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BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS AND INCLUSIONS WITH NONLOCAL AND RIEMANN-LIOUVILLE INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we study a new class of boundary value problems for fractional differential equations and inclusions with nonlocal and integral boundary conditions. Some new existence and uniqueness results are obtained by using fixed point theorems. Illustrative examples are also presented.

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1. INTRODUCTION

Fractional differential equations have attracted the attention of many researchers in a variety of directions, due to the development and applications in many fields such as engineering, mathematics, physics, chemistry, etc. (see [12, 18, 19, 20]). Different aspects of fractional differential equations are studied and being developed, but one of the most important area of research in the field of fractional order differential equations is the theory of existence and uniqueness of solutions of nonlinear fractional order differential equations. The recent development of the subject can be found in a series of papers [1, 2, 3, 4, 5, 7, 9, 10] and the references therein.

Here we refer to some boundary value problems which motivated us for the present work. By the help of fixed-point theorems, in [22] Tariboon *et al.* investigated the existence and uniqueness of solutions for a new class of fractional boundary value

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problems involving three-point nonlocal Riemann-Liouville integral boundary conditions of the form:

$$\begin{cases} D^\alpha x(t) = f(t, x(t)), \quad 1 < \alpha \le 2,\ 0 < t < T, \\ x(\eta) = 0, \quad I^p x(T) \equiv \displaystyle\int_0^T \frac{(T-s)^{p-1}}{\Gamma(p)} x(s)\, ds = 0, \end{cases} \tag{1.1}$$

where D^α denotes the Riemann-Liouville fractional derivative of order α, $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$ and $\eta \in (0,T)$ is a given constant. Note that in problem (1.1) instead of the value $x(0)$, which appeared usually in references, there exists the value $x(\eta)$ for some $\eta \in (0,T)$ and an "average type" boundary condition $I^p x(T) = 0$ was introduced.

Ntouyas, in [14], studied the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations of order $q \in (1,2]$ with nonlocal and integral boundary conditions given by:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)),\ 0 < t < 1,\ 1 < q \le 2, \\ x(0) = x_0 + g(x), \quad x(1) = \alpha I^p x(\theta),\ 0 < \theta < 1, \end{cases} \tag{1.2}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $g : C([0,1], \mathbb{R}) \to \mathbb{R}$, $\alpha \in \mathbb{R}$ is such that $\alpha \neq \Gamma(p+2)/\theta^{p+1}$, Γ is the Euler gamma function and I^p is the Riemann-Liouville fractional integral of order p.

Nonlocal conditions can be more useful than the standard initial condition to describe some physical phenomena. For example, $g(x)$ may be given by $g(x) = \sum_{i=1}^m c_i x(t_i)$ where $c_i, i = 1, \dots, m$, are given constants and $0 < t_1 < \cdots < t_m \le T$. For recent papers on nonlocal fractional boundary value problems, the interested reader is referred to [6], [8], [21], [23] and the references cited therein.

In this paper, we discuss the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations of order $q \in (1,2]$ with nonlocal and integral boundary conditions given by:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad 0 \le t \le T,\ 1 < q \le 2, \\ x(\eta) = g(x), \quad AI^p x(T) + Bx(\xi) = c,\ 0 < \eta < \xi < T, \end{cases} \tag{1.3}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, $g : C([0,T], \mathbb{R}) \to \mathbb{R}$ are given continuous functions, I^p is the Riemann-Liouville fractional integral of order p and A, B, c are real constants.

In Section 3 we give some sufficient conditions for the existence and uniqueness of solutions and for the existence of at least one solution of problem (1.3). The first result is based on Banach's contraction principle and the second on a fixed point theorem due to D. O'Regan. Concrete examples are also provided to illustrate the possible applications of the established analytical results.

In Section 4, we extend the results to cover the multi-valued case, considering the following boundary value problem for fractional order differential inclusions with nonlocal and fractional integral boundary conditions:

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \quad 0 \le t \le T,\ 1 < q \le 2, \\ x(\eta) = g(x), \quad AI^p x(T) + Bx(\xi) = c, \end{cases} \tag{1.4}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q and $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

We give an existence result for the problem (1.4) in the case when the right hand side is convex valued by using the nonlinear alternative for contractive maps.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel, in Section 3 we prove our results for single-valued case and in Section 4 our results for multi-valued case.

2. PRELIMINARIES

Let us recall some basic definitions of fractional calculus [12, 18, 20].

Definition 2.1. For at least n-times differentiable function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n,\ n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Lemma 2.3. *For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R},\ i = 0, 1, 2, \ldots, n-1$ ($n = [q] + 1$).

In view of Lemma 2.3, it follows that

$$I^q\ {}^cD^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.1}$$

for some $c_i \in \mathbb{R},\ i = 0, 1, 2, \ldots, n-1$ ($n = [q] + 1$).

To define the solution for the problem (1.3), we find the solution for its associated linear problem.

