Communications in Applied Analysis **19** (2015), 565–578

POSITIVE SOLUTIONS OF SINGULAR ALGEBRAIC SYSTEMS WITH A PARAMETER

HAIRONG LIAN[1], HAIYAN WANG[2], MENGNIEN WU[3], AND TANGAN GAO[4]

[1]School of Science, China University of Geosciences
Beijing, 100083, PR China
E-mail: lianhr@cugb.edu.cn

[2]Department of Mathematics, Northwest Normal University
Lanzhou, 730070, P.R. China
[2]Department of Mathematics, North University of China
Taiyuan, Shanxi, 030051, P.R. China
[2]School of Mathematical and Natural Sciences, Arizona State University
Phoenix, AZ 85069, USA
E-mail: haiyan.wang@asu.edu

[3]Department of Mathematics, Tamkang University, Danshui, Taiwan 251
E-mail: mwu@mail.tku.edu.tw

[4]Department of Mathematics and Statistics, California State University
Long Beach, CA 90840, USA
E-mail: tgao@csulb.edu

ABSTRACT. In this paper, we study positive solutions of singular systems of nonlinear algebraic equations with a parameter. With singularity at the origin, superlinearity or sublinearity at infinity, the existence and multiplicity of positive solutions are established. The proof of the results is based on Krasnoselskii fixed point theorem. Some examples are gave to illusive the results.

AMS (MOS) Subject Classification. 39A10, 65H10.

1. Introduction

When solving certain mathematical problems from applications such as difference equations, numerical solutions of differential equations, steady states of complex dynamical systems, the problems can be often transformed into the study of the existence of positive solutions to systems of algebraic equations, see [6, 9] and the references therein. Knowing the number of solutions of systems of algebraic educations often often is a key step to understand the systems of algebraic equations.

Received February 4, 2015

In [9], Zhang and Feng study the following nonlinear algebraic system

$$\mathbf{x} = \lambda A\mathbf{F}(\mathbf{x}) \tag{1.1}$$

where $\lambda > 0$ is a parameter, $\mathbf{x} = (x_1, \ldots, x_n)^T$ is a vector, $A = (a_{ij})$ is an $n \times n$ real matrix, and $\mathbf{F}(\mathbf{x}) = \big(f_1(x_1), \ldots, f_n(x_n)\big)^T$. Under the assumptions that the single variable functions $f_i(x_i)$, $i = 1, \ldots, n$, are nonnegative, superlinear or sublinear at infinity, and $a_{ij} > 0$, $i, j = 1, \ldots, n$, they prove the existence, multiplicity, and nonexistence of positive solutions by using the fixed point theorem on cones. Later, Wang etc in [6] generalize the results in [9] for $\mathbf{F}(\mathbf{x}) = \big(f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})\big)^T$ and establish the existence, multiplicity, and nonexistence of positive solutions of (1.1) under weaker condition that $a_{ij} \geq 0$ and every column of A has at least one positive element. The authors in [1] further extend the results in [6] to the system (1.1) when A may have negative elements and $\mathbf{F}$ may have negative terms.

There are similar algebraic systems discussed in literature, see [7, 8, 10]. In [7], the authors discuss the algebraic equations

$$B\mathbf{x} = \lambda \mathbf{F}(\mathbf{x}) \tag{1.2}$$

where B is an $n \times n$ positive definite matrix. By using the variational method and the critical point theory, they obtain the existence of the solutions to (1.2). Obviously, (1.2) can be transformed to (1.1) by multiplying the inverse of B.

Meanwhile, there is considerable amount of work done about the existence of positive solutions for singular systems. We refer to [4] and [5] on singular ordinary differential equations. It has been shown that many existence and multiplicity results for nonsingular differential systems are still valid for the singular case. So we would like to know whether the existence and multiplicity results for nonsingular algebraic systems still hold for the singular ones. In this paper, we shall study the existence and multiplicity of positive solutions for the singular nonlinear algebraic system of (1.1) with $\mathbf{F}(\mathbf{x}) = \big(f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})\big)^T$. Note that the singularity in this paper means that at least one of $|f_1(\mathbf{x})|, \ldots, |f_n(\mathbf{x})|$ tends to infinity when $\mathbf{x}$ goes to $\mathbf{0}$. Our results show the existence of positive solutions of (1.1) when λ is chosen appropriately. In addition, the conditions for our results allow the matrix A containing negative elements and the function F containing negative terms. What we obtain in this paper essentially extend and improve the results in [1], [6] and [9].

In this paper, let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^n = \prod_{i=1}^n \mathbb{R}$, $\mathbb{R}_+ = [0, \infty)$, and $\mathbb{R}_+^n = \prod_{i=1}^n \mathbb{R}_+$. We will use the standard 1-norm $\|\mathbf{x}\| = \sum_{i=1}^n |x_i|$ over $\mathbb{R}^n$ since all norms are equivalent in a finite dimensional space. By a positive solution of the algebraic system (1.1), we mean an $\mathbf{x} \in \mathbb{R}^n$ which satisfies the equation (1.1) and $\mathbf{x} > 0$. Here $\mathbf{x} = (x_1, \ldots, x_n)^T > \mathbf{0}$ means that $x_i > 0$ for all $i = 1, 2, \ldots, n$. The assumptions used in this paper are:

(H$_1$) $\mathbf{F} : \mathbb{R}^n_+ \setminus \{\mathbf{0}\} \to \mathbb{R}^n$ is continuous and $\mathbf{F}(\mathbf{x}) \neq \mathbf{0}$ for any $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$.

