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A FOURTH-ORDER SEMIPOSITONE BOUNDARY VALUE PROBLEM

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ABSTRACT. We apply Krasnosel'skiĭ's fixed point theorem [6] to study the semipositone eigenvalue problem

$$u^{(4)}(t) + \omega^2 u''(t) = \lambda f(t, u(t)), \quad 0 < t < 1,$$
$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

We show that there exist at least two positive solutions for a sufficiently small value of $\lambda > 0$.

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1. INTRODUCTION

In this paper, we are interested in the fourth order nonlinear boundary-value problem

$$u^{(4)}(t) + \omega^2 u''(t) = \lambda f(t, u(t)), \quad 0 < t < 1, \tag{1.1}$$
$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.2}$$

which serves as a nonlinear model describing deformations of elastic beams with axial force effects.

Due to numerous applications [10], solvability of fourth order both local and nonlocal boundary value problems has been discussed in many papers. Various methods were applied in [5, 1, 3, 4, 11] to obtain the existence of a unique or multiple solutions of fourth-order boundary value problems including the result for semipositone problems [9, 12, 8, 7, 2].

In the next section we present the properties of Green's function of the homogeneous analogue of (1.1) with (1.2), and state Krasnosel'skiĭ's fixed point theorem [6], which will be used to show the existence of at least two positive solutions. The main result is obtained in Section 3.

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2. GREEN'S FUNCTION

First, we state Green's functions of

$$L_1u(t) = -u''(t) - \omega^2 u(t) = 0, \quad t \in (0,1),$$

with $0 < \omega < \pi$, and

$$L_2u(t) = -u''(t) = 0, \quad t \in (0,1),$$

both satisfying $u(0) = u(1) = 0$. These functions are well-known and are given, respectively, by

$$\mathcal{G}(t,s) = \frac{1}{\omega \sin\omega} \begin{cases} \sin\omega s \sin\omega(1-t), & 0 \le s \le t \le 1, \\ \sin\omega t \sin\omega(1-s), & 0 \le t \le s \le 1, \end{cases}$$

and

$$\mathcal{H}(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

In particular,

$$p(t)H_0(s) \le \mathcal{H}(t,s) \le H_0(s), \quad H_0(s) = s(1-s), \quad p(t) = \min\{t, 1-t\}. \tag{2.1}$$

Using the Green functions $\mathcal{G}$ and $\mathcal{H}$, we can see that Green's function of

$$L_1L_2u(t) = u^{(4)}(t) + \omega^2 u''(t) = 0$$

satisfying (1.2) is

$$\begin{aligned} G(t,s) &= \int_0^1 \mathcal{H}(t,\tau)\mathcal{G}(\tau,s)\,d\tau \\ &= \frac{1}{\omega^3 \sin\omega} \begin{cases} \sin\omega s \sin\omega(1-t) - s(1-t)\omega\sin\omega, & 0 \le s \le t \le 1, \\ \sin\omega t \sin\omega(1-s) - t(1-s)\omega\sin\omega, & 0 \le t \le s \le 1. \end{cases} \end{aligned} \tag{2.2}$$

It is clear that $\mathcal{G}(t,s)$ and $G(t,s)$ are nonnegative valued in $[0,1]\times[0,1]$. As a result, $G(t,s) \ge 0$ for $(t,s) \in [0,1]\times[0,1]$. The next two lemmas concerning $G(t,s)$ are useful whenever one would like to apply a cone-theoretic result such as Theorem 2.5. The first lemma can be found in [11]. The second lemma is similar to Lemma 2.2 in [11], so we omit the proof.