Lemma 2.4. *Let $\eta \neq \frac{T}{p+1}$. For a given $y \in C([0,T],\mathbb{R})$, the problem*

$$\begin{cases} {}^cD^q x(t) = y(t), \ 1 < q \le 2, \quad t \in [0,T], \\ x(\eta) = y_0, \quad AI^p x(T) + Bx(\xi) = c, \quad y_0 \in \mathbb{R}, \ 0 < \eta < \xi < T, \end{cases} \tag{2.2}$$

is equivalent to an integral equation:

$$\begin{aligned} x(t) &= I^q y(t) + \frac{t-\eta}{\Omega}[c - AI^{p+q}y(T) - BI^q y(\xi)] \\ &\quad + \frac{1}{\Omega}\Big[\frac{AT^p}{\Gamma(p+1)}\Big(t - \frac{T}{p+1}\Big) + B(t-\eta)\Big] I^q y(\eta) \\ &\quad - \frac{1}{\Omega}\Big[\frac{AT^p}{\Gamma(p+1)}\Big(t - \frac{T}{p+1}\Big) + B(t-\eta)\Big] y_0, \quad t \in [0,T], \end{aligned} \tag{2.3}$$

where

$$\Omega = \frac{AT^p}{\Gamma(p+1)}\Big(\frac{T}{p+1} - \eta\Big) + B(\xi - \eta).$$

Proof. For some constants $c_1, c_2 \in \mathbb{R}$, we have [12]

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds + c_1 t + c_2, \quad t \in [0,T]. \tag{2.4}$$

Using the Riemann-Liouville integral of order p for (2.4), we have

$$\begin{aligned} I^p x(t) &= \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} \left[\int_0^s \frac{(s-r)^{q-1}}{\Gamma(q)} y(r)dr + c_1 s + c_2\right] ds \\ &= \frac{1}{\Gamma(p)}\frac{1}{\Gamma(q)}\int_0^t\int_0^s (t-s)^{p-1}(s-r)^{q-1}y(r)drds + c_1\frac{t^{p+1}}{\Gamma(p+2)} + c_2\frac{t^p}{\Gamma(p+1)} \\ &= I^p I^q y(t) + c_1\frac{t^{p+1}}{\Gamma(p+2)} + c_2\frac{t^p}{\Gamma(p+1)} \\ &= I^{p+q} y(t) + c_1\frac{t^{p+1}}{\Gamma(p+2)} + c_2\frac{t^p}{\Gamma(p+1)}. \end{aligned}$$

Applying the boundary conditions, we get the system

$$\begin{aligned} c_1\eta + c_2 &= y_0 - I^q y(\eta) \\ \Big(\frac{AT^{p+1}}{\Gamma(p+2)} + B\xi\Big)c_1 + \Big(\frac{AT^p}{\Gamma(p+1)} + B\Big)c_2 &= c - AI^{p+q}y(T) - BI^q y(\xi), \end{aligned}$$

from which we get

$$c_1 = \frac{1}{\Omega}\left\{c - AI^{p+q}y(T) - BI^q y(\xi) - \Big(\frac{AT^p}{\Gamma(p+1)} + B\Big)(y_0 - I^q y(\eta))\right\},$$

$$c_2 = \frac{1}{\Omega}\left\{\Big(\frac{AT^{p+1}}{\Gamma(p+2)} + B\eta\Big)(y_0 - I^q y(\eta)) - \eta\left(c - AI^{p+q}y(T) - BI^q y(\xi)\right)\right\}.$$

Substituting into (2.4) the values of c_1 and c_2 gives (2.3). Conversely, applying the operator ${}^cD^q$ on (2.3) and taking into account the fact that ${}^cD^\beta I^\beta f(t) = f(t)$, it follows that $({}^cD^q)x(t) = y(t)$. From (2.3), it is easy to verify that the boundary conditions $x(\eta) = y_0,\ AI^p x(T) + Bx(\xi) = c$ are satisfied. This establishes the equivalence between (2.2) and (2.3). $\square$

3. EXISTENCE RESULTS FOR THE SINGLE-VALUED CASE

We denote by $\mathcal{C} = C([0,T],\mathbb{R})$ the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,T]\}$.

In the forthcoming analysis we assume that $\eta \neq \frac{T}{p+1}$, which implies that $\Omega \neq 0$.

In view of Lemma 2.4, we define an operator $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ by

$$
\begin{aligned}
(\mathcal{Q}x)(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\
& + \frac{t-\eta}{\Omega}\Bigg[c - A\int_0^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)} f(s,x(s))ds \\
& - B\int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\Bigg] \\
& + \frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t - \frac{T}{p+1}\Big) + B(t-\eta)\right]\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\
& - \frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t - \frac{T}{p+1}\Big) + B(t-\eta)\right] g(x),\ \ t \in [0,T].
\end{aligned}
\tag{3.1}
$$

For convenience, we set:

$$
k_0 = \frac{1}{|\Omega|}\left[\frac{|A|T^{p+1}(p+2)}{\Gamma(p+2)} + |B|(T+\eta)\right], \tag{3.2}
$$

and

$$
p_0 = \frac{T^q}{\Gamma(q+1)} + \frac{T+\eta}{|\Omega|}\left[\frac{|A|T^{p+q}}{\Gamma(p+q+1)} + \frac{|B|\xi^q}{\Gamma(q+1)}\right] + k_0\frac{\eta^q}{\Gamma(q+1)}. \tag{3.3}
$$

Theorem 3.1. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $g : C([0,T],\mathbb{R}) \to \mathbb{R}$ be continuous functions. Assume that:*

(A_1) $|f(t,x) - f(t,y)| \le L|x-y|$, $\forall t \in [0,T]$, $L > 0$, $x, y \in \mathbb{R}$;
(A_2) $|g(u) - g(v)| \le \ell\|u-v\|$, $\ell < k_0^{-1}$ *for all* $u, v \in C([0,T],\mathbb{R})$;
(A_3) $\gamma := Lp_0 + \ell k_0 < 1$.