(H$_2$) There exist two positive constants m and M such that for any $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$

$$m\|\mathbf{F}(\mathbf{x})\| \le \sum_{j=1}^{n} a_{ij} f_j(\mathbf{x}) \le M\|\mathbf{F}(\mathbf{x})\|, \quad i = 1, 2, \ldots, n.$$

We use the notations as in [6] to state our main results. Let

$$f_\infty^{(i)} = \lim_{\|\mathbf{x}\|\to\infty} \frac{|f_i(\mathbf{x})|}{\|\mathbf{x}\|} \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}, \quad i = 1, 2, \ldots, n$$

and

$$\mathbf{F}_\infty = \sum_{i=1}^{n} f_\infty^{(i)}.$$

Our main results are:

Theorem 1.1. *Let* (H$_1$) *and* (H$_2$) *hold. Assume that* $\lim_{\|\boldsymbol{x}\|\to 0} |f_k(\boldsymbol{x})| = \infty$ *for an index* k *with* $1 \le k \le n$.

- *(a) If* $\boldsymbol{F}_\infty = 0$, *then for all* $\lambda > 0$, (1.1) *has at least one positive solution.*
- *(b) If* $\boldsymbol{F}_\infty = \infty$, *then there exists a* $\lambda_0 > 0$ *such that for any* $0 < \lambda < \lambda_0$, (1.1) *has at least two positive solutions.*
- *(c) There exists a* $\lambda_0 > 0$ *such that for any* $0 < \lambda < \lambda_0$, (1.1) *has at least one positive solution.*
- *(d) If* $\boldsymbol{F}_\infty > 0$, *then there exists a* $\lambda_0 > 0$ *such that for any* $\lambda > \lambda_0$, (1.1) *has no positive solution.*

When the matrix A is positive, we can directly obtain the existence results with a weaker condition on $\mathbf{F}$, that is, replacing conditions (H$_1$) and (H$_2$) by

(H$_3$) A is positive and $f_i : \mathbb{R}^n_+ \setminus \{\mathbf{0}\} \to (0, +\infty)$ is continuous, $i = 1, 2, \ldots, n$.

Corollary 1.2. *Let* (H$_3$) *hold. Assume that* $\lim_{\|\boldsymbol{x}\|\to 0} |f_k(\boldsymbol{x})| = \infty$ *for an index* k *with* $1 \le k \le n$.

- *(a) If* $\boldsymbol{F}_\infty = 0$, *then for all* $\lambda > 0$, (1.1) *has at least one positive solution.*
- *(b) If* $\boldsymbol{F}_\infty = \infty$, *then there exists a* $\lambda_0 > 0$ *such that for any* $0 < \lambda < \lambda_0$, (1.1) *has at least two positive solutions.*
- *(c) There exists a* $\lambda_0 > 0$ *such that for any* $0 < \lambda < \lambda_0$, (1.1) *has at least one positive solution.*
- *(d) If* $\boldsymbol{F}_\infty > 0$, *then there exists a* $\lambda_0 > 0$ *such that for any* $\lambda > \lambda_0$, (1.1) *has no positive solution.*

2. Preliminaries

In this section, we recall some basic concepts and Krasnoselskii fixed point theorem and then prove a few lemmas which will be used to show our main results.

Definition 2.1. Let $(X, \|\cdot\|)$ be a Banach space and K be a closed nonempty subset of X. K is said to be a cone of X if

(i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and $\alpha, \beta \geq 0$;
(ii) $u, -u \in K$ implies that $u = 0$.

The following well-known Krasnoselskii fixed point theorem on compression and expansion of cones has been effectively used to solve singular problems, see [4], [5] and [6] for examples.

Theorem 2.2 (Krasnoselskii Fixed Point Theorem, [2, 3])**.** *Let $(X, \|\cdot\|)$ be a Banach space and K be a cone. Assume that Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and*

$$T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$$

is completely continuous such that either

(i) $\|Tu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|, \quad u \in K \cap \partial\Omega_2$; *or*
(ii) $\|Tu\| \leq \|u\|, \quad u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

In order to apply Krasnoselskii fixed point theorem to solve the singular algebraic system (1.1), we shall let $X = \mathbb{R}^n$ with the standard 1-norm. Let $\sigma = \frac{m}{nM}$ with m and M given in the condition (H_2). Define the cone K of X by

$$K = \{\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n_+ : x_i \geq \sigma\|\mathbf{x}\|, \ i = 1, \ldots, n\}.$$

For $r > 0$, define

$$\Omega_r = \{\mathbf{x} \in K : \ \|\mathbf{x}\| \leq r\}$$

with boundary $\partial\Omega_r = \{\mathbf{x} \in K : \ \|\mathbf{x}\| = r\}$ and

$$\widetilde{M}(r) \ = \ \max\{|f_i(\mathbf{x})| : \mathbf{x} \in \partial\Omega_r, \ i = 1, \ldots, n\}. \tag{2.1}$$

Note that $\widetilde{M}(r) > 0$ by the condition (H_1). Let $T^\lambda = (T_1^\lambda, \ldots, T_n^\lambda)^T : K \setminus \{\mathbf{0}\} \to X$ be the map with

$$T_i^\lambda(\mathbf{x}) = \lambda \sum_{j=1}^n a_{ij} f_j(\mathbf{x}), \quad i = 1, \ldots, n. \tag{2.2}$$

Note that for an $\mathbf{x} \in K \setminus \{\mathbf{0}\}$, $\mathbf{x}$ is a fixed point of $T^\lambda(\mathbf{x})$ if and only if it is a positive solution of (1.1). So, we only need to discuss the existence of fixed points of $T^\lambda(\mathbf{x})$ in $K \setminus \{\mathbf{0}\}$. In order to apply Krasnoselskii fixed point theorem, it is necessary that T^λ maps $K \setminus \{\mathbf{0}\}$ into K, which is proved in the next theorem.

