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INTEGRATING PATH-DEPENDENT FUNCTIONALS ON YEH-WIENER SPACE

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ABSTRACT. Denote by $C_{a,b}(Q)$ the generalized two-parameter Yeh-Wiener space with associated Gaussian measure. We investigate several scenarios in which integrals of functionals on this space can be reduced to integrals of related functionals over an appropriate single-parameter generalized Wiener space $C_{\hat{a},\hat{b}}[0,T]$. This extends some interesting results of R. H. Cameron and D. A. Storvick.

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1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space (named after Norbert Wiener who did some of the earliest work in this area); that is the space of all continuous real-valued functions x on $[0,T]$ with $x(0)=0$. During the past 75 years many people have made substantial contributions in studying this space with important applications to both physics and mathematics. In particular there is a considerable body of work relating to what we now call generalized Wiener spaces. Earlier discussions of these spaces can be found in [8, 22] and more recent developments can be found in [5, 6, 7, 9, 10]; the references listed in these papers led to a very large collection of other results.

The space which we will refer to as Yeh-Wiener space was introduced in [25] with further results in [23, 24]. In these papers, Yeh explored the structure and behavior of a Gaussian measure analogous to the classical Weiner measure but defined on the space of continuous functions of two variables. The associated stochastic process is often called the Brownian sheet. See [3, 12, 13, 14, 15, 16, 17, 18] for more information and examples.

The setting for this paper involves what we term a generalized Yeh-Wiener space. This function space extends the ordinary Yeh-Wiener space in a similar manner to

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the single-parameter case. The main ideas and results of this paper follow and expand on those found in [2]. The primary goal is to relate certain integrals on a general two-parameter Wiener space with corresponding integrals on a general single-parameter space.

2. Definitions and Preliminaries

In [25], Yeh described the properties of a measure similar to Wiener measure on the space of continuous functions of two variables defined on the unit square. We are concerned with a family of similar measures.

Let Q denote the rectangle $[0,S]\times[0,T]$ in $\mathbb{R}^2$ and let $\leq$ be the usual partial order on Q such that $s\leq t$ if and only if each $s_j\leq t_j$. Also let $a(s,t)$ be an absolutely continuous function with derivative $\frac{\partial^2 a}{\partial s\partial t}\in L^2(Q)$ and $a(0,0)=0$, and let $b(s,t)$ be an absolutely continuous function with a continuous derivative $\frac{\partial^2 b(s,t)}{\partial s\partial t}>0$ on Q.

The functions a and b act to determine the center (or mean), and variance of the generalized Yeh-Wiener measure. We list several properties and useful basic results in this section; see Chapter 3 of [19] for a more detailed discussion of these matters.

A generalized Yeh-Wiener measure is a Gaussian measure on the space of continous functions $C(Q)$. Therefore, the distribution of finite-dimensional projections of this space with respect to this measure are Gaussian, with the following basic form. For $0<s_1<\cdots<s_m\leq S$ and $0<t_1<\cdots<t_n\leq T$, the distribution of a finite-dimensional projection onto $\mathbb{R}^{mn}$ with component projections $\{\delta_{(s_i,t_j)}:1\leq i\leq m, 1\leq j\leq n\}$ (the generalized Yeh-Wiener kernel) is given by

$$W_{m,n}(\mathbf{u},\mathbf{s},\mathbf{t})=\left(\prod_{i=1}^{m}\prod_{j=1}^{n}2\pi\Delta_i\Delta_j b(s,t)\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{(\Delta_i\Delta_j(u-a(s,t)))^2}{\Delta_i\Delta_j b(s,t)}\right),$$

where $\Delta_i\Delta_j u=u_{i,j}-u_{i-1,j}-u_{i,j-1}+u_{i-1,j-1}$ and $u_{i,0}=u_{0,j}=0$ for all $i,j\geq 0$.

We will let $\mathfrak{m}$ denote the generalized Yeh-Wiener measure on $C_0(Q)$ determined by the functions a and b and will write $C_{a,b}(Q)$ for the resulting measure space.

A function f on Q is said to be of bounded variation in the sense of Hardy-Krause provided that $\sup_{P\in\mathcal{P}}\left\{\sum_{R_j\in P}\left|\Delta_{R_j}f\right|\right\}<\infty$ over all finite partitions P of Q into non-degenerate rectangles $\{R_j\}$ and the restriction of f to any vertical or horizontal line in Q yields a single-variable function of bounded variation in the usual sense. We refer to the collection of such functions as $BV(Q)$. For a more detailed discussion of these functions and their properties see [1].