Then the boundary value problem (1.3) *has a unique solution.*

Proof. For $x, y \in \mathcal{C}$ and for each $t \in [0,T]$, from the definition of $\mathcal{Q}$, see (3.1), and assumptions (A_1) and (A_2), we obtain

$|(\mathcal{Q}x)(t) - (\mathcal{Q}y)(t)|$

$$
\begin{aligned}
&\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds \\
&\quad + \frac{|t-\eta||A|}{|\Omega|}\int_0^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s))-f(s,y(s))|ds \\
&\quad + \frac{|t-\eta||B|}{|\Omega|}\int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds \\
&\quad + \frac{1}{|\Omega|}\left|\frac{AT^p}{\Gamma(p+1)}\left(t-\frac{T}{p+1}\right)+B(t-\eta)\right|\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds \\
&\quad + \frac{1}{|\Omega|}\left|\frac{AT^p}{\Gamma(p+1)}\left(t-\frac{T}{p+1}\right)+B(t-\eta)\right||g(x)-g(y)| \\
&\leq L\|x-y\|\left[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds+\frac{(T+\eta)|A|}{|\Omega|}\int_0^T\frac{(T-s)^{p+q-1}}{\Gamma(p+q)}ds\right. \\
&\quad + \frac{(T+\eta)|B|}{|\Omega|}\int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)}ds \\
&\quad \left.+ \frac{1}{|\Omega|}\left(\frac{|A|T^{p+1}(p+2)}{\Gamma(p+2)}+|B|(T+\eta)\right)\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}ds\right] \\
&\quad + \frac{1}{|\Omega|}\left[\frac{|A|T^{p+1}(p+2)}{\Gamma(p+2)}+|B|(T+\eta)\right]\ell\|x-y\| \\
&\leq L\|x-y\|\left\{\frac{T^q}{\Gamma(q+1)}+\frac{T+\eta}{|\Omega|}\left[\frac{|A|T^{p+q}}{\Gamma(p+q+1)}+\frac{|B|\xi^q}{\Gamma(q+1)}\right]\right. \\
&\quad \left.+ \frac{1}{|\Omega|}\left(\frac{|A|T^{p+1}(p+2)}{\Gamma(p+2)}+|B|(T+\eta)\right)\frac{\eta^q}{\Gamma(q+1)}\right\} \\
&\quad + \frac{1}{|\Omega|}\left[\frac{|A|T^{p+1}(p+2)}{\Gamma(p+2)}+|B|(T+\eta)\right]\ell\|x-y\| \\
&= (Lp_0+\ell k_0)\|x-y\|.
\end{aligned}
$$

Hence

$$\|\mathcal{Q}x-\mathcal{Q}y\| \leq \gamma\|x-y\|.$$

As $\gamma < 1$, by (A_3), F is a contraction map from the Banach space $\mathcal{C}$ into itself. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Example 3.2. Consider the following fractional boundary value problem

$$
\begin{cases}
{}^cD^{3/2}x(t) = \dfrac{1}{(t+3)^2}\dfrac{|x|}{1+|x|}+1+\sin^2 t, \quad t\in[0,1], \\
x\left(\dfrac{1}{4}\right) = \dfrac{1}{5}x\left(\dfrac{1}{2}\right), \quad \dfrac{2}{3}I^{\frac{5}{2}}x(1)+\dfrac{3}{5}x\left(\dfrac{3}{4}\right)=\dfrac{7}{2}.
\end{cases}
\tag{3.4}
$$

Here, $q = 3/2$, $T = 1$, $p = 5/2$, $\eta = 1/4$, $A = 2/3$, $B = 3/5$, $\xi = 3/4$, $c = 7/2$, $f(t,x) = (1/(t+3)^2)(|x|/(1+|x|)) + 1 + \sin^2 t$ and $g(x) = (1/5)x$. It is easy to verify that $\Omega \approx 0.307164 \neq 0$, $k_0 \approx 3.281355$ and $p_0 \approx 2.366861$. As $|f(t,x) - f(t,y)| \leq (1/9)|x-y|$ and $|g(x) - g(y)| \leq (1/5)|x-y|$ therefore, (A_1) and (A_2) are satisfied with $L = 1/9$ and $\ell = 1/5$ such that $\ell k_0 \approx 0.656271 < 1$. Since $\gamma \approx 0.919256 < 1$, by the conclusion of Theorem 3.1, the boundary value problem (3.4) has a solution on $[0,1]$.

Next, we introduce the fixed point theorem which was established by O'Regan in [15]. This theorem will be adopted to prove the next main result.

Lemma 3.3. *Let U be an open set in a closed, convex set C of a Banach space E. Assume $0 \in U$. Also assume that $\mathcal{F}(\bar{U})$ is bounded and that $\mathcal{F} : \bar{U} \to C$ is given by $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, in which $\mathcal{F}_1 : \bar{U} \to E$ is continuous and completely continuous and $\mathcal{F}_2 : \bar{U} \to E$ is a nonlinear contraction (i.e., there exists a continuous nondecreasing function $\vartheta : [0,\infty) \to [0,\infty)$ satisfying $\vartheta(z) < z$ for $z > 0$, such that $\|\mathcal{F}_2(x) - \mathcal{F}_2(y)\| \leq \vartheta(\|x-y\|)$ for all $x, y \in \bar{U}$). Then, either*

(C1) *$\mathcal{F}$ has a fixed point $u \in \bar{U}$; or*

(C2) *there exist a point $u \in \partial U$ and $\kappa \in (0,1)$ with $u = \kappa\mathcal{F}(u)$, where $\bar{U}$ and ∂U, respectively, represent the closure and boundary of U on C.*

In the sequel, we will use Lemma 3.3 by taking C to be E. For more details of such fixed point theorems, we refer a paper [16] by Petryshyn.