Lemma 2.1. *The Green function $\mathcal{G}(t,s)$, $(t,s) \in [0,1]\times[0,1]$, satisfies*

$$\mathcal{G}_i(s) \ge \mathcal{G}(t,s) \ge q_i(t)\mathcal{G}_i(s),$$

where

$$q_1(t) = \frac{1}{\sin\omega}\min\{\sin\omega t, \sin\omega(1-t)\}, \quad \mathcal{G}_1(s) = \mathcal{G}(s,s),$$

for $0 < \omega \le \pi/2$, and

$$q_2(t) = \min\{\sin\omega t, \sin\omega(1-t)\},$$

$$\mathcal{G}_2(s) = \frac{1}{\sin\omega}\begin{cases} \sin\omega s, & 0 \le s \le 1 - \frac{\pi}{2\omega}, \\ \sin\omega s \sin\omega(1-s), & 1 - \frac{\pi}{2\omega} < s < \frac{\pi}{2\omega}, \\ \sin\omega(1-s), & \frac{\pi}{2\omega} \le s < 1, \end{cases}$$

for $\pi/2 < \omega < \pi$.

Since $p(t) \le q_i(t)$, we prefer to use $p(t)$ to define the cone, which is done with the help of the next lemma.

Lemma 2.2. *The Green function* $G(t,s)$ *satisfies*

$$H(s) \ge G(t,s) \ge p(t)H(s), \quad (t,s) \in [0,1] \times [0,1],$$

where

$$H(s) = \int_0^1 H_0(\tau)\mathcal{G}(\tau,s)\,d\tau.$$

In the Banach space $X = C[0,1]$ with the max-norm, we define a cone by

$$\mathcal{C} = \{v \in X : v(t) \ge p(t)\|v\|,\ t \in [0,1]\}.$$

In particular, if $0 < \alpha < 1/2$,

$$u(t) \ge \gamma\|u\|, \quad t \in [\alpha, 1-\alpha], \tag{2.3}$$

where $\gamma = \min_{t\in[\alpha,1-\alpha]} p(t) = \alpha$.

Lemma 2.3. *If* $g_0 \in C[0,1]$, $g_0(t) \ge 0$ *in* $[0,1]$, $g_0(t_0) > 0$ *for some* $t_0 \in [0,1]$, *then there exists* $\mu > 0$ *such that the inequality*

$$p(t) \ge \mu u_0(t), \quad t \in [0,1], \tag{2.4}$$

holds where $u_0(t) = \int_0^1 G(t,s)g_0(s)ds$.

Proof. Consider first the case $0 < \omega \le \pi/2$. We have

$$\begin{aligned} u_0(t) &= \int_0^1 G(t,s)g_0(s)\,ds \\ &= \frac{1}{\omega^3\sin\omega}\Bigg(\int_0^t (\sin\omega s\sin\omega(1-t) - s(1-t)\omega\sin\omega)g_0(s)\,ds \\ &\quad + \int_t^1 (\sin\omega t\sin\omega(1-s) - t(1-s)\omega\sin\omega)g_0(s)\,ds)\Bigg) \\ &\le \frac{1}{\omega^3\sin\omega}\Bigg(\int_0^t (\sin\omega s\sin\omega(1-t)g_0(s)\,ds + \int_t^1 \sin\omega t\sin\omega(1-s)g_0(s)\,ds)\Bigg) \qquad (2.5) \\ &\le \frac{1}{\omega^3\sin\omega}\sin\omega t\sin\omega(1-t)\int_0^1 g_0(s)\,ds \end{aligned}$$

$$\begin{aligned}&\le \frac{1}{\omega^3 \sin\omega}\,\omega t\,\sin\omega \int_0^1 g_0(s)\,ds\\ &= \frac{\|g_0\|_1}{\omega^2} t\end{aligned}$$

and at the same time

$$u_0(t) \le \frac{\|g_0\|_1}{\omega^2}(1-t).$$

Thus,

$$p(t) \ge \frac{\omega^2}{\|g_0\|_1} u_0(t).$$

If

$$\mu \le \mu_1 = \frac{\omega^2}{\|g_0\|_1}, \tag{2.6}$$

then the inequality (2.4) is fulfilled.