Theorem 2.3. *Assume* (H$_1$) *and* (H$_2$) *hold. Then* $T^\lambda(K \setminus \{\mathbf{0}\}) \subset K$ *and* $T^\lambda : K \setminus \{0\} \to K$ *is completely continuous.*

Proof. If $\mathbf{x} = (x_1, \dots, x_n)^T \in K \setminus \{\mathbf{0}\}$, then $x_i \geq \sigma\|\mathbf{x}\| > 0,\ i = 1, \dots, n$. From the condition (H$_2$), we have $T_i^\lambda(\mathbf{x}) \geq 0$ and

$$T_i^\lambda(\mathbf{x}) = \lambda \sum_{j=1}^n a_{ij} f_j(\mathbf{x}) \leq \lambda M \|\mathbf{F}(\mathbf{x})\| = \lambda M \sum_{j=1}^n |f_j(\mathbf{x})|, \quad i = 1, \dots, n$$

which led to

$$\lambda \sum_{j=1}^n |f_j(\mathbf{x})| \geq \frac{1}{nM} \sum_{i=1}^n T_i^\lambda(\mathbf{x}).$$

Therefore,

$$T_i^\lambda(\mathbf{x}) = \lambda \sum_{j=1}^n a_{ij} f_j(\mathbf{x}) \geq \lambda m \|\mathbf{F}(\mathbf{x})\| = \lambda m \sum_{j=1}^n |f_j(\mathbf{x})| \geq \frac{m}{nM} \sum_{i=1}^n T_i^\lambda(\mathbf{x}) = \sigma \|T^\lambda(\mathbf{x})\|$$

for $i = 1, \dots, n$ and $T^\lambda(K \setminus \{\mathbf{0}\}) \subset K$. It is easy to see that $T^\lambda : K \setminus \{\mathbf{0}\} \to K$ is continuous and compact. □

Lemma 2.4. *Assume* (H$_1$) *and* (H$_2$) *hold. For any* $\eta > 0$ *and* $\boldsymbol{x} \in K \setminus \{\boldsymbol{0}\}$*, if there exists a component* f_k *of* $\boldsymbol{F}$ *such that*

$$|f_k(\boldsymbol{x})| \geq \eta \|\boldsymbol{x}\|,$$

then $\|T^\lambda(\boldsymbol{x})\| \geq \lambda n m \eta \|\boldsymbol{x}\|$.

Proof. Since $\mathbf{x} \in K \setminus \{\mathbf{0}\}$ and $|f_k(\mathbf{x})| \geq \eta\|\mathbf{x}\|$, by the condition (H$_2$), we have that for any $i = 1, \dots, n$

$$T_i^\lambda(\mathbf{x}) = \lambda \sum_{j=1}^n a_{ij} f_j(\mathbf{x}) \geq \lambda m \sum_{j=1}^n |f_j(\mathbf{x})| \geq \lambda m |f_k(\mathbf{x})| \geq \lambda m \eta \|\mathbf{x}\|$$

and thus, $\|T^\lambda(\mathbf{x})\| = \sum_{i=1}^n T_i^\lambda(\mathbf{x}) \geq \lambda n m \eta \|\mathbf{x}\|$. □

Lemma 2.5. *Assume* (H$_1$) *and* (H$_2$) *hold. For any* $\varepsilon > 0$ *and* $\boldsymbol{x} \in K \setminus \{\boldsymbol{0}\}$*, if all components of* $\boldsymbol{F}$ *satisfy*

$$|f_j(\boldsymbol{x})| \leq \varepsilon \|\boldsymbol{x}\|, \quad j = 1, 2, \dots, n,$$

then $\|T^\lambda(\boldsymbol{x})\| \leq \lambda n^2 M \varepsilon \|\boldsymbol{x}\|$.

Proof. By the condition (H$_2$), $T_i^\lambda(\mathbf{x}) \geq 0$ for any $x \in K \setminus \{\mathbf{0}\}$ for $i = 1, \dots, n$ and

$$\|T^\lambda(\mathbf{x})\| = \sum_{i=1}^n T_i^\lambda(\mathbf{x}) \leq \sum_{i=1}^n \left(\lambda M \sum_{j=1}^n |f_j(\mathbf{x})| \right) \leq \lambda n^2 M \varepsilon \|\mathbf{x}\|.$$

□

Lemma 2.6. *Assume* (H_1) *and* (H_2) *hold. If* $\boldsymbol{x} \in \partial\Omega_r$, $r > 0$, *then*

$$\|T^\lambda(\boldsymbol{x})\| \leq \lambda n M \widetilde{M}(r).$$

Proof. For any $\mathbf{x} \in \partial\Omega_r$, the definition of $\widetilde{M}(r)$ in (2.1) implies that $|f_j(\mathbf{x})| \leq \widetilde{M}(r)$ for $j = 1, 2, \ldots, n$. It follows that

$$T_i^\lambda(\mathbf{x}) \leq \lambda M \sum_{j=1}^{n} |f_j(\mathbf{x})| \leq \lambda n M \widetilde{M}(r).$$