Let

$$\Omega_r = \{x \in C([0,T],\mathbb{R}) : \|x\| < r\}.$$

Theorem 3.4. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that (A_2) holds. In addition, we assume that:*

(A_4) $g(0) = 0$;

(A_5) *there exists a nonnegative function $p \in C([0,T],\mathbb{R})$ and a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that*

$$|f(t,u)| \leq p(t)\psi(|u|) \text{ for any } (t,u) \in [0,T] \times \mathbb{R};$$

(A_6) $\sup_{r \in (0,\infty)} \frac{r}{p_0\|p\|\psi(r) + \frac{T+\eta}{|\Omega|}|c|} > \frac{1}{1-k_0\ell}$, *where p_0 and k_0 are defined in (3.3) and (3.2) respectively.*

Then the boundary value problem (1.3) has at least one solution on $[0,T]$.

Proof. Consider the operator $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ as defined in (3.1). We decompose $\mathcal{Q}$ into a sum of two operators

$$(\mathcal{Q}x)(t) = (\mathcal{Q}_1 x)(t) + (\mathcal{Q}_2 x)(t), \quad t \in [0,T], \tag{3.5}$$

where

$$
\begin{aligned}
(\mathcal{Q}_1x)(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\
& + \frac{t-\eta}{\Omega}\left[c - A\int_0^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)} f(s,x(s))ds \right.\\
& \left. - B\int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\right] \\
& + \frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t - \frac{T}{p+1}\Big) + B(t-\eta)\right]\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds,
\end{aligned} \tag{3.6}
$$

and

$$
(\mathcal{Q}_2x)(t) = -\frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t - \frac{T}{p+1}\Big) + B(t-\eta)\right] g(x), \quad t\in[0,T]. \tag{3.7}
$$

From (A_6) there exists a number $r_0 > 0$ such that

$$
\frac{r_0}{p_0\|p\|\psi(r_0) + \frac{T+\eta}{|\Omega|}|c|} > \frac{1}{1-k_0\ell}. \tag{3.8}
$$

We shall prove that the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy the conditions in Lemma 3.3.

Step 1. *The operator* $\mathcal{Q}_2 : \bar{\Omega}_{r_0} \to C([0,T],\mathbb{R})$ *is contractive.* Indeed, we have:

$$
\begin{aligned}
|(\mathcal{Q}_2x)(t) - (\mathcal{Q}_2y)(t)| & \le \frac{1}{|\Omega|}\left|\frac{AT^p}{\Gamma(p+1)}\Big(t - \frac{T}{p+1}\Big) + B(t-\eta)\right| |g(x)-g(y)| \\
& \le k_0\ell\|x-y\|,
\end{aligned}
$$

and hence by (A_2), $\mathcal{Q}_2$ is contractive.

Step 2. *The operator* $\mathcal{Q}_1$ *is continuous and completely continuous.* We first show that $\mathcal{Q}_1(\bar{\Omega}_{r_0})$ is bounded. For any $x \in \bar{\Omega}_{r_0}$ we have

$$
\begin{aligned}
\|\mathcal{Q}_1x\| \le & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))|ds \\
& + \frac{|t-\eta|}{|\Omega|}\left[|c| + |A|\int_0^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)} |f(s,x(s))|ds \right.\\
& \left. + |B|\int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} |f(s,x(s))|ds\right] \\
& \left. + \frac{1}{|\Omega|}\left(\frac{|A|T^{p+1}(p+2)}{\Gamma(p+2)} + |B|(T+\eta)\right)\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} |f(s,x(s))|ds\right] \\
\le & \|p\|\psi(r_0)\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{T+\eta}{|\Omega|}\left[|c| + |A|\int_0^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)} ds \right.\right.\\
& \left. + |B|\int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} ds\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{|\Omega|}\left(\frac{|A|T^{p+1}(p+2)}{\Gamma(p+2)}+|B|(T+\eta)\right)\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}ds\Bigg\}\\
&\le p_0\|p\|\psi(r_0)+\frac{T+\eta}{|\Omega|}|c|.
\end{aligned}
$$

This proves that $\mathcal{Q}_1(\bar{\Omega}_{r_0})$ is uniformly bounded.

In addition for any $t_1, t_2 \in [0,T]$, $t_1 < t_2$, we have:

$$
\begin{aligned}
&|(\mathcal{Q}_1x)(t_2)-(\mathcal{Q}_1x)(t_1)|\\
&\quad\le \int_0^{t_1}\frac{[(t_2-s)^{q-1}-(t_1-s)^{q-1}]}{\Gamma(q)}|f(s,x(s))|ds+\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\\
&\qquad+\frac{|t_2-t_1|}{|\Omega|}\Bigg\{|c|+|A|\int_0^T\frac{(T-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s))|ds\\
&\qquad+|B|\int_0^\xi\frac{(\xi-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\Bigg\}\\
&\qquad+\frac{|t_2-t_1|}{|\Omega|}\left(\frac{|A|T^p}{\Gamma(p+1)}+|B|\right)\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\\
&\quad\le\frac{\|p\|\psi(r_0)}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]ds+\frac{\|p\|\psi(r_0)}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}ds\\
&\qquad+\frac{\|p\|\psi(r_0)|t_2-t_1|}{|\Omega|}\Bigg\{|c|+|A|\frac{T^{p+q}}{\Gamma(p+q+1)}+|B|\frac{\xi^q}{\Gamma(q+1)}\\
&\qquad+\left(\frac{|A|T^p}{\Gamma(p+1)}+|B|\right)\frac{\eta^q}{\Gamma(q+1)}\Bigg\},
\end{aligned}
$$