By $L^2_{a,b}(Q)$ we denote the collection of functions on Q that are square integrable with respect to the Lebesgue-Stieltjes measure induced by the functions a and b. That is,

$$L^2_{a,b}(Q)=\left\{f:Q\to\mathbb{R}:\int_Q f^2(s,t)\mathrm{d}(b(s,t)+|a|(s,t))<\infty\right\}.$$

The space $L^2_{a,b}(Q)$ is in fact a Hilbert space (as our notation suggests), and has the obvious inner product

$$(f,g)_{a,b} = \int_Q f(s,t)g(s,t)\mathrm{d}(b(s,t) + |a|(s,t)).$$

In addition, by $\|\cdot\|_b$ and $(\cdot,\cdot)_b$ we denote, respectively, the L^2-norm of a function and the inner product with respect to the Lebesgue-Stieltjes measure induced by b; that is

$$\|f\|_b^2 = \int_Q f^2(s,t)\mathrm{d}b(s,t).$$

and

$$(f,g)_b = \int_Q f(s,t)g(s,t)\mathrm{d}b(s,t).$$

Note that the conditions on $b(s,t)$ ensure that $L^2_b(Q)$ is equivalent to $L^2(Q)$ and further note that $BV(Q)$ is a subset of both $L^2_b(Q)$ and $L^2_{a,b}(Q)$.

We next define the Paley-Wiener-Zygmund stochastic integral of a function $f \in L^2_{a,b}(Q)$, which is a basic tool in understanding how the measure works.

Definition 2.1. Let $\{\phi_j\}$ be a complete orthonormal set of functions of bounded variation in $L^2_{a,b}(Q)$. For $f \in L^2_{a,b}(Q)$, put

$$I_n f(x) = \sum_{j=1}^{n} (f,\phi_j)_{a,b} \int_Q \phi_j(u)\mathrm{d}x(u),$$

Define the Paley-Wiener-Zygmund (PWZ) stochastic integral $If(x) = \lim_{n\to\infty} I_n f(x)$ for all $x \in C_{a,b}(Q)$ for which this limit exists.

The following theorem is fundamental in computing integrals over $C_{a,b}(Q)$. It gives some essential properties of the PWZ integral.

Theorem 2.2. 1. *If $f \in L^2_{a,b}(Q)$, then the PWZ stochastic integral $If(x)$ exists for a.e. $x \in C_{a,b}(Q)$ and is essentially independent of the choice of orthonormal basis in Definition* 2.1.

2. *If $f \in L^2_{a,b}(Q)$, then If is a Gaussian random variable with mean $If(a)$ and variance $\|f\|^2_{L^2_b(Q)}$.*
3. *If f and g are in $L^2_{a,b}(Q)$, then the covariance of the random variables If and Ig is $(f,g)_{L^2_b(Q)}$.*

Remark 2.3. We pause briefly to note that the order of measurability assumptions in the following theorems, where the Lebesgue measurability of f is assumed and the μ-measurability of F is a conclusion, is not actually necessary. By similar arguments to those found in [4, 11, 21], the hypothesis of measurability can be either that F is μ-measurable on $C_{a,b}(Q)$ or that f is Lebesgue measurable, and the measurability of one of these will imply the measurability of the other.

Theorem 2.4 (Tame Functionals)**.** *Let $0 < s_1 < \cdots < s_m \le S$ and $0 < t_1 < \cdots < t_n \le T$ and let $f : \mathbb{R}^{mn} \to \mathbb{C}$ and $F : C_{a,b}(Q) \to \mathbb{C}$ be defined by $F(x) = f(x(s_1,t_1),\dots,x(s_m,t_n))$. Then F is measurable if and only if f is Lebesgue measurable, and in this case,*

$$\int_{C_{a,b}(Q)} F(x)\mathfrak{m}(dx) \overset{*}{=} \int_{\mathbb{R}^{mn}} f(u_{1,1},\dots u_{m,n})W_{m,n}(\mathbf{u},\mathbf{s},\mathbf{t})\mathrm{d}\mathbf{u}, \tag{2.1}$$

where the equality $(*)$ is in the sense that if one of the integrals exists then the other exists and the equality holds.