□

3. Proof of Theorem 1.1

Part (a) Consider any fixed $\lambda > 0$. From the assumption $\lim_{\|\mathbf{x}\|\to 0} |f_k(\mathbf{x})| = \infty$, for any $\eta_1 > \frac{1}{\lambda n m}$, there exists a constant $r_1 > 0$ such that for all $\mathbf{x} \in K \setminus \{\mathbf{0}\}$ and $0 < \|\mathbf{x}\| \leq r_1$,

$$|f_k(\mathbf{x})| \geq \eta_1 \|\mathbf{x}\|. \tag{3.1}$$

Note also that $\lambda n m \eta_1 > 1$. Lemma 2.4 implies that for any $x \in \partial\Omega_{r_1}$,

$$\|T^\lambda(\mathbf{x})\| \geq \lambda n m \eta_1 \|\mathbf{x}\| > \|\mathbf{x}\|. \tag{3.2}$$

Now we determine the region Ω_{r_2}. Since $\mathbf{F}_\infty = 0$, for any $\varepsilon < \frac{1}{\lambda n^2 M}$, there exists a constant $r_2 > r_1$ such that

$$|f_j(\mathbf{x})| \leq \varepsilon \|\mathbf{x}\|$$

for all $j = 1, \ldots, n$ and all $\mathbf{x} \in K$ with $\|\mathbf{x}\| \geq r_2$. Note also that $\lambda n^2 M \varepsilon < 1$. Lemma 2.5 implies that for any $x \in \partial\Omega_{r_2}$,

$$\|T^\lambda(\mathbf{x})\| \leq \lambda n^2 M \varepsilon \|\mathbf{x}\| < \|\mathbf{x}\|. \tag{3.3}$$

Therefore, for any fixed $\lambda > 0$, from the inequalities (3.2) on $\partial\Omega_{r_1}$ and (3.3) on $\partial\Omega_{r_2}$, Krasnoselskii fixed point theorem implies that T^λ has a fixed point in $\Omega_{r_2} \backslash \overline{\Omega}_{r_1}$, which is a positive solution of the algebraic system (1.1).

Part (b) Note that the inequality (3.2) still holds on $\partial\Omega_{r_1}$. Fix a constant $r_3 > r_1$. Let $\lambda_0 = \frac{r_3}{n M \widetilde{M}(r_3)} > 0$. Then, Lemma 2.6 implies that for any $\mathbf{x} \in \partial\Omega_{r_3}$ and $0 < \lambda < \lambda_0$,

$$\|T^\lambda(\mathbf{x})\| \leq \lambda n M \widetilde{M}(r_3) < \lambda_0 n M \widetilde{M}(r_3) = r_3 = \|\mathbf{x}\|. \tag{3.4}$$

On the other hand, from $\mathbf{F}_\infty = \sum_{i=1}^{n} f_\infty^{(i)} = \infty$, there exists an index k such that $f_\infty^{(k)} = \lim_{\mathbf{x}\to\infty} \frac{|f_k(\mathbf{x})|}{\|\mathbf{x}\|} = \infty$. Therefore, for any $\eta > \frac{1}{\lambda n m}$, there exists a constant $r_4 > r_3$ such that

$$|f_k(\mathbf{x})| \geq \eta \|\mathbf{x}\|$$

for all $\mathbf{x} \in K \setminus \{\mathbf{0}\}$ with $\|\mathbf{x}\| \geq r_4$. Note that $\lambda n m \eta > 1$. If follows from Lemma 2.4 that for any $\mathbf{x} \in \partial\Omega_{r_4}$,

$$\|T^\lambda(\mathbf{x})\| \geq \lambda n m \eta \|\mathbf{x}\| > \|\mathbf{x}\|. \tag{3.5}$$

With the inequalities (3.2) on $\partial\Omega_{r_1}$, (3.4) on $\partial\Omega_{r_3}$ and (3.5) on $\partial\Omega_{r_4}$, Krsonselskii's fixed point theorem implies that T^λ has two fixed points $\mathbf{x}_1$ and $\mathbf{x}_2$ with $\mathbf{x}_1 \in \Omega_{r_3} \setminus \overline{\Omega}_{r_1}$ and $\mathbf{x}_2 \in \Omega_{r_4} \setminus \overline{\Omega}_{r_3}$, which are two distinct positive solutions of the system (1.1) and satisfy

$$r_1 < \|\mathbf{x}_1\| < r_3 < \|\mathbf{x}_2\| < r_4.$$

Part (c) Note that the inequality (3.2) still holds on $\partial\Omega_{r_1}$. Fix a constant $r_5 > r_1 > 0$ and let $\lambda_0 = \frac{r_5}{nM\widetilde{M}(r_5)} > 0$. Then, Lemma 2.6 implies that for any $\mathbf{x} \in \partial\Omega_{r_5}$ and any $0 < \lambda < \lambda_0$,

$$\|T^\lambda(\mathbf{x})\| \leq \lambda n M \widetilde{M}(r_5) < \lambda_0 n M \widetilde{M}(r_5) = r_5 = \|\mathbf{x}\|.$$

With this inequality on $\partial\Omega_{r_5}$ and the inequality (3.2) on $\partial\Omega_{r_1}$, Krsonselskii fixed point theorem implies that T^λ has a fixed point in $\Omega_{r_5} \setminus \overline{\Omega}_{r_1}$ for any $0 < \lambda < \lambda_0$. This fixed point of T^λ is a positive solution to the system (1.1).

