for fractional differential equations*

$$I^{q\,c}D^q y(t) = y(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

for arbitrary $c_i \in R$, $i = 0,1,2,\ldots,n-1$.

Definition 2.5. The Kuratowski measure of noncompactness $\alpha : \mathbb{B} \to \mathbb{R}_+$ is defined as

$$\alpha(B) = \inf\{d > 0\},$$

where $B \in \mathbb{B}$ admits a finite cover by sets of diameter $\le d$.

Proposition 2.6. *The Kuratowski measure α satisfies the following properties:*
(i) $\alpha(B) = 0$ iff B is relatively compact.
(ii) α is a seminorm, that is, $\alpha(\lambda B) = |\lambda|\alpha(B), \lambda \in \mathbb{R}$ and $\alpha(B_1 + B_2) \le \alpha(B_1) + \alpha(B_2)$.
(iii) $B_1 \subset B_2$ implies $\alpha(B_1) \le \alpha(B_2); \alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}$.
(iv) $\alpha(conv B) = \alpha(B)$.
(v) $\alpha(\bar{B}) = \alpha(B)$.

Definition 2.7. Let the function $F : \Omega \to X$ be a continuous bounded map, where $\Omega \subset X$. Then F is α-Lipschitz if there exists $K \ge 0$ such that

$$\alpha(F(B)) \le K\alpha(B), \forall \quad B \subset \Omega \quad \text{bounded}.$$

Further, F will be strict α-contraction if $K < 1$.

Definition 2.8. The function F is α-condensing if

$$\alpha(F(B)) < \alpha(B), \forall \quad B \subset \Omega \quad \text{bounded with} \quad \alpha(B) > 0.$$

In other words, $\alpha(F(B)) \ge \alpha(B)$ implies $\alpha(B) = 0$.

Recently, Isaia [18] developed the following results using the degree theory arguments.

Theorem 2.9. *Let $F : X \to X$ be α-condensing and*

$$\Theta = \{x \in X : \exists \quad \lambda \in [0,1] \quad \textit{such that} \quad x = \lambda F x\}.$$

If Θ is a bounded set in X, so there exists $r > 0$ such that $\Theta \subset B_r(0)$, then the degree

$$D(I - \lambda F, B_r(0), 0) = 1, \quad \forall \quad \lambda \in [0,1].$$

Consequently, F has at least one fixed point and the set of the fixed points of F lies in $B_r(0)$.

We recall the following propositions from the book by K. Deimling [12].

Proposition 2.10. *If $F, G : \Omega \to X$ are α-Lipschitz maps with constants K and K' respectively, then $F + G : \Omega \to X$ is α-Lipschitz with constant $K + K'$.*

Proposition 2.11. *If $F : \Omega \to X$ is compact, then F is α-Lipschitz with constant $K = 0$.*

Proposition 2.12. *If $F : \Omega \to X$ is Lipschitz with constant K, then F is α-Lipschitz with the same constant K.*

3. Main Results

In this section, we discuss the existence and uniqueness of solutions to the BVP (1.1). In view of the definition (2.1), we write

$$I^q f(t, u(t)) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))\, ds \tag{3.1}$$

and use the notations $A = 1 - \sum_{i=1}^{m-2} \lambda_i$, $\Delta = 1 - \sum_{i=1}^{m-2} \lambda_i \eta_i > 0$ and

$$\begin{aligned} I^q f(1, u(1)) &= \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s, u(s))\, ds, \\ I^q f(\eta_i, u(\eta_i)) &= \frac{1}{\Gamma(q)} \int_0^{\eta_i} (\eta_i - s)^{q-1} f(s, u(s))\, ds. \end{aligned} \tag{3.2}$$

Lemma 3.1. *If the BVP for the fractional differential equation* (1.1) *has a solution u, then u has the form*

$$\begin{aligned} u(t) = \left(1 - \frac{tA}{\Delta}\right) g(u) + \frac{t}{\Delta} h(u) + \frac{t}{\Delta}\left[I^q f(1, u(1)) - \sum_{i=1}^{m-2} \lambda_i I^q f(\eta_i, u(\eta_i)) \right] \\ - I^q f(t, u(t)) = \left(1 - \frac{tA}{\Delta}\right) g(u) + \frac{t}{\Delta} h(u) + \int_0^1 G(t,s) f(s, u(s))\, ds, \end{aligned} \tag{3.3}$$