Proof. Let $\phi_{i,j} = \chi_{[s_{i-1},s_i]\times[t_{j-1},t_j]}(u,v)$. It is easy to calculate

$$x(s_k,t_l) = \sum_{\substack{0\le i\le k\\ 0\le j\le l}} \Delta_i\Delta_j x(s,t)$$

for any (s_k,t_l). It is not difficult to see that

$$\Delta_i\Delta_j x(s,t) = I\phi_{i,j}(x) = \int_Q \chi_{[s_{i-1},s_i]\times[t_{j-1},t_j]}(u,v)\mathrm{d}x(u,v),$$

whence we have

$$\begin{aligned} F(x) &= f(x(s_1,t_1),\dots,x(s_m,t_n)) \\ &= f\left(I\phi_{1,1}(x),\dots,\sum_{\substack{0\le i\le k\\ 0\le j\le l}} I\phi_{i,j}(x),\dots,\sum_{\substack{0\le i\le m\\ 0\le j\le n}} I\phi_{i,j}(x)\right). \end{aligned}$$

As $\phi_{i,j} \in BV(Q)$, we can use Theorem 2.2 to see that

$$\int_{C_{a,b}(Q)} I\phi_{i,j}(x)\mu(dx) = I\phi_{i,j}(a) = \Delta_i\Delta_j a(s,t),$$

and also observe that

$$\begin{aligned} &\int_{C_{a,b}(Q)} \left(I\phi_{i,j}(x) - I\phi_{i,j}(a)\right)\left(I\phi_{l,m}(x) - I\phi_{l,m}(a)\right)\mathfrak{m}(dx) \\ &\qquad = \int_Q \phi_{i,j}(u,v)\phi_{l,m}(u,v)\mathrm{d}b(u,v) \\ &\qquad = \begin{cases} \Delta_i\Delta_j b(s,t) & \text{if } i=l, j=m \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Accordingly, we see that the covariance matrix M for the collection $\{\phi_{i,j}\}$ is a diagonal matrix whose nonzero entries are $\Delta_i\Delta_j b(s,t)$. Now we can apply Theorem 2.2 to complete the proof. □

3. Integrals Over Paths

We will first be concerned with integrating functionals defined in terms of certain paths in Q. We confine our discussion to paths $\phi : [0, S] \to Q$ for which $\phi(\tau) = (\phi_1(\tau), \phi_2(\tau))$ satisfies the condition that its component functions ϕ_1 and ϕ_2 are each piecewise continuously differentiable. We will say that ϕ is an increasing path in Q if $\phi' \cdot \phi' > 0$ on $[0, S]$ and $\phi(\tau_1) \le \phi(\tau_2)$ whenever $\tau_1 \le \tau_2$.

The first theorem in this section establishes a special case of the tame functionals theorem in the case that one defines the functional in terms of a sequence of points lying on an increasing path.

Theorem 3.1. *Let $0 = s_0 < s_1 \le \cdots \le s_n \le S$ and $0 = t_0 < t_1 \le \cdots \le t_n \le T$ and let $f : \mathbb{R}^n \to \mathbb{C}$ be Lebesgue measurable. If $F : C_{a,b}(Q) \to \mathbb{C}$ is defined by $F(x) = f(x(s_1, t_1)), x(s_2, t_2), \ldots, x(x_n, t_n))$, then F is μ-measurable and*

$$\int_{C_{a,b}(Q)} F(x)\mathfrak{m}(dx) \overset{*}{=} \left(\prod_{j=1}^{n} 2\pi(b(s_j, t_j) - b(s_{j-1}, t_{j-1})\right)^{-\frac{1}{2}} \int_{\mathbb{R}^n} f(u_1, \ldots, u_n) \tag{3.1}$$
$$\exp\left(-\frac{1}{2}\sum_{j=1}^{n} \frac{(u_j - a(s_j, t_j) - u_{j-1} + a(s_{j-1}, t_{j-1}))^2}{b(s_j, t_j) - b(s_{j-1}, t_{j-1})}\right) \mathrm{d}u_1 \cdots \mathrm{d}u_n,$$

where the equality ($\overset{*}{=}$) is in the sense that if one of the integrals exists then the other exists and the equality holds.