Part (d) Note that the inequality (3.1) still holds on $\Omega_{r_1}\setminus\{\mathbf{0}\}$ for the index k. It follows from Lemma 2.4 that for any $\mathbf{x} \in \Omega_{r_1}\setminus\{\mathbf{0}\}$,

$$\|T^\lambda(\mathbf{x})\| \geq \lambda n m \eta_1 \|\mathbf{x}\|.$$

Since $\mathbf{F}_\infty > 0$, there exists an index l such that $f_\infty^{(l)} > 0$. Thus, for any $0 < \eta_2 < f_\infty^{(l)}$, there exists a constant $r_6 > r_1$ such that for any $\mathbf{x} \in K$ with $\|\mathbf{x}\| \geq r_6$,

$$|f_l(\mathbf{x})| \geq \eta_2 \|\mathbf{x}\|.$$

It follows from Lemma 2.4 that for any $\mathbf{x} \in K$ with $\|\mathbf{x}\| \geq r_6$,

$$\|T^\lambda(\mathbf{x})\| \geq \lambda n m \eta_2 \|\mathbf{x}\|.$$

Let

$$\eta_3 = \min\left\{\frac{\|\mathbf{F}(\mathbf{x})\|}{n\|\mathbf{x}\|} : \mathbf{x} \in K, r_1 \leq \|\mathbf{x}\| \leq r_6\right\}$$

which is positive by the condition (H$_1$). Then, by the condition (H$_2$), for any $\mathbf{x} \in K$ with $r_1 \leq \|\mathbf{x}\| \leq r_6$,

$$\|T^\lambda(\mathbf{x})\| \geq \lambda m \|\mathbf{F}(\mathbf{x})\| \geq \lambda n m \eta_3 \|\mathbf{x}\|.$$

Now, let

$$\lambda_0 = \max\left\{\frac{1}{nm\eta_1}, \frac{1}{nm\eta_2}, \frac{1}{nm\eta_3}\right\}.$$

Then, for any $\lambda > \lambda_0$ and any $\mathbf{x} \in K\setminus\{\mathbf{0}\}$,

$$\|T^\lambda(\mathbf{x})\| \geq \lambda n m \eta \|\mathbf{x}\| > \lambda_0 n m \eta \|\mathbf{x}\| = \|\mathbf{x}\|.$$

Therefore, $T^\lambda(\mathbf{x})$ has no fixed point in K for $\lambda > \lambda_0$ as we always can choose r_1 as small as possible. The proof is complete.

Remark 3.1. When A is positive, we may choose

$$m = \min_{i,j=1,\dots,n} a_{i,j} > 0 \quad \text{and} \quad M = \max_{i,j=1,\dots,n} a_{i,j} > 0.$$

Then, for positive functions $f_1(\mathbf{x}), \dots, f_n(\mathbf{x})$,

$$\begin{aligned} m\|\mathbf{F}(\mathbf{x})\| = m(f_1(\mathbf{x}) + \cdots + f_n(\mathbf{x})) &\leq \sum_{j=1}^{n} a_{i,j} f_j(\mathbf{x}) \\ &\leq M(f_1(\mathbf{x}) + \cdots + f_n(\mathbf{x})) = M\|\mathbf{F}(\mathbf{x})\|. \end{aligned}$$

Thus, the condition (H_3) implies that both conditions (H_1) and (H_2) are true. Therefore, Corollary 1.2 can be obtained directly from Theorem 1.1.

4. **Applications**

A large number of problems can be converted into the algebraic system (1.1). In this section, we show by some examples for the use of our results.

Example 4.1. We seek positive solutions of the fractional equation

$$x = \lambda \left(x^{-\alpha} + x^{\beta} \right), \quad \lambda > 0,\ \alpha > 0,\ \beta > 0. \tag{4.1}$$

This is a scalar equation with $f(x) = x^{-\alpha} + x^{\beta}$. It is easy to see that

$$\lim_{x\to 0} f(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} x^{\beta-1} = \begin{cases} 0 & \text{if } \beta < 1, \\ 1 & \text{if } \beta = 1, \\ \infty & \text{if } \beta > 1. \end{cases}$$

According to Theorem 1.1, we have

(1) When $\beta < 1$, for all $\lambda > 0$, (4.1) has at least one positive solution. In fact, there is only one positive solution for each λ. This result can be verified by the facts that the equation (4.1) can be viewed λ as a function of x

$$\lambda = x/(x^{-\alpha} + x^{\beta}), \tag{4.2}$$

λ is strictly increasing, $\lim_{x\to 0} \lambda = 0$, and $\lim_{x\to\infty} \lambda = \infty$.

(2) When $\beta = 1$, there exists a $\lambda_0 > 0$ such that for any $0 < \lambda < \lambda_0$, (4.1) has at least one positive solution. In fact, there is only one positive solution for each $\lambda < \lambda_0 = 1$. This result can be seen directly from that λ is strictly increasing, $\lim_{x\to 0} \lambda = 0$, and $\lim_{x\to\infty} \lambda = 1$.