which is independent of x and tends to zero as $t_2-t_1\to 0$. Thus, $\mathcal{Q}_1$ is equicontinuous. Hence, by the Arzelà-Ascoli Theorem, $\mathcal{Q}_1(\bar{\Omega}_{r_0})$ is a relatively compact set. Now, let $x_n \subset \bar{\Omega}_{r_0}$ with $\|x_n - x\| \to 0$. Then the limit $|x_n(t)-x(t)|\to 0$ is uniformly valid on $[0,T]$. From the uniform continuity of $f(t,x)$ on the compact set $[0,T]\times[-r_0,r_0]$, it follows that $\|f(t,x_n(t))-f(t,x(t))\|\to 0$ is uniformly valid on $[0,T]$. Hence $\|\mathcal{Q}_1x_n-\mathcal{Q}_1x\|\to 0$ as $n\to\infty$ which proves the continuity of $\mathcal{Q}_1$. Hence Step 2 is completely proved.

Step 3. *The set $F(\bar{\Omega}_{r_0})$ is bounded.* (A_2) and (A_4) imply that

$$\|\mathcal{Q}_2(x)\|\le k_0\ell r_0,$$

for any $x\in\bar{\Omega}_{r_0}$. This, with the boundedness of the set $\mathcal{Q}_1(\bar{\Omega}_{r_0})$, implies that the set $\mathcal{Q}(\bar{\Omega}_{r_0})$ is bounded.

Step 4. *Finally, we show that the case* (C2) *in Lemma* 3.3 *does not occur.* To this end, we suppose that (C2) holds. Then, we have that there exist $\lambda\in(0,1)$ and

$x \in \partial\Omega_{r_0}$ such that $x = \lambda \mathcal{Q}x$. So, we have $\|x\| = r_0$ and

$$\begin{aligned} x(t) &= \lambda(\mathcal{Q}x)(t) \\ &= \lambda\Bigg\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &\quad + \frac{t-\eta}{\Omega}\left[c - A\int_0^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)} f(s,x(s))ds - B\int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\right] \\ &\quad + \frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\left(t - \frac{T}{p+1}\right) + B(t-\eta)\right]\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &\quad - \frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\left(t - \frac{T}{p+1}\right) + B(t-\eta)\right] g(x)\Bigg\}, \quad t \in [0,T]. \end{aligned}$$

With hypotheses (A_4)-(A_6), and similar computations, as in Step 2, we have

$$|x(t)| \le \psi(\|x\|)\|p\|p_0 + \frac{T+\eta}{|\Omega|}|c| + k_0\ell\|x\|.$$

Taking the supremum over all t gives

$$\|x\| \le \psi(\|x\|)\|p\|p_0 + \frac{T+\eta}{|\Omega|}|c| + k_0\ell\|x\|,$$

which implies

$$\frac{r_0}{p_0\|p\|\psi(r_0) + \frac{T+\eta}{|\Omega|}|c|} \le \frac{1}{1-k_0\ell},$$

contradicting (3.8). Consequently, we have proved that the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy all the conditions in Lemma 3.3. Hence, the operator $\mathcal{Q}$ has at least one fixed point $x \in \bar{\Omega}_{r_0}$, which is the solution of the boundary value problem (1.3). The proof is completed. □

Example 3.5. Consider the following fractional boundary value problem

$$\begin{cases} {}^cD^{3/2}x(t) = \dfrac{t}{20}\left(|x| + \dfrac{1}{1+|x|}\right), \quad t \in [0,1], \\ x\left(\dfrac{1}{3}\right) = \dfrac{1}{19}\sin\left(x\left(\dfrac{1}{2}\right)\right), \quad \dfrac{4}{5}I^{5/2}x(1) + \dfrac{3}{4}x\left(\dfrac{2}{3}\right) = \dfrac{1}{25}. \end{cases} \tag{3.9}$$

Here $q = 3/2$, $T = 1$, $p = 5/2$, $\eta = 1/3$, $\xi = 2/3$, $A = 4/5$, $B = 3/4$, $c = 1/25$, $f(t,x) = (t/20)(|x| + (1/(1+|x|)))$ and $g(x) = (1/19)\sin x$ with $g(0) = 0$. Since $|f(t,x)| = |(t/20)(|x| + (1/(1+|x|)))| \le (t/20)(x^2 + |x| + 1)$, we choose $p(t) = t/20$ and $\psi(|x|) = x^2 + |x| + 1$. It is easy to verify that $\Omega \approx 0.238537 \neq 0$, $k_0 \approx 4.840962$, $p_0 \approx 3.356013$. Since $|g(x) - g(y)| \le (1/19)|x-y|$, we have $\ell = (1/19)$ such that $\ell k_0 \approx 0.254787 < 1$. We find that

$$\sup_{r\in(0,\infty)} \frac{r}{p_0\|p\|\psi(r) + \frac{T+\eta}{|\Omega|}|c|} \approx 1.469846 > (1/(1-k_0\ell)) \approx 1.341899.$$

Hence, by Theorem 3.4, the boundary value problem (3.9) has a solution on $[0,1]$.

4. EXISTENCE RESULTS FOR THE MULTI-VALUED CASE

In this section, we deal with the problem (1.4). Let $L^1([0,T],\mathbb{R})$ be the Banach space of measurable functions $x:[0,T]\to\mathbb{R}$ which are Lebesgue integrable with the norm $\|x\|_{L^1}=\int_0^T|x(t)|dt$. In the following, $AC^1([0,T],\mathbb{R})$ will denote the space of functions $x:[0,T]\to\mathbb{R}$ that are absolutely continuous and whose first derivatives are absolutely continuous.