Proof. The proof is by induction on n. The theorem is certainly true for $n = 1$ because by (2.1),

$$\int_{C_{a,b}(Q)} f(x(s_1, t_1))\mathfrak{m}(dx)$$
$$= \left(\prod_{i=1}^{1}\prod_{j=1}^{1} 2\pi\Delta_i\Delta_j b(s, t)\right)^{-\frac{1}{2}} \int_{\mathbb{R}} f(u_{1,1}) \exp\left(-\frac{1}{2}\sum_{i=1}^{1}\sum_{j=1}^{1} \frac{(\Delta_i\Delta_j(u - a(s, t)))^2}{\Delta_i\Delta_j b(s, t)}\right) \mathrm{d}u_{1,1}$$
$$= \frac{1}{\sqrt{2\pi b(s_1, t_1)}} \int_{\mathbb{R}} f(u_1) \exp\left(-\frac{1}{2}\sum_{j=1}^{1} \frac{(u_j - a(s_j, t_j))^2}{b(s_j, t_j) - b(s_{j-1}, t_{j-1})}\right) \mathrm{d}u_1,$$

and thus (3.1) holds.

Assume that the result holds for $n = k \ge 1$. Then for $n = k + 1$ we have

$$\int_{C_{a,b}(Q)} f(x(s_1, t_1), \ldots, x(s_n, t_n))\mu(dx) =$$
$$\left(\prod_{i=1}^{k+1}\prod_{j=1}^{k+1} 2\pi\Delta_i\Delta_j b(s, t)\right)^{-\frac{1}{2}} \int_{\mathbb{R}^{(k+1)^2}} f(u_{1,1}, \ldots, u_{k+1,k+1})$$

$$\exp\left(-\frac{1}{2}\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\frac{(\Delta_i\Delta_j(u-a(s,t)))^2}{\Delta_i\Delta_j b(s,t)}\right)\mathrm{d}u_{1,1}du_{1,2}du_{2,1}\cdots \mathrm{d}u_{k+1,k+1}. \quad (3.2)$$

Note that for $j=1,\dots,k$ the variables $u_{k+1,j}$ and $u_{j,k+1}$ appear in (3.2) only in the kernel as the functional $F(x)$ does not depend on the values of x at these points. Also observe that $b(s_{k+1},t_1)-b(s_k,t_1)=\Delta_{k+1}\Delta_1 b(s,t)$, $b(s_{k+1},t_2)-b(s_k,t_2)=\Delta_{k+1}\Delta_2 b(s,t)+\Delta_{k+1}\Delta_1 b(s,t)$, and eventually $b(s_{k+1},t_k)-b(s_k,t_k)=\Delta_{k+1}\Delta_k b(s,t)+\cdots+\Delta_{k+1}\Delta_1 b(s,t)$. In addition observe that

$$\begin{aligned}
&u_{k+1,1}-a(s_{k+1},t_1)-u_{k,1}+a(s_k,t_1)=\Delta_{k+1}\Delta_1(u-a(s,t)),\\
&u_{k+1,2}-a(s_{k+1},t_2)-u_{k,2}+a(s_k,t_2)=\Delta_{k+1}\Delta_2(u-a(s,t))+\Delta_{k+1}\Delta_1(u-a(s,t)),\\
&\qquad\vdots\\
&u_{k+1,k}-a(s_{k+1},t_k)-u_{k,k}+a(s_k,t_k)\\
&\qquad=\Delta_{k+1}\Delta_k(u-a(s,t))+\cdots+\Delta_{k+1}\Delta_1(u-a(s,t)).
\end{aligned}$$

Applying the Chapman-Kolmogorov equation $2k-2$ times to the right side of (3.2) yields

$$\left(\prod_{i=1}^{k}\prod_{j=1}^{k}2\pi\Delta_i\Delta_j b(s,t)\right)^{-\frac{1}{2}} \quad (3.3)$$

$$\int_{\mathbb{R}^{k^2+1}} f(u_{1,1},\dots,u_{k,k})\exp\left(-\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{(\Delta_i\Delta_j(u-a(s,t)))^2}{\Delta_i\Delta_j b(s,t)}\right)$$

$$\left(\frac{1}{\sqrt{2\pi\Delta_{k+1}\Delta_{k+1}b(s,t)}}\frac{1}{\sqrt{2\pi(b(s_k,t_{k+1})-b(s_k,t_k))}}\frac{1}{\sqrt{2\pi(b(s_k,t_{k+1})-b(s_k,t_k))}}\right)$$