where $G(t,s)$ is the Green function and is given by

$$G(t,s)=\begin{cases} s\le t\begin{cases}\frac{t}{\Delta\Gamma(q)}\left[(1-s)^{q-1}-\sum\limits_{i=1}^{m-2}\lambda_i(\eta_i-s)^{q-1}\right]-\frac{(t-s)^{q-1}}{\Gamma(q)},\ s\in[0,\eta_1]\\ \frac{t}{\Delta\Gamma(q)}\left[(1-s)^{q-1}-\sum\limits_{i=j}^{m-2}\lambda_i(\eta_i-s)^{q-1}\right]-\frac{(t-s)^{q-1}}{\Gamma(q)},\ s\in[\eta_{j-1},\eta_j],\\ j=2,3,\dots,m-2,\\ \frac{t}{\Delta\Gamma(q)}(1-s)^{q-1}-\frac{(t-s)^{q-1}}{\Gamma(q)},\ s\in[\eta_{m-2},1]\end{cases}\\ t\le s\begin{cases}\frac{t}{\Delta\Gamma(q)}\left[(1-s)^{q-1}-\sum\limits_{i=1}^{m-2}\lambda_i(\eta_i-s)^{q-1}\right],\ s\in[0,\eta_1]\\ \frac{t}{\Delta\Gamma(q)}\left[(1-s)^{q-1}-\sum\limits_{i=j}^{m-2}\lambda_i(\eta_i-s)^{q-1}\right],\ s\in[\eta_{j-1},\eta_j],\\ j=2,3,\dots,m-2,\\ \frac{t}{\Delta\Gamma(q)}(1-s)^{q-1},\quad s\in[\eta_{m-2},1].\end{cases}\end{cases} \tag{3.4}$$

Proof. Applying I^q on the differential equation in (1.1) and using Lemma (2.4), we obtain

$$u(t)=-I^qf(t,u(t))+c_0+c_1t,\quad c_0,c_1\in\mathbb{R} \tag{3.5}$$

The boundary conditions $u(0)=g(u)$ and $u(1)-\sum\limits_{i=1}^{m-2}\lambda_iu(\eta_i)=h(u)$ yield

$$c_0=g(u),\ c_1=\frac{1}{\Delta}\big[h(u)+I^qf(1,u(1))+Ag(u)-\sum_{j=i}^{m-2}\lambda_iI^qf(\eta_i,u(\eta_i))\big].$$

Hence, it follows that

$$\begin{aligned} u(t)&=-I^qf(t,u(t))+g(u)+\frac{t}{\Delta}\Bigg[h(u)+I^qf(1,u(1))+Ag(u) \\ &\quad-\sum_{i=1}^{m-2}\lambda_iI^qf(\eta_i,u(\eta_i))\Bigg]=-I^qf(t,u(t))+\left(1+\frac{tA}{\Delta}\right)g(u)+\frac{t}{\Delta}h(u)\\ &\quad+\frac{t}{\Delta}\left[I^qf(1,u(1))-\sum_{i=1}^{m-2}\lambda_iI^qf(\eta_i,u(\eta_i))\right]=\left(1+\frac{tA}{\Delta}\right)g(u)+\frac{t}{\Delta}h(u)+U_1(t),\end{aligned} \tag{3.6}$$

where

$$\begin{aligned} U_1(t)&=\frac{t}{\Delta}\left[I^qf(1,u(1))-\sum_{i=1}^{m-2}\lambda_iI^qf(\eta_i,u(\eta_i))\right]\\ &=\frac{t}{\Delta\Gamma(q)}\left[\int_0^1(1-s)^{q-1}f(s,u(s))ds-\sum_{i=1}^{m-2}\lambda_i\int_0^{\eta_i}(\eta_i-s)^{q-1}f(s,u(s))ds\right]\\ &\quad-\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,u(s))ds=\int_0^1G(t,s)f(s,u(s))ds.\end{aligned}$$

$\square$

Define the following operators F, G, $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$(Fu)(t) = g(u) - \frac{tA}{\Delta}g(u) + \frac{t}{\Delta}h(u) = \left(1 - \frac{tA}{\Delta}\right) g(u) + \frac{t}{\Delta}h(u),$$

$$(Gu)(t) = \frac{t}{\Delta\Gamma(q)}\left[\int_0^1 (1-s)^{q-1} f(s,u(s))\,ds - \sum_{j=i}^{m-2}\lambda_i \int_0^{\eta_i} (\eta_i - s)^{q-1} f(s,u(s))\,ds\right]$$
$$- \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,u(s))\,ds, \text{ and } \mathbb{T}u = Fu + Gu$$

The operator $\mathbb{T}$ is well defined as g, h, f are continuous functions. We can write (3.6) as an operator equation

$$u = \mathbb{T}u = Fu + Gu, \tag{3.7}$$

and solutions of the BVP (1.1) mean solutions of the operator equation, that is, fixed points of T.