Definition 4.1. A function $x\in AC^1([0,T],\mathbb{R})$ is a solution of the problem (1.4) if $x(\eta)=g(x)$, $AI^px(T)+Bx(\xi)=c$, and there exists a function $f\in L^1([0,T],\mathbb{R})$ such that $f(t)\in F(t,x(t))$ a.e. on $[0,T]$ and

$$\begin{aligned}
x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds \qquad (4.1)\\
&\quad+\frac{t-\eta}{\Omega}\left[c-A\int_0^T\frac{(T-s)^{p+q-1}}{\Gamma(p+q)}f(s)ds-B\int_0^\xi\frac{(\xi-s)^{q-1}}{\Gamma(q)}f(s)ds\right]\\
&\quad+\frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t-\frac{T}{p+1}\Big)+B(t-\eta)\right]\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}f(s)ds\\
&\quad-\frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t-\frac{T}{p+1}\Big)+B(t-\eta)\right]g(x),\quad t\in[0,T].
\end{aligned}$$

To prove our main result in this section we will use the following form of the Nonlinear Alternative for contractive maps [17, Corollary 3.8].

Lemma 4.2. *Let X be a Banach space, and D a bounded neighborhood of $0\in X$. Let $Z_1:X\to\mathcal{P}_{cp,c}(X)$ (here $\mathcal{P}_{cp,c}(X)$ denotes the family of all nonempty, compact and convex subsets of X) and $Z_2:\bar{D}\to\mathcal{P}_{cp,c}(X)$ two multi-valued operators satisfying*

(a) *Z_1 is contraction, and*
(b) *Z_2 is upper semicontinuous (u.s.c) and compact.*

Then, if $G=Z_1+Z_2$, either

(i) *G has a fixed point in $\bar{D}$, or*
(ii) *there is a point $u\in\partial D$ and $\lambda\in(0,1)$ with $u\in\lambda G(u)$.*

Definition 4.3. A multivalued map $F:[0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t\longmapsto F(t,x)$ is measurable for each $x\in\mathbb{R}$;
(ii) $x\longmapsto F(t,x)$ is upper semicontinuous for almost all $t\in[0,T]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha>0$, there exists $\varphi_\alpha\in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\|=\sup\{|v|:v\in F(t,x)\}\le\varphi_\alpha(t)$$

for all $\|x\| \leq \alpha$ and for a.e. $t \in [0,T]$.

For each $y \in C([0,T],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,T],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,T]\}.$$

The following lemma will be also used in the sequel.

Lemma 4.4 ([13]). *Let X be a Banach space. Let $F : [0,T] \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,T],X)$ to $C([0,T],X)$. Then the operator*

$$\Theta \circ S_F : C([0,T],X) \to \mathcal{P}_{cp,c}(C([0,T],X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,T],X) \times C([0,T],X)$.

Theorem 4.5. *Assume that (A_2) holds. In addition we suppose that:*

(H_1) *$F : [0,T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory multivalued map;*

(H_2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,T],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq p(t)\psi(\|x\|) \text{ for each } (t,x) \in [0,T] \times \mathbb{R};$$

and

(H_3) *there exists a number $M > 0$ such that*

$$\frac{(1-k_0\ell)M}{p_0\|p\|\psi(M) + \frac{T+\eta}{|\Omega|}|c|} > 1, \tag{4.2}$$

where p_0 and k_0 are defined in (3.3) and (3.2), respectively.

Then the boundary value problem (1.4) has at least one solution on $[0,T]$.

Proof. Transform the problem (1.4) into a fixed point problem. Consider the operator $\mathcal{N} : C([0,T],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,T],\mathbb{R}))$ defined by:

$$\begin{aligned}
\mathcal{N}(x) = \{&h \in C([0,T],\mathbb{R}) : \\
h(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\
&+ \frac{t-\eta}{\Omega}\left[c - A\int_0^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)} f(s)ds \right. \\
&\left. - B\int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f(s)ds\right] \\
&+ \frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t - \frac{T}{p+1}\Big) + B(t-\eta)\right]\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s)ds
\end{aligned}$$

$$-\frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t-\frac{T}{p+1}\Big)+B(t-\eta)\right]g(x)\Bigg\},$$

for $f \in S_{F,x}$.

Now, we define two operators as follows: $\mathcal{A} : C([0,T],\mathbb{R}) \longrightarrow C([0,T],\mathbb{R})$ by

$$\mathcal{A}x(t) = -\frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t-\frac{T}{p+1}\Big)+B(t-\eta)\right]g(x), \tag{4.3}$$

and the multi-valued operator $\mathcal{B} : C([0,T],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,T],\mathbb{R}))$ by

$$\begin{aligned}\mathcal{B}(x) = \Bigg\{ & h \in C([0,T],\mathbb{R}) : \\ h(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds \\ & + \frac{t-\eta}{\Omega}\left[c - A\int_0^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)}f(s)ds - B\int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)}f(s)ds\right] \\ & + \frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t-\frac{T}{p+1}\Big)+B(t-\eta)\right]\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}f(s)ds\Bigg\},\end{aligned}$$

for $f \in S_{F,x}$. Then $\mathcal{N} = \mathcal{A} + \mathcal{B}$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.2 on $[0,T]$. For better readability, we break the proof into a sequence of steps and claims.

Step 1: We show that $\mathcal{A}$ *is a contraction on* $C([0,T],\mathbb{R})$. The proof is similar to the one for the operator $\mathcal{Q}_2$ in Step 2 of Theorem 3.4.

Step 2: We shall show that the operator $\mathcal{B}$ *is compact and convex valued and it is completely continuous.* This will be given in several claims.