(3) When $\beta > 1$, there exists a $\lambda_0 > 0$ such that for any $0 < \lambda < \lambda_0$, (4.1) has at least two positive solutions. In fact, there are exactly two positive solutions for each $\lambda \leq \lambda_0$ with

$$\lambda_0 = \frac{(\alpha+1)^{\frac{\alpha+1}{\alpha+\beta}} (\beta-1)^{\frac{\beta-1}{\alpha+\beta}}}{\alpha+\beta}.$$

This result can be seen from that the maximum of the function λ in (4.2) occurs at

$$x = \left(\frac{1+\alpha}{\beta-1}\right)^{\frac{1}{\alpha+\beta}}$$

and this maximum gives the value of λ_0.

The two graphs in Figure 1 shows the single positive solution of the equation (4.1) for each $0 < \lambda < \lambda_0 = 1$ for the case $\beta = 1$ and two positive solutions (one in blue and the other in red) of the equation (4.1) for each $0 < \lambda < \lambda_0$ for the case $\beta = 2$.

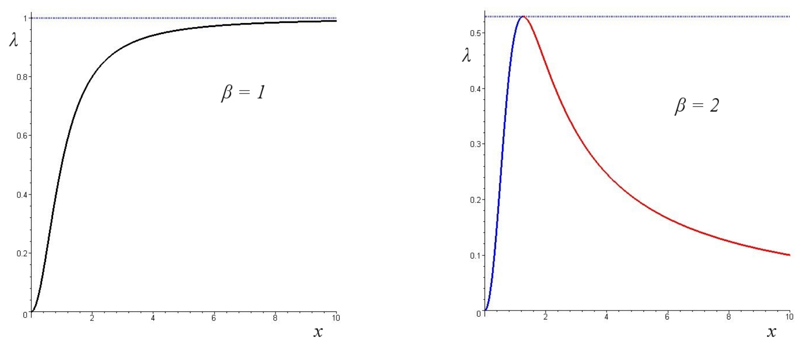

FIGURE 1. Positive solutions of (4.1) for $\beta = 1$ and $\beta = 2$.

(4) When $\beta \geq 1$, there exists a constant $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, (4.1) has no positive solution. In fact,
   (i) when $\beta = 1$, $\lambda_0 = 1$ as explained in part (2) above;
   (ii) when $\beta > 1$,

$$\lambda_0 = \frac{(\alpha+1)^{\frac{\alpha+1}{\alpha+\beta}}(\beta-1)^{\frac{\beta-1}{\alpha+\beta}}}{\alpha+\beta}$$

   as shown in part (3) above.

Example 4.2. Consider

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{x_1 x_2} \\ x_1 x_2 \end{pmatrix}, \quad \lambda > 0. \tag{4.3}$$

Let $f_1(x_1, x_2) = \frac{1}{x_1 x_2}$, $f_2(x_1, x_2) = x_1 x_2$. Choose $m = 1$ and $M = 3$, then (H_2) holds for this example. It is easy to see that $\sigma = \frac{m}{nM} = \frac{1}{6}$ and

$$K = \left\{ \mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2 : \ x_i \geq 0, \ x_i \geq \frac{1}{6}(x_1 + x_2), \ i = 1, 2 \right\}.$$

K is the region between the lines $5x_1 = x_2$ and $x_1 = 5x_2$. For $\mathbf{x} = (x_1, x_2)^T \in K$, we can see that $\lim_{x_1+x_2 \to 0} f_1(x_1, x_2) = \infty$ and

$$f_\infty^{(2)} = \lim_{x_1+x_2\to\infty} \frac{x_1 x_2}{x_1 + x_2} \geq \lim_{x_1+x_2\to\infty} \frac{(x_1+x_2)^2}{36(x_1+x_2)} = \lim_{x_1+x_2\to\infty} \frac{x_1+x_2}{36} = \infty.$$

According to Theorem 1.1 (b), (4.3) has two positive solutions for $\lambda > 0$ small enough and has no positive solution for $\lambda > 0$ sufficiently large. In fact, multiplying the two equations together, we have $x_1x_2 = \lambda^2(2(x_1x_2)^{-2} + 3(x_1x_2)^2 + 7)$, which suggests that we consider the scalar equation

$$u = \lambda^2(2u^{-2} + 3u^2 + 7). \tag{4.4}$$

Because $\lambda^2 = \frac{u^3}{3u^4+7u^2+2}$ attains its maximum $\frac{3\sqrt{3}}{50}$ on $[0, +\infty)$, let $\lambda_0 = \sqrt{\frac{3\sqrt{3}}{50}}$. Then, for $\lambda > \lambda_0$, (4.4) has no positive solution; for $\lambda = \lambda_0$, (4.4) has one positive solution with multiplicity two; for $\lambda < \lambda_0$, (4.4) has two positive solutions. For any $0 < \lambda \leq \lambda_0$, let u be a positive solution of (4.4). Then $(x_1, x_2)^T$ defined by

$$x_1 = \lambda(u^{-1} + 3u), \qquad x_2 = \lambda(2u^{-1} + u)$$

is a positive solution of (4.3). In conclusion, for $\lambda > \lambda_0$, (4.3) has no positive solution; for $\lambda = \lambda_0$, (4.3) has one positive solution with multiplicity two; for $\lambda < \lambda_0$, (4.3) has two positive solutions. Figure 2 shows the two positive solutions (one in red and the other in blue) of (4.3) for each $0 < \lambda \leq \lambda_0$ with the red solution curve going to infinity when λ goes to zero.