Assume that the following hold.

(A_1) There exist constants K_g, C_g, M_g, $q_1 \in [0,1)$ such that

$$|g(u) - g(v)| \le K_g\|u - v\|,\ |g(u)| \le C_g\|u\|^{q_1} + M_g \text{ for } u, v \in C(J,\mathbb{R}).$$

(A_2) There exist constants K_h, C_h, q_1, $M_h \in [0,1)$ such that

$$|h(u) - h(v)| \le K_h\|u - v\|,\ |h(u)| \le C_h\|u\|^{q_1} + M_h, \quad \text{for } u,\ v \in C(J,\mathbb{R}).$$

(A_3) There exist constants C_f, M_f, $q_2 \in [0,1)$ such that

$$|f(t,u(s)| \le C_f\|u\|^{q_2} + M_f \text{ for } (t,u) \in J \times C(J,\mathbb{R}).$$

Lemma 3.2. *Under the assumptions* (A_1), (A_2), *the operator* $F : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ *satisfies the Lipschitz conditions with constant* K *and the growth condition*

$$\|Fu\| \le C_g\|u\|^{q_1} + C_h\|u\|^{q_1}, \quad \textit{for every} \quad u \in C(J,\mathbb{R}). \tag{3.8}$$

Proof. Using the assumptions (A_1) and (A_2), we obtain

$$|Fu(t) - Fv(t)| = \left|\left(1 - \frac{tA}{\Delta}\right)(g(u) - g(v)) + \frac{t}{\Delta}(h(u) - h(v))\right|$$
$$\le K_g\|u - v\| + K_h\|u - v\| = K\|u - v\|, \quad \text{where } K = K_g + K_h.$$

By Proposition 2.12, F is also α-Lipschitz with constant K. For a growth condition, we get

$$\|Fu\| \le C_g\|u\|^{q_1} + C_h\|u\|^{q_1}.$$

□

Lemma 3.3. *The operator $G : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is continuous and under the assumption (A_3) satisfies the growth condition*

$$\|Gu\| \le \frac{2(C_f\|u\|^{q_2} + M_f)}{\Delta\Gamma(q+1)}, \quad u \in C(J,\mathbb{R}). \tag{3.9}$$

Proof. Let $\{u_n\}$ be a sequence of a bounded set $B_k = \{\|u\| \le r\colon\ u \in C(J,\mathbb{R})\}$ such that $u_n \to u$ in B_k. Since f is continuous and $u_n \to u$ as $n \to \infty$, it follows that $f(s,u_n(s)) \to f(s,u(s))$ as $n \to \infty$. Now consider

$$\begin{aligned}
|(Gu_n)(t) - (Gu)(t)| \le{}& \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,u_n(s)) - f(s,u(s))|ds \\
&+ \frac{t}{\Delta\Gamma(q)}\sum_{i=1}^{m-2}\lambda_i\int_0^{\eta_i}(\eta_i - s)^{q-1}|f(s,u_n(s)) - f(s,u(s))|ds \\
&+ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,u_n(s)) - f(s,u(s))|ds,
\end{aligned}$$

which in view of the Lebesgue Dominated Convergence theorem implies that

$$\|(Gu_n)(t) - (Gu)(t)\| \to 0 \quad \text{as} \quad n \to \infty.$$

For growth conditions on G, using the triangle inequality and (A_3), we obtain

$$\begin{aligned}
|(Gu)(t)| ={}& |\frac{t}{\Delta\Gamma(q)}\left(\int_0^1 (1-s)^{q-1}f(s,u(s))\,ds - \sum_{i=1}^{m-2}\lambda_i\int_0^{\eta_i}(\eta_i - s)^{q-1}f(s,u(s))\,ds\right) \\
&- \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s,u(s))\,ds| \\
\le{}& |\frac{t}{\Delta\Gamma(q)}\left(\int_0^1 (1-s)^{q-1}f(s,u(s))\,ds - \sum_{i=1}^{m-2}\lambda_i\int_0^{\eta_i}(\eta_i - s)^{q-1}f(s,u(s))\,ds\right)| \\
&+ |\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s,u(s))\,ds| \le \frac{1}{\Delta\Gamma(q)}\int_0^1 (1-s)^{q-1}|f(s,u(s))|\,ds \\
&+ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,u(s))|\,ds \\
\le{}& \frac{1}{\Delta\Gamma(q+1)}(C_f|u(t)|^{q_2} + M_f) + \frac{1}{\Gamma(q+1)}(C_f|u(t)|^{q_2} + M_f).
\end{aligned}$$