CLAIM I: $\mathcal{B}$ *maps bounded sets into bounded sets in* $C([0,T],\mathbb{R})$. To see this, let $B_\rho = \{x \in C([0,T],\mathbb{R}) : \|x\| \le \rho\}$ be a bounded set in $C([0,T],\mathbb{R})$. Then, for each $h \in \mathcal{B}(x)$, $x \in B_\rho$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned}h(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds \\ & + \frac{t-\eta}{\Omega}\left[c - A\int_0^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)}f(s)ds - B\int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)}f(s)ds\right] \\ & + \frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t-\frac{T}{p+1}\Big)+B(t-\eta)\right]\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}f(s)ds.\end{aligned}$$

Then for $t \in [0,T]$ we have

$$|h(t)| \le \|p\|\psi(\|x\|)\Bigg\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds + \frac{T+\eta}{|\Omega|}\left[|A|\int_0^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)}ds\right.$$

$$+|B|\int_0^{\xi}\frac{(\xi-s)^{q-1}}{\Gamma(q)}ds\Bigg]$$
$$+\frac{1}{|\Omega|}\left(\frac{|A|T^{p+1}(p+2)}{\Gamma(p+2)}+|B|(T+\eta)\right)\int_0^{\eta}\frac{(\eta-s)^{q-1}}{\Gamma(q)}ds\Bigg\}+\frac{T+\eta}{|\Omega|}|c|.$$

Thus,

$$\|h\|\le\|p\|\psi(\rho)p_0+\frac{T+\eta}{|\Omega|}|c|.$$

Claim II: Next we show that $\mathcal{B}$ *maps bounded sets into equicontinuous sets.* Let $t_1,t_2\in[0,T]$ with $t_1<t_2$ and $x\in B_\rho$. For each $h\in\mathcal{B}(x)$, we obtain

$$\begin{aligned}
&|h(t_2)-h(t_1)|\\
&\le\frac{\|p\|\psi(\rho)}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]ds+\frac{\|p\|\psi(\rho)}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}ds\\
&\quad+\frac{\|p\|\psi(\rho)(t_2-t_1)}{|\Omega|}\Bigg\{|c|+|A|\frac{T^{p+q}}{\Gamma(p+q+1)}+|B|\frac{\xi^q}{\Gamma(q+1)}\\
&\quad+\left(\frac{|A|T^p}{\Gamma(p+1)}+|B|\right)\frac{\eta^q}{\Gamma(q+1)}\Bigg\}.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_\rho$ as $t_2-t_1\to 0$. Therefore it follows by the Arzelà-Ascoli theorem that $\mathcal{B}:C([0,T],\mathbb{R})\to\mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.

Claim III: It is well known [11, Proposition 3.1] that if an operator is completely continuous and has a closed graph, then it is u.s.c. Thus, we prove that $\mathcal{B}$ *has a closed graph.* Let $x_n\to x_*, h_n\in\mathcal{B}(x_n)$ and $h_n\to h_*$. Then we need to show that $h_*\in\mathcal{B}(x_*)$. Associated with $h_n\in\mathcal{B}(x_n)$, there exists $f_n\in S_{F,x_n}$ such that for each $t\in[0,T]$,

$$\begin{aligned}
h_n(t)=&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f_n(s)ds\\
&+\frac{t-\eta}{\Omega}\left[c-A\int_0^T\frac{(T-s)^{p+q-1}}{\Gamma(p+q)}f_n(s)ds-B\int_0^{\xi}\frac{(\xi-s)^{q-1}}{\Gamma(q)}f_n(s)ds\right]\\
&+\frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t-\frac{T}{p+1}\Big)+B(t-\eta)\right]\int_0^{\eta}\frac{(\eta-s)^{q-1}}{\Gamma(q)}f_n(s)ds.
\end{aligned}$$

Thus, it suffices to show that there exists $f_*\in S_{F,x_*}$ such that for each $t\in[0,T]$,

$$\begin{aligned}
h_*(t)=&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f_*(s)ds\\
&+\frac{t-\eta}{\Omega}\left[c-A\int_0^T\frac{(T-s)^{p+q-1}}{\Gamma(p+q)}f_*(s)ds-B\int_0^{\xi}\frac{(\xi-s)^{q-1}}{\Gamma(q)}f_*(s)ds\right]
\end{aligned}$$

$$+\frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t-\frac{T}{p+1}\Big)+B(t-\eta)\right]\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}f_*(s)ds.$$

Let us consider the linear operator $\Theta : L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ given by

$$\begin{aligned}
f \mapsto \Theta(f)(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds \\
&+\frac{t-\eta}{\Omega}\left[c-A\int_0^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)}f(s)ds - B\int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)}f(s)ds\right] \\
&+\frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t-\frac{T}{p+1}\Big)+B(t-\eta)\right]\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}f(s)ds.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\|h_n(t)-h_*(t)\| \\
&=\Bigg\|\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}(f_n(s)-f_*(s))ds \\
&\quad+\frac{t-\eta}{\Omega}\left[c-A\int_0^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)}(f_n(s)-f_*(s))ds\right. \\
&\quad\left.-B\int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)}(f_n(s)-f_*(s))ds\right] \\
&\quad+\frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t-\frac{T}{p+1}\Big)+B(t-\eta)\right]\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}(f_n(s)-f_*(s))ds\Bigg\| \to 0,
\end{aligned}$$

as $n\to\infty$. Thus, it follows by Lemma 4.4 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, therefore, we have

$$\begin{aligned}
h_*(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f_*(s)ds \\
&+\frac{t-\eta}{\Omega}\left[c-A\int_0^T \frac{(T-s)^{p+q-1}}{\Gamma(p+q)}f_*(s)ds - B\int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)}f_*(s)ds\right] \\
&+\frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t-\frac{T}{p+1}\Big)+B(t-\eta)\right]\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}f_*(s)ds,
\end{aligned}$$

for some $f_*\in S_{F,x_*}$. Hence $\mathcal{B}$ has a closed graph (and therefore has closed values). As a result $\mathcal{B}$ is compact valued. Therefore the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.2 and hence an application of it yields that either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x\in\lambda\mathcal{A}(x)+\lambda\mathcal{B}(x)$ for $\lambda\in(0,1)$, then there exists $f\in S_{F,x}$ such that