For $\pi/2 < \omega < \pi$, we note that (2.5) still applies and obtain, for $t \in [0, 1/2]$,

$$\begin{aligned}u_0(t) &\le \frac{1}{\omega^3 \sin\omega}\left(\sin\omega(1-t)\int_0^t (\sin\omega s g_0(s)\,ds + \sin\omega t\int_t^1 \sin\omega(1-s)g_0(s)\,ds)\right)\\ &\le \frac{1}{\omega^3 \sin\omega}\left(\int_0^t (\sin\omega s g_0(s)\,ds + \sin\omega t\int_t^1 g_0(s)\,ds)\right)\\ &\le \frac{1}{\omega^3 \sin\omega}\sin\omega t\|g_0\|_1\\ &\le \frac{\|g_0\|_1}{\omega^2 \sin\omega} t\\ &= \frac{\|g_0\|_1}{\omega^2 \sin\omega} p(t).\end{aligned}$$

One can easily arrive at the same inequality for $t \in [1/2, 1]$. Again, the inequality (2.4) holds provided

$$\mu \le \mu_2 = \frac{\omega^2 \sin\omega}{\|g_0\|_1}. \tag{2.7}$$

□

We will also need the constants

$$\begin{aligned}D &= \max_{t\in[0,1]} \int_0^1 G(t,s)p(s)\,ds\\ &= \frac{1}{\omega^3 \sin\omega}\left(\sin\frac{\omega}{2}\left(\frac{2}{\omega^2}\sin\frac{\omega}{2} - \frac{1}{\omega}\cos\frac{\omega}{2}\right) - \frac{1}{24}\omega\sin\omega\right),\end{aligned} \tag{2.8}$$

$$\begin{aligned}L &= \max_{t\in[0,1]} \int_0^1 G(t,s)\,ds\\ &= \frac{1}{8\omega^4 \sin\omega}\left(8(2\sin\frac{\omega}{2} - \sin\omega) - \omega^2 \sin\omega\right),\end{aligned} \tag{2.9}$$

and, for $0 < \alpha < 1/2$,

$$C = \max_{t\in[0,1]} \int_{\alpha}^{1-\alpha} G(t,s)\,ds$$
$$= \frac{1}{8\omega^4 \sin\omega}\left(8(2\cos\omega\alpha \sin\frac{\omega}{2} - \sin\omega) - \omega^2 \sin\omega(1-4\alpha^2)\right). \tag{2.10}$$

Define

$$f_p(t,z) = \begin{cases} f(t,z) + g_0(t), & (t,z) \in [0,1]\times[0,\infty), \\ f(t,0) + g_0(t), & (t,z) \in [0,1]\times(-\infty,0), \end{cases}$$

and consider the equation

$$v^{(4)}(t) + \omega^2 v''(t) = \lambda f_p(t, v(t) - \lambda u_0(t)), \quad t \in (0,1), \tag{2.11}$$

under the boundary conditions (1.2).

Lemma 2.4. *The function u is a positive solution of the boundary value problem* (1.1), (1.2) *if and only if the function $v = u + \lambda u_0$ is a solution of the boundary value problem* (2.11), (1.2) *satisfying $v(t) \ge \lambda u_0(t)$ in* $[0,1]$.

Suppose that the function f in (1.1) satisfies

(H_1) $f \in C([0,1]\times \mathbf{R}_+, \mathbf{R})$;

(H_2) there exists a function $g_0 \in C[0,1]$ such that $g_0(t) \ge 0$ in $[0,1]$, $g(t_0) > 0$ for some $t_0 \in [0,1]$ and $f(t,z) + g_0(t) \ge 0$ in $[0,1]\times\mathbf{R}_+$;

In the Banach space $X = C[0,1]$ endowed with usual max-norm, we consider the operator

$$Tv(t) = \lambda \int_0^1 G(t,s) f_p(s, v(s) - \lambda u_0(s))\,ds, \tag{2.12}$$

where $G(t,s)$ is given by (2.2). By (H_1), $T : X \to X$ is completely continuous.