$$\int_{\mathbb{R}^2}\exp\left(\frac{(\Delta_{k+1}\Delta_{k+1}(u-a(a,t)))^2}{-2\Delta_{k+1}\Delta_{k+1}b(s,t)}\right)\exp\left(\frac{(u_{k+1,k}-a(s_{k+1},t_k)-u_{k,k}+a(s_k,t_k))^2}{-(b(s_{k+1},t_k)-b(s_k,t_k))}\right)$$

$$\exp\left(\frac{(u_{k,k+1}-a(s_k,t_{k+1})-u_{k,k}+a(s_k,t_k))^2}{-(b(s_k,t_{k+1})-b(s_k,t_k))}\right)\mathrm{d}u_{k+1,k}\mathrm{d}u_{k,k+1}\mathrm{d}u_{k+1,k+1}\mathrm{d}u_{k,k}\dots\mathrm{d}u_{1,1}$$

Now notice that

$$\begin{aligned}
&\Delta_{k+1}\Delta_{k+1}(\mathbf{u}-a(s,t))\\
&=u_{k+1,k+1}-a(s_{k+1},t_{k+1})-u_{k+1,k}+a(s_{k+1},t_k)\\
&\qquad -u_{k,k+1}+a(s_k,t_{k+1})+u_{k,k}-a(s_k,t_k)\\
&=[(u_{k+1,k+1}-a(s_{k+1},t_{k+1}))-(u_{k,k}-a(s_k,t_k))]\\
&\qquad -[(u_{k,k+1}-a(s_k,t_{k+1}))-(u_{k,k}-a(s_k,t_k))]\\
&\qquad\qquad -[(u_{k+1,k}-a(s_{k+1},t_k))-(u_{k,k}-a(s_k,t_k))],
\end{aligned}$$

and also that

$$\begin{aligned}\Delta_{k+1}\Delta_{k+1}b(s,t) &= b(s_{k+1},t_{k+1}) - b(s_k,t_{k+1}) - b(s_{k+1},t_k) + b(s_k,t_k)\\ &= [b(s_{k+1},t_{k+1}) - b(s_k,t_k)] - [b(s_k,t_{k+1}) - b(s_k,t_k)] - [b(s_{k+1},t_k) - b(s_k,t_k)]\,,\end{aligned}$$

and apply the Chapman-Kolmogorov equation twice to the inner double integral in (3.3),

$$\begin{aligned}&\left(\frac{1}{\sqrt{2\pi\Delta_{k+1}\Delta_{k+1}b(s,t)}}\frac{1}{\sqrt{2\pi(b(s_k,t_{k+1})-b(s_k,t_k))}}\frac{1}{\sqrt{2\pi(b(s_k,t_{k+1})-b(s_k,t_k))}}\right)\\ &\qquad\int_{\mathbb{R}^2}\exp\left(-\frac{(\Delta_{k+1}\Delta_{k+1}(u-a(a,t)))^2}{2\Delta_{k+1}\Delta_{k+1}b(s,t)}\right)\\ &\qquad\quad\exp\left(-\frac{(u_{k+1,k}-a(s_{k+1},t_k)-u_{k,k}+a(s_k,t_k))^2}{b(s_{k+1},t_k)-b(s_k,t_k)}\right)\\ &\qquad\quad\exp\left(-\frac{(u_{k,k+1}-a(s_k,t_{k+1})-u_{k,k}+a(s_k,t_k))^2}{b(s_k,t_{k+1})-b(s_k,t_k)}\right)\mathrm{d}u_{k+1,k}\mathrm{d}u_{k,k+1}.\end{aligned}$$

This yields

$$\left(\frac{1}{2\pi(b(s_{k+1},t_{k+1})-b(s_k,t_k))}\right)^{\frac12}\exp\left(\frac{(u_{k+1,k+1}-a(s_{k+1},t_{k+1})-u_{k,k}+a(s_k,t_k))^2}{-2(b(s_{k+1},t_{k+1})-b(s_k,t_k))}\right)$$