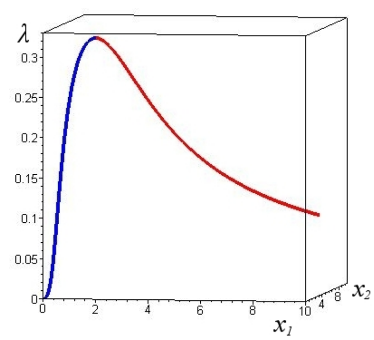

FIGURE 2. Two positive solutions of (4.3) for each $\lambda \leq \lambda_0$.

Example 4.3. Consider the singular second order difference equation of the form

$$\triangle^2 x(k-1) + \lambda f(x(k)) = 0, \quad k \in [1, N],\ \lambda > 0, \tag{4.5}$$

with the boundary value condition

$$x(0) = x(N+1) = 0, \tag{4.6}$$

where N is an integer greater than or equal to 1, $[1, N]$ is the discrete interval $\{1, \ldots, N\}$, $\triangle x(k) = x(k+1) - x(k)$ is the forward difference operator and $f \in C([1, N] \times (0, \infty), [0, \infty))$ may be singular at $x = 0$.

Let $\mathbf{x} = (x(1), x(2), \ldots, x(N))^T$ and $\mathbf{F}(\mathbf{x}) = (f(x(1)), f(x(2)), \ldots, f(x(N)))^T$. Then the problem (4.5) with (4.6) is equivalent to the system

$$B\mathbf{x} = \lambda \mathbf{F}(\mathbf{x})$$

where the tridiagonal matrix

$$B = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

B is positive definite and invertible. Denote $A = (a_{ij}) = B^{-1}$, then for any $i, j \in [1, N]$

$$a_{ij} = \begin{cases} \frac{j(N+1-i)}{N+1}, & \text{if } 1 \le j \le i \le N, \\ \frac{i(N+1-j)}{N+1}, & \text{if } 1 \le i \le j \le N. \end{cases}$$

If we choose

$$m = \frac{1}{N+1} \quad \text{and} \quad M = \frac{\left[\frac{N+1}{2}\right]\left(N+1-\left[\frac{N+1}{2}\right]\right)}{N+1},$$

where $[\cdot]$ is the greatest integer function, then the condition (H_2) holds for this example. So our main results are valid for such singular discrete Dirichlet boundary value problems as long as $f(x) > 0$ for any $x > 0$. For example, when $N = 3$ and $f(x) = \frac{1}{x} + x - 1$, there is a λ_0 such that for any $0 < \lambda < \lambda_0$, the system $B\mathbf{x} = \lambda\mathbf{F}(\mathbf{x})$ has at least one positive solution. The three graphs in Figure 3 show the components x_1, x_2, and x_3 of the positive solutions $\mathbf{x}$ with respect to λ and it is true that $x_1 = x_3$ for the positive solutions. For the values of λ below the dotted line, the system has

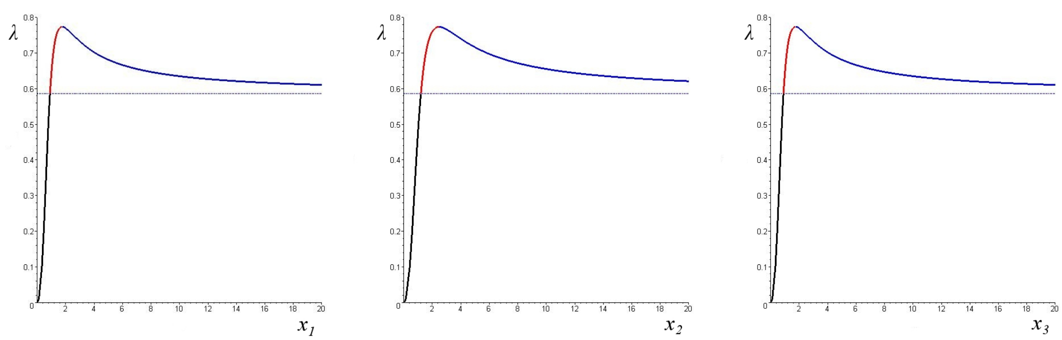

FIGURE 3. Positive solutions of (4.5) with $N = 3$ and $f(x) = \frac{1}{x} + x - 1$.

one positive solution (in black); For the values of λ above the dotted line, the system has two positive solutions (one in red and the other in blue).

Example 4.4. Consider the system of equations

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \cos x_1 \\ \sin x_2 \\ f_3(x_3) \end{pmatrix} \tag{4.7}$$

with continuous function $f_3(x_3) \ge 3$ for $x_3 > 0$ and $\lim_{x_3 \to 0^+} f_3(x_3) = \infty$.

Note that the matrix A contains negative elements and the first two components of $\mathbf{F}(\mathbf{x})$ can be negative. Since $f_3(x_3) \geq 3$,

$$\begin{aligned}
&\pm\cos x_1 \pm \sin x_2 + f_3(x_3) - \frac{1}{5}(|\cos x_1| + |\sin x_2| + f_3(x_3)) \\
&= \left(\pm\cos x_1 - \frac{|\cos x_1|}{5}\right) + \left(\pm\sin x_2 - \frac{|\sin x_2|}{5}\right) + \frac{4}{5}f_3(x_3) \\
&\geq \left(-\frac{6}{5}\right) + \left(-\frac{6}{5}\right) + \frac{12}{5} = 0
\end{aligned}$$

and

$$\pm\cos x_1 \pm \sin x_2 + f_3(x_3) \leq |\cos x_1| + |\sin x_2| + f_3(x_3).$$

The condition (H_2) holds when $m = \frac{1}{5}$ and $M = 1$. First, consider the case of $f_3(x_3) = \frac{1}{x_3} + x_3 + 1$ which attains its minimum value 3 at $x_3 = 1$. By Theorem 1.1 (c), there is a positive λ_0 such that for any $0 < \lambda < \lambda_0$ the system of equations (4.7) has at least one positive solution. The three graphs in Figure 4 show the components x_1, x_2, and x_3 of the positive solutions $\mathbf{x}$ with respect to λ.