Using the choice of $\Delta < 1$, we obtain

$$\|(Gu)(t)\| \le \frac{2}{\Delta\Gamma(q+1)}(C_f\|u\|^{q_2} + M_f).$$

□

Lemma 3.4. *The operator $G : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is compact. Consequently, G is α-Lipschitz with zero constant.*

Proof. Take a bounded set $\mathcal{D} \subset B_k \subseteq C(J, \mathbb{R})$ and a sequence $\{u_n\}$ in $\mathcal{D}$, then using (3.9), we have

$$\|Gu_n\| \leq \frac{2(C_f||u||^{q_2} + M_f)}{\Delta\Gamma(q+1)},$$

which implies that $G(\mathcal{D})$ is bounded. Now, for $0 \leq t_1 < t_2 \leq 1$, consider

$$|Gu_n(t_1) - Gu_n(t_2)| = |\frac{1}{\Gamma(q)} \int_0^{t_1} ((t_2 - s)^{q-1} - (t_1 - s)^{q-1}) f(s, u_n(s)) ds$$

$$+ \frac{(t_1 - t_2)}{\Delta\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s, u_n(s)) ds - \frac{(t_1 - t_2)}{\Delta\Gamma(q)} \sum_{i=1}^{m-2} \lambda_i \int_0^{\eta_i} (\eta_i - s)^{q-1} f(s, u_n(s)) ds|$$

$$\leq \frac{|t_2 - t_1|}{\Delta\Gamma(q+1)} (C_f||u_n||^{q_2} + M_f) + \frac{|t_2 - t_1|}{\Delta\Gamma(q+1)} (C_f||u_n||^{q_2} + M_f)$$

$$+ \frac{1}{\Gamma(q+1)} (t_2^q - t_1^q)(C_f||u_n||^{q_2} + M_f)$$

where we have used the assumption $\sum_{i=1}^{m-2} \lambda_i \eta_i^q < 1$ in the second term. Hence it follows that

$$|(Gu_n)(t_1) - (Gu_n)(t_2)| \leq \frac{3}{\Delta\Gamma(q+1)} (C_f||u_n||^{q_2} + M_f) \left[(t_2^q - t_1^q) + (t_2 - t_1) \right].$$

The right side of the above inequality tends to zero as $t_2 \to t_1$. Hence, $\{Gu_n\}$ is equicontinuous. Therefore, $G(\mathcal{D})$ is relatively compact in $C(J, \mathbb{R})$ by Arzela-Ascoli theorem. Furthermore, by Proposition 2.11, G is α-Lipschitz with constant zero. ☐

Theorem 3.5. *Under the assumptions (A1)–(A3), the BVP* (1.1) *has at least one solution $u \in C(J, \mathbb{R})$. Moreover, the set of solutions of* (1.1) *is bounded in $C(J, \mathbb{R})$.*

Proof. From Proposition 2.10, the operator $\mathbb{T}$ is a strict α-contraction with constant K. Now set

$$S_0 = \{u \in C(J, \mathbb{R}) : \exists \quad \lambda \in [0,1] \text{ such that } u = \lambda \mathbb{T} u\}.$$

We need to prove that S_0 is bounded in $C(J, \mathbb{R})$. For this, consider

$$||u|| = ||\lambda \mathbb{T} u|| = \lambda ||Tu|| \leq \lambda (\|Fu\| + \|Gu\|)$$

which in view of (3.8) and (3.9) together with $q_1 < 1$, $q_2 < 1$, implies that S_0 is bounded in $C(J, \mathbb{R})$. Therefore, by Theorem 2.9, $\mathbb{T}$ has at least one fixed point and the set of fixed points is bounded in $C(J, \mathbb{R})$. ☐

Assume that the following holds:

(A_4) There exists a constant $L_f > 0$ such that

$$|f(t,u) - f(t,v)| \leq L_f |u - v|, \quad \text{for each} \quad t \in J, \quad \text{and} \quad \forall\, u, v \in \mathbb{R}.$$