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds$$

$$
\begin{aligned}
&+\frac{t-\eta}{\Omega}\left[c-A\int_0^T\frac{(T-s)^{p+q-1}}{\Gamma(p+q)}f(s)ds-B\int_0^\xi\frac{(\xi-s)^{q-1}}{\Gamma(q)}f(s)ds\right]\\
&+\frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t-\frac{T}{p+1}\Big)+B(t-\eta)\right]\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}f(s)ds\\
&-\frac{1}{\Omega}\left[\frac{AT^p}{\Gamma(p+1)}\Big(t-\frac{T}{p+1}\Big)+B(t-\eta)\right]g(x),\quad t\in[0,T].
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
|x(t)|\le\ &\|p\|\psi(\|x\|)\Bigg\{\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}ds+\frac{T+\eta}{|\Omega|}\left[|A|\int_0^T\frac{(T-s)^{p+q-1}}{\Gamma(p+q)}ds\right.\\
&\left.+|B|\int_0^\xi\frac{(\xi-s)^{q-1}}{\Gamma(q)}ds\right]\\
&+\frac{1}{|\Omega|}\left(\frac{|A|T^{p+1}(p+2)}{\Gamma(p+2)}+|B|(T+\eta)\right)\int_0^\eta\frac{(\eta-s)^{q-1}}{\Gamma(q)}ds\Bigg\}+\frac{T+\eta}{|\Omega|}|c|\\
&+\frac{1}{|\Omega|}\left(\frac{|A|T^{p+1}(p+2)}{\Gamma(p+2)}+|B|(T+\eta)\right)\ell\|x\|\\
\le\ & p_0\|p\|\psi(\|x\|)+\frac{T+\eta}{|\Omega|}|c|+k_0\ell\|x\|,
\end{aligned}
$$

or taking the supremum over t,

$$\|x\|\le p_0\|p\|\psi(\|x\|)+\frac{T+\eta}{|\Omega|}|c|+k_0\ell\|x\|.$$

If condition (ii) of Theorem 4.2 holds, then there exists $\lambda\in(0,1)$ and $x\in\partial B_M$ with $x=\lambda\mathcal{N}(x)$. Then, x is a solution of (3.5) with $\|x\|=M$. Now, the previous inequality implies

$$\frac{(1-k_0\ell)M}{p_0\|p\|\psi(M)+\frac{T+\eta}{|\Omega|}|c|}\le 1$$

which contradicts (4.2). Hence, $\mathcal{N}$ has a fixed point in $[0,T]$ by Theorem 4.2, and consequently the boundary value problem (1.4) has a solution. This completes the proof. □

Remark 4.6. We note that the following boundary value problem

$$
\begin{cases}
{}^cD^qx(t)=f(t,x(t)),\quad 0\le t\le T,\ 1<q\le 2,\\
x(\eta)=g(x),\quad AI^px(\xi)+Bx(T)=c,\ 0<\eta<\xi<T,
\end{cases}
\tag{4.4}
$$

can be treated in a similar way.

Example 4.7. Consider the following fractional boundary value problem

$$
\begin{cases}
{}^cD^{3/2}x(t)\in F(t,x),\ t\in[0,1],\\
x\left(\dfrac{1}{2}\right)=\dfrac{|x(8/9)|}{16(1+|x(8/9)|)},\quad \dfrac{3}{8}I^{5/2}x(1)+\dfrac{5}{6}x\left(\dfrac{3}{4}\right)=\dfrac{1}{32}.
\end{cases}
\tag{4.5}
$$

Here $q = 3/2$, $T = 1$, $p = 5/2$, $\eta = 1/2$, $\xi = 3/4$, $A = 3/8$, $B = 5/6$, $c = 1/32$, $g(x) = (|x|/(16(1+|x|)))$ and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$x \to F(t,x) = \left[\frac{t|x|(\cos^2 x + 1)}{2(|x|+1)}, \frac{(t+1)x^2}{x^2+1}\right].$$

We find that $\Omega \approx 0.184154 \neq 0$, $k_0 \approx 7.575611$ and $p_0 \approx 6.210884$. It is easy to verify that $|g(x) - g(y)| \leq (1/16)|x - y|$ with $\ell = 1/16$ such that $\ell k_0 \approx 0.473476 < 1$.

For $f \in F$, we have

$$|f| \leq \max\left(\frac{t|x|(\cos^2 x + 1)}{2(|x|+1)}, \frac{(t+1)x^2}{x^2+1}\right) \leq t + 1, \quad x \in \mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| \, : \, y \in F(t,x)\} \leq p(t)\psi(\|x\|), \quad x \in \mathbb{R},$$

with $p(t) = t + 1$, $\psi(\|x\|) = 1$. With the given data, it found that $M > 18.177459$. Clearly, all the conditions of Theorem 4.5 are satisfied. Hence, the boundary value problem (4.5) has at least one solution on $[0,1]$.

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