Obviously, a fixed point of $T : \mathcal{C} \to \mathcal{C}$ is a positive solution of (2.11), (1.2). The existence of the former will be shown using Krasnosel'skiĭ's fixed point theorem:

Theorem 2.5. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{C} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1, Ω_2 are open with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$T\colon \mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{C}$$

be a completely continuous operator such that either

(i) $\|Tu\| \le \|u\|$, $u \in \mathcal{C}\cap\partial\Omega_1$, *and* $\|Tu\| \ge \|u\|$, $u \in \mathcal{C}\cap\partial\Omega_2$, *or*

(ii) $\|Tu\| \ge \|u\|$, $u \in \mathcal{C}\cap\partial\Omega_1$, *and* $\|Tu\| \le \|u\|$, $u \in \mathcal{C}\cap\partial\Omega_2$.

Then, T has a fixed point in $\mathcal{C}\cap(\overline{\Omega}_2 \setminus \Omega_1)$.

Subsequently, (1.1), (1.2) has a positive solution provided the inequality of Lemma 2.4 holds.

3. POSITIVE SOLUTIONS

We present our main result for $0 < \omega < \pi$ since the only difference between the cases $0 < \omega \le \pi/2$ and $\pi/2 < \omega < \pi$ is that between the constants μ_1 and μ_2 in Lemma 2.3. The presence of the parameter $\lambda > 0$ provides an additional control on the growth of the right side. We will need the following assumptions:

(M_1) there exists an interval $[\alpha, 1-\alpha] \subset (0,1)$ such that

$$\lim_{u\to\infty} \frac{f(t,u)}{u} = \infty,$$

uniformly in $[\alpha, 1-\alpha]$.

(M_2) $f(t,0) > 0$, $t \in [0,1]$.

Our next result is a multiplicity criterion. We introduce

$$\phi(r) = \max\{f(t, z - u_0(t)) + g_0(t) : t \in [0,1],\, z \in [0,r]\} \tag{3.1}$$

Theorem 3.1. *Assume that* (H_1), (H_2), (M_1), (M_2) *hold. Then, the boundary value problem* (1.1), (1.2) *has at least two positive solutions provided* $\lambda > 0$ *is small enough.*

Proof. We will construct open nonempty subsets $\Omega_i = \{v \in \mathcal{C} : \|v\| < R_i\}$, $i = 1, ..., 4$.

Let the $R_1 > 0$. Then, using (3.1),

$$\|Tv\| = \max_{t\in[0,1]} \lambda \int_0^1 G(t,s) f_p(s, v(s) - \lambda u_0(s))\, ds \le \lambda L\phi(R_1) \le R_1$$

for all $v \in \mathcal{C} \cap \partial\Omega_1$, provided

$$\lambda \le \frac{R_1}{L\phi(R_1)}. \tag{3.2}$$

Let $v \in \mathcal{C} \cap \partial\Omega_2$, where $R_2 > R_1$. We choose $\mu = \mu_i$ according to Lemma 2.3. Note that the equation in (M_2) holds with f_p in place of f. Thus, given $A > 0$ satisfying

$$\frac{1}{2}\lambda C\gamma A \ge 1, \tag{3.3}$$

where C is given by by (2.9), there exists $h \ge \frac{\gamma}{2} R_2$ such that $f_p(t,z) > Az$ for all $z \ge h$ and $t \in [\alpha, 1-\alpha]$. For every λ in (3.2), there exists a constant $A > 0$ such that (3.3) is satisfied. Since $p(s) \ge \mu u_0(s)$ in $[0,1]$, for all $s \in [\alpha, 1-\alpha]$, we have

$$\lambda u_0(s) \le \frac{\lambda}{\mu} p(s) \le \frac{\lambda}{\mu R_2} v(s).$$

So,

$$v(s) - \lambda u_0(s) \ge \left(1 - \frac{\lambda}{\mu R_2}\right) v(s) \ge \left(1 - \frac{\lambda}{\mu R_2}\right)\gamma R_2 \ge \frac{1}{2}\gamma R_2$$

provided

$$\lambda \le \frac{\mu R_2}{2}. \tag{3.4}$$

Hence,

$$f_p(s, v(s) - u_0(s)) \geq A(v(s) - \lambda u_0(s)) \geq \frac{\gamma A}{2} R_2, \quad s \in [\alpha, 1-\alpha].$$