Theorem 3.6. *In addition to the assumption* (A_1)–(A_4)*, assume that there exists a constant* $M > 0$ *such that*

$$M = \left(K_g + K_h + \frac{3L_f}{\Delta\Gamma(q+1)} \right) < 1. \tag{3.10}$$

Then the BVP (1.1) *has a unique solution.*

Proof. We use the Banach contraction principle. For u, $v \in C(J, \mathbb{R})$, using the assumptions (A_1)–(A_4), we have

$$\begin{aligned}
|(Tu)(t) - (Tv)(t)| &= \Big| \left(1 - \frac{tA}{\Delta}\right)(g(u) - g(v)) + \frac{t}{\Delta}(h(u) - h(v)) \\
&+ \frac{t}{\Delta\Gamma(q)} \int_0^1 (1-s)^{q-1}(f(s,u(s)) - f(s,v(s)))ds \\
&- \frac{t \sum_{i=1}^{m-2} \lambda_i}{\Delta\Gamma(q)} \int_0^{\eta_i} (\eta_i - s)^{q-1}(f(s,u(s)) - f(s,v(s)))ds \\
&+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}(f(s,u(s)) - f(s,v(s)))ds \Big| \\
&\leq \left(1 + \frac{A}{\Delta}\right)|g(u) - g(v)| + \frac{1}{\Delta}\big|h(u) - h(v)| + \frac{1}{\Delta\Gamma(q+1)} L_f|u(t)) - v(t)| \\
&+ \frac{1}{\Delta\Gamma(q+1)} L_f|u(t)) - v(t)| + \frac{1}{\Gamma(q+1)} L_f|u(t)) - v(t)|.
\end{aligned}$$

Hence, it follows that

$$|(Tu)(t) - (Tv)(t)| \leq K_g\|u - v\| + K_h\|u - v\| + \frac{3L_f}{\Delta\Gamma(q+1)}\|u - v\|,$$

which implies that

$$|(Tu)(t) - (Tv)(t)| \leq \left(K_g + K_h + \frac{3L_f}{\Delta\Gamma(q+1)}\right)\|u - v\| = M\|u - v\|.$$

Hence, the BVP (1.1) has a unique solution. □

4. **Example**

Example 4.1. Consider the following multi-point BVP

$$\begin{aligned}
{}^cD^{\frac{2}{3}}u(t) &= \frac{|u(t)|^{\frac{1}{2}}}{(1+12e^{2t})(1+|u(t)|^{\frac{1}{2}})} = y(t, u(t)), \quad t \in [0,1], \\
u(0) &= g(u) \\
u(1) &= \sum_{i=1}^{m-2} \lambda_i u(\eta_i) + h(u), \sum_{i=1}^{m-2} \lambda_i u(\eta_i) = \frac{1}{5}.
\end{aligned} \tag{4.1}$$

Take $q = \frac{2}{3}$, $q_1 = 1$, $q_2 = \lambda = \frac{1}{2}$, $r = 2 \in (1, 3)$, $\lambda = \frac{1}{2} \in [0, \frac{1}{2}]$, $L_f = C_f = \frac{1}{10}$, $m_f = 0$, $K_g = K_h = \frac{1}{4}$; the assumptions (A_1)–(A_4) are satisfied. The solution of the BVP (4.1) is given by

$$u(t) = g(u)[1 - t] + t\left(\frac{2}{5} + h(u)\right) - I^{\frac{2}{3}} y(t, u(t)) + I^{\frac{2}{3}} y(1, u(1)).$$

Here

$$Fu(t) = (1 - t)g(u) + t\left(\frac{2}{5} + h(u)\right) \text{ and } Gu(t) = -I^{\frac{2}{3}} y(t, u(t)) + I^{\frac{2}{3}} y(1, u(1)).$$

Since F, G are continuous and bounded so also $T = F + G$ is continuous and bounded. Further

$$||Fu - Fv|| \leq \frac{1}{2}||u - v||,$$

that is F is α Lipschitz and G is α Lipschitz with zero constant implies that T is strict-α- contraction with constant $\frac{1}{2}$. By Theorem 2.9 the BVP has a solution u in $C[J, R]$.

REFERENCES

[1] S. Abbas, M. Benchohra and J. Henderson, Global asymptotic stability of solutions of nonlinear quadratic Volterra integral equations of fractional order, *Comm. Appl. Nonlinear Anal.*, **19** (2012), 79–89.