Thus (3.2) becomes

$$\begin{aligned}&\int_{C_{a,b}(Q)} f(x(s_1,t_1),\ldots,x(s_n,t_n))\mathfrak{m}(dx)\\ &\quad=\left(\prod_{i=1}^{k}\prod_{j=1}^{k}2\pi\Delta_i\Delta_j b(s,t)\right)^{-\frac12}\left(\frac{1}{2\pi(b(s_{k+1},t_{k+1})-b(s_k,t_k))}\right)^{\frac12}\\ &\qquad\int_{\mathbb{R}^{k^2}}\int_{\mathbb{R}} f(u_{1,1},\ldots,u_{k+1,k+1})\\ &\qquad\quad\exp\left(-\frac{(u_{k+1,k+1}-a(s_{k+1},t_{k+1})-u_{k,k}+a(s_k,t_k))^2}{2(b(s_{k+1},t_{k+1})-b(s_k,t_k))}\right)\mathrm{d}u_{k+1,k+1}\\ &\qquad\quad\exp\left(-\frac12\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{(\Delta_i\Delta_j(u-a(s,t)))^2}{\Delta_i\Delta_j b(s,t)}\right)\prod_{\substack{i=1\\j=1}}^{k}\mathrm{d}u_{i,j}.\end{aligned}\tag{3.4}$$

Define the function $g:\mathbb{R}^{k^2}\to\mathbb{C}$ so that $g(u_{1,1},\ldots,u_{k,k})$ is equal to

$$\begin{aligned}&\int_{\mathbb{R}} f(u_{1,1},\ldots,u_{k+1,k+1})\\ &\qquad\exp\left(-\frac{(u_{k+1,k+1}-a(s_{k+1},t_{k+1})-u_{k,k}+a(s_k,t_k))^2}{2(b(s_{k+1},t_{k+1})-b(s_k,t_k))}\right)\mathrm{d}u_{k+1,k+1}\end{aligned}\tag{3.5}$$

and define a tame functional $G(x):C_{a,b}(Q)\to\mathbb{R}$ by

$$G(x)=g(x(s_1,t_1),\ldots,x(s_k,t_k)).\tag{3.6}$$

Combine (3.4) and (3.6) to obtain

$$\int_{C_{a,b}(Q)} f(x(s_1,t_1),\ldots,x(s_n,t_n))\mathfrak{m}(dx) =$$

$$\left(\prod_{i=1}^{k}\prod_{j=1}^{k} 2\pi\Delta_i\Delta_j b(s,t)\right)^{-\frac{1}{2}} \int_{\mathbb{R}^{k^2}} g(u_{1,1},\ldots,u_{k,k})$$

$$\exp\left(-\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{(\Delta_i\Delta_j(u-a(s,t)))^2}{\Delta_i\Delta_j b(s,t)}\right)\prod_{\substack{i=1\\j=1}}^{k} \mathrm{d}u_{i,j}$$

$$= \int_{C_{a,b}(Q)} G(x)\mathfrak{m}(x). \tag{3.7}$$

Apply the induction hypothesis to the functional G. Put $u_{k+1,k+1} = u_{k+1}$ and $u_{k,k} = u_k$ in equation (3.5) and then use (3.7) to obtain

$$\int_{C_{a,b}(Q)} f(x(s_1,t_1),\ldots,x(s_n,t_n))\mathfrak{m}(dx)$$

$$= \left(\prod_{j=1}^{k} 2\pi(b(s_j,t_j)-b(s_{j-1},t_{j-1}))\right)^{-\frac{1}{2}} \int_{\mathbb{R}^k} g(u_1,\ldots,u_k)$$

$$\exp\left(-\frac{1}{2}\sum_{j=1}^{k}\frac{(u_j-a(s_j,t_j)-u_{j-1}+a(s_{j-1},t_{j-1}))^2}{b(s_j,t_j)-b(s_{j-1},t_{j-1})}\right)\mathrm{d}u_k\cdots\mathrm{d}u_1$$

$$= \left(\prod_{j=1}^{k+1} 2\pi(b(s_j,t_j)-b(s_{j-1},t_{j-1}))\right)^{-\frac{1}{2}} \int_{\mathbb{R}^{k+1}} f(u_1,\ldots,u_{k+1})$$

$$\exp\left(-\frac{1}{2}\sum_{j=1}^{k+1}\frac{(u_j-a(s_j,t_j)-u_{j-1}+a(s_{j-1},t_{j-1}))^2}{b(s_j,t_j)-b(s_{j-1},t_{j-1})}\right)\mathrm{d}u_{k+1}\cdots\mathrm{d}u_1,$$

and so for $n = k+1$, equation (3.1) holds by induction. □

4. One-line Theorems

We are now equipped to investigate formulas for the integration of functionals depending only on the values of x on certain well-behaved paths in Q. The following theorem permits reduction of certain integrals over $C_{a,b}(Q)$ to integrals over an appropriately chosen single-parameter function space $C_{\tilde{a},\tilde{b}}[0,S]$.