Then, by (3.3),

$$\begin{aligned}\|Tv\| = \max_{t\in[0,1]} \lambda \int_0^1 G(t,s) f_p(s, v(s) - \lambda u_0(s))\, ds &\geq \lambda \max_{t\in[0,1]} \int_\alpha^{1-\alpha} G(t,s)\, ds\, \frac{\gamma A}{2} R_2 \\ &= \lambda \max_{t\in[0,1]} \int_\alpha^{1-\alpha} G(t,s)\, ds\, \frac{\gamma}{2} A R_2 \\ &= \lambda C \frac{\gamma}{2} A R_2 \\ &\geq R_2.\end{aligned}$$

That is, $\|Tv\| \geq \|v\|$ for all $v \in \mathcal{C} \cap \partial\Omega_2$. By Theorem 2.5, we have a solution v_1 such that $R_1 \leq \|v_1\| \leq R_2$ for every

$$0 < \lambda \leq \lambda_0 = \min\left\{ \frac{R_1}{L\phi(R_1)}, \frac{\mu R_2}{2} \right\}.$$

In order to make use of the assumption (M_2), we note that there exist $a, b > 0$ such that $f(t,z) \geq b$ for all $t \in [0,1]$ and $z \in [0,a]$ and introduce a "truncation" of f given by

$$f_t(t,z) = \begin{cases} f(t,z), & (t,z) \in [0,1] \times [0,a]), \\ f(t,a), & (t,z) \in [0,1] \times (a,\infty). \end{cases}$$

Consider now,

$$v^{(4)}(t) + \omega^2 v''(t) = \lambda f_t(t, v(t)), \quad 0 < t < 1,$$

subject to (1.2). The operator, whose fixed point will be shown to be (a second) solution of (1.1), (1.2), is

$$Tv(s) = \lambda \int_0^1 G(t,s) f_t(s, v(s))\, ds.$$

Choose $R_3 < \min\{R_1, a\}$. Then, as in the first part of the proof,

$$\|Tv\| \leq \lambda L \phi(R_3),$$

where $\phi(R_3) = \max\{f(t,z) : t \in [0,1],\ z \in [0, R_3]\}$. Choose

$$\lambda < \min\left\{ \frac{R_3}{L\phi(R_3)}, \lambda_0 \right\}, \tag{3.5}$$

then $\|Tv\| \leq \|v\|$ for all $v \in \mathcal{C} \cap \partial\Omega_3$. Choose λ according to (3.5). Since

$$\lim_{z\to 0^+} \frac{f_t(t,z)}{z} \geq \lim_{z\to 0^+} \frac{b}{z} = \infty$$

uniformly in $[0,1]$, there exists $0 < R_4 < R_3$ such that

$$f_t(t,z) \geq Bz, \quad t \in [0,1], \quad z \in [0, R_4],$$

where

$$\lambda BD \geq 1, \quad D = \max_{t\in[0,1]} \int_0^1 G(t,s)p(s)\,ds,$$

and D is defined by (2.8). Then, for all $v \in \mathcal{C} \cap \partial\Omega_4$,

$$\begin{aligned}
\|Tv\| = \max_{t\in[0,1]} \lambda \int_0^1 G(t,s)f_t(s,v(s))\,ds &\geq \max_{t\in[0,1]} \lambda B \int_0^1 G(t,s)v(s)\,ds \\
&\geq \lambda B \max_{t\in[0,1]} \int_0^1 G(t,s)p(s)R_4\,ds \\
&= \lambda BDR_4 \\
&\geq \|v\|.
\end{aligned}$$

Thus, there exists a positive solution v_2 with $R_4 \leq \|v_2\| \leq R_3 < R_1 \leq \|v_1\| \leq R_2$ for every $\lambda > 0$ satisfying (3.5). □

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