[2] R. P. Agarwal, M. Belmekki, and M. Benchohra, A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative, *Adv. Difference Equ.,* (2009) Article ID 981728, 47 pp.

[3] R. P. Agarwal, M. Benchohra, and S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta Appl. Math.,* **109** (2010), 973–1033.

[4] B. Ahmad and J. J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, *Comput. Math. Appl.,* **58** (2009), 1838–1843.

[5] B. Ahmad and J. J. Nieto, Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory, *Topol. Methods Nonlinear Anal.,* **35** (2010), 295–304.

[6] K. Balachandran, S. Kiruthika, and J. J. Trujillo, Existence results for fractional impulsive integrodifferential equations in Banach spaces, *Commun. Nonlinear Sci. Numer. Simul.,* **16** (2011), 1970–1977.

[7] M. Benchohra, J. R. Graef, and S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations, *Appl. Anal.,* **87** (2008), 851–863.

[8] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahab, Existence Results for Fractional Functional Differential Inclusions with Infinite Delay and Applications to Control Theory, *Fract. Calc. Appl. Anal.*, **11** (2008), 35–55.

[9] M. Benchohra, J. Henderson and D. Seba, Measure of noncompactness and fractional differential equations in Banach spaces, *Commun. Appl. Anal.* **12** (2008), 419–428.

[10] T. Chen W. Liu, Z. Hu, A boundary value problem for fractional differential equation with p-Laplacian operator at resonance, *Nonlinear Anal.,* **75** (2012), 3210–3217.

[11] Z. Cui, P. Yu and Z. Mao, Existence of solutions for nonlocal boundary value problems of nonlinear fractional differential equations, *Adv. Dyn. Sys. Appl.,* **7** (2012), 31–40.

[12] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.

[13] M. El-Shahed and J. J. Nieto, Nontrivial solutions for a nonlinear multi-point boundary value problem of fractional order, *Comput. Math. Appl.,* **59** (2010), 3438–3443.

[14] M. El-Shahed and W. M. Shammakh, Existence of positive solutions for m-point boundary value problem for nonlinear fractional differential equation, *Abstr. Appl. Anal.,* **2011** Article ID 986575 (2011) 20 pp.

[15] A. M. A. EL-Sayed and E.O.Bin-Taher, Positive solutions for a nonlocal multi-point boundary-value problem of fractional and second order, *Electron. J. Differential Equations,* **64** (2013), 1–8.

[16] X. Han and T. Wang, The existence of solutions for a nonlinear fractional multi-point boundary value problem at resonance, *Int. J. Differ. Equ.,* Article ID 401803 (2011) 14 pp.

[17] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[18] F. Isaia, On a nonlinear integral equation without compactness. *Acta. Math. Univ. Comenian.,* **75** (2006), 233–240.

[19] R. A. Khan, Three-point boundary value problems for higher order nonlinear fractional differential equations, *J. Appl. Math. Inform.* **31** (2013), 221–228.

[20] A. A. Kilbas, O.I. Marichev, and S. G. Samko, *Fractional Integrals and Derivatives (Theory and Applications)*, Gordon and Breach, Switzerland, 1993.

[21] A. A. Kilbas, H. M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland athematics Studies, Elsevier, Amsterdam, 2006.

[22] V. Lakshmikantham, S. Leela and J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, UK, 2009.

[23] C. F. Li, X. N. Luo, and Y. Zhou, Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations, *Comput. Math. Appl.,* **59** (2010), 1363–1375.

[24] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations,*, Wiley, New York, 1993.

[25] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*, Academic Press, New York, 1999.

[26] M. Rehman and R. A. Khan, Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations, *Appl. Math. Lett.,* **23** (2010), 1038–1044.

[27] X. Tang, C. Yan, and Q. Liu, Existence of solutions of two-point boundary value problems for fractional p-laplace differential equations at resonance, *J. Appl. Math. Comput.,* **41** (2013), 119–131.

[28] J. Wang , Y. Zhou and W. Wei, Study in fractional differential equations by means of topological degree methods, *Numer. Funct. Anal. Optim.,* **33** (2012), 216–238.

[29] A. Yang and W. Ge, Positive solutions of multi-point boundary value problems of nonlinear fractional differential equation at resonance, *J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.*, **16** (2009), 213–225.

[30] W. Zhong and W. Lin, Nonlocal and multiple-point boundary value problem for fractional differential equations, *Comput. Math. Appl.,* **59** (2010), 1345–1351.