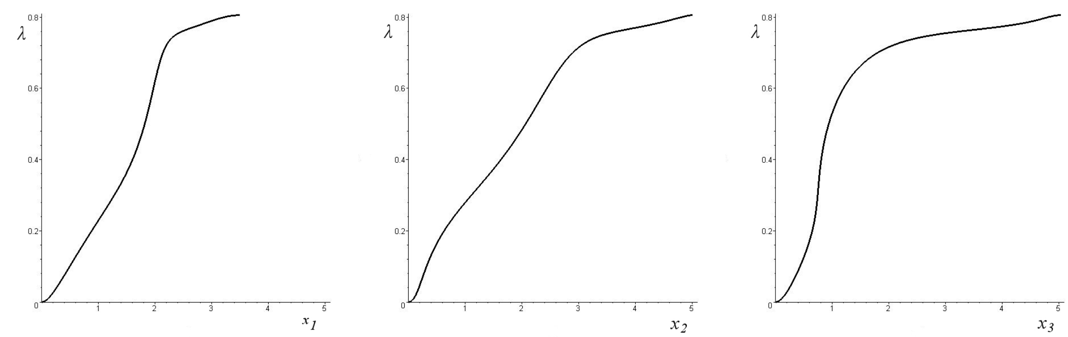

FIGURE 4. Positive solutions of (4.7) with $f_3(x_3) = \frac{1}{x_3} + x_3 + 1$.

Now, consider the case of $f_3(x_3) = \frac{1}{x_3} + 3 \geq 3$. Since $\mathbf{F}_\infty = 0$, by Theorem 1.1 (a), the system of equations (4.7) has at least one positive solution for any $\lambda > 0$. The three graphs in Figure 5 show the components x_1, x_2, and x_3 of the positive solutions $\mathbf{x}$ with respect to λ.

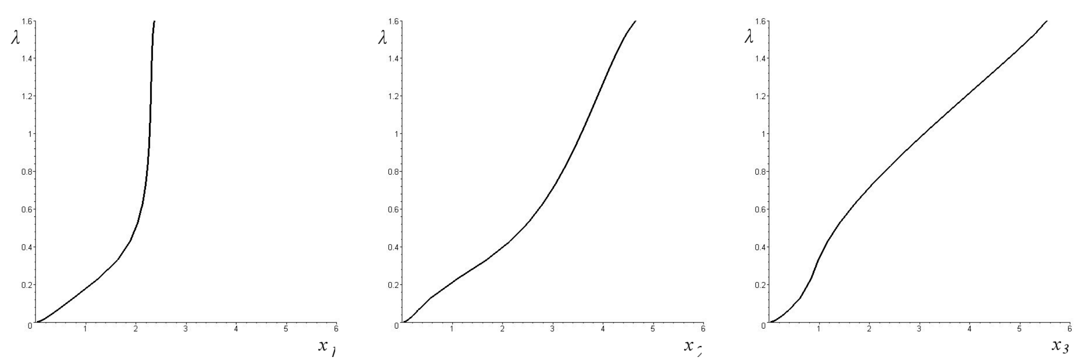

FIGURE 5. Positive solutions of (4.7) with $f_3(x_3) = \frac{1}{x_3} + 3$.

ACKNOWLEDGMENTS

This research is supported by the Fundamental Research Funds for the Central Universities of China.

REFERENCES

[1] T. Gao, H. Wang, and M. Wu, Nonnegative solutions of a class of systems of algebraic equations, *Dynam. Systems Appl.* Accepted for publication.

[2] D. Guo and V. Lakshmikantham, *Nonlinear Problem in Abstract Cones*, Academic Press, Orlando, 1988.

[3] M. Krasnoselskii, *Positive Solution of Operator Equations*, Noordhoff, Groningen, 1964.

[4] H. Wang, Positive periodic solutions of singular systems with a parameter, *J. Differential Equations* **249** (2012), 2986–3002.

[5] H. Wang, Positive periodic solutions of singular systems of first order ordinary differential equations, *Appl. Math. Comput.*, **218** (2011), 1605–1610.

[6] H. Wang, M. Wang and E. Wang, An application of the Krasnoselskii theorem to systems of algebraic equations, *J. Appl. Math. Comput.*, **38** (2012), 585–600.

[7] Y. Yang and J. Zhang, Existence results for a nonlinear system with a parameter, *J. Math. Anal. Appl.*, **340** (2008), 658–668.

[8] G. Zhang, S.Cheng, Existence of solutions for a nonlinear system with parameter, *J. Math. Anal. Appl.* **314** (2006), 311–319.

[9] G. Zhang and W. Feng, On the number of positive solutions of a nonlinear algebraic system, *Linear Algebra Appl.*, **422** (2007), 404–421.

[10] G. Zhang, Existence of non-zero solutions for a nonlinear system with a parameter,*Nonlinear Anal.*, **66** (2007), 140–146.