Theorem 4.1. *Let $\phi : [0,S] \to Q$ be an increasing path. Let $a_\phi(\tau) = a(\phi(\tau)) - a(\phi(0))$ and $b_\phi(\tau) = b(\phi(\tau)) - b(\phi(0))$, and let $\mathfrak{m}_\phi$ be the Gaussian measure on $C_0[0,S]$ subordinate to a_ϕ and b_ϕ. If $F(x) = f(x(\phi(\cdot)))$ is a measurable functional on $C_{a,b}(Q)$,* *then*

$$\int_{C_{a,b}(Q)} F(x)\mathfrak{m}(dx) \overset{*}{=} \int_{C_{a_\phi,b_\phi}[0,S]} f(w)\mathfrak{m}_\phi(dw), \tag{4.1}$$

where the equality ($\overset{*}{=}$) *is in the sense that if one of the integrals exists then the other exists and the equality holds.*

Proof. Let $0 = \tau_0 < \tau_1 < \cdots < \tau_j < \cdots < \tau_n \le S$ and let $I = \{x \in C_{a,b}(Q) : \alpha_j < x(\phi(\tau_j)) < \beta_j\}$and $J = \{w \in C_{a_\phi,b_\phi}[0,S] : \alpha_j < w(\tau_j) < \beta_j\}$. Note that by the conditions on γ we have $\phi_1(0) \le \phi_1(\tau_1) \le \cdots \le \phi_1(\tau_n)$ and $\phi_2(0) \le \phi_2(\tau_1) \le \cdots \le \phi_2(\tau_n)$. Then by Theorem 3.1,

$$\begin{aligned}
\mathfrak{m}(I) &= \int_{C_{a,b}(Q)} \chi_I(x)(dx) \\
&= \left(\prod_{j=1}^n 2\pi(b(\phi(\tau_j)) - b(\phi(\tau_{j-1})))\right)^{-\frac{1}{2}} \\
&\qquad \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\sum_{j=1}^n \frac{(u_j - a(\phi(\tau_j)) - u_{j-1} + a(\phi(\tau_{j-1})))^2}{b(\phi(\tau_j)) - b(\phi(\tau_{j-1}))}\right) \mathrm{d}u_n \cdots \mathrm{d}u_1 \\
&= \left(\prod_{j=1}^n 2\pi(b_\phi(\tau_j) - b_\phi(\tau_{j-1}))\right)^{-\frac{1}{2}} \\
&\qquad \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\sum_{j=1}^n \frac{(u_j - a_\phi(\tau_j) - u_{j-1} + a_\phi(\tau_{j-1}))^2}{b_\phi(\tau_j) - b_\phi(\tau_{j-1})}\right) \mathrm{d}u_n \cdots \mathrm{d}u_1 \\
&= \int_{C_{a_\phi,b_\phi}[0,S]} \chi_J(y)\mathfrak{m}_\phi(dy) \\
&= \mathfrak{m}_\phi(J).
\end{aligned}$$

Hence the result holds for characteristic functions of sets of the form $\{x \in C_{a,b}(Q) : \alpha_j < x(\phi(\tau_j)) < \beta_j\}$. The theorem follows by taking the function f to successively be the characteristic function of a Borel set, and then to be a simple function. From here, by monotone convergence the theorem holds for positive functions, and hence for general functions by taking positive and negative and real and imaginary parts. □

As a corollary to Theorem 4.1 we have the following one-line theorem of Cameron and Storvick from [2].

Corollary 4.2. *Let* $0 < \beta \le T$ *and let* $f(\cdot)$ *be a real or complex valued functional defined on* $C_0[0,S]$ *such that* $f(\sqrt{\beta}w)$ *is a Wiener measurable functional on* $C_0[0,S]$*. Then* $f(x(\cdot,\beta))$ *is a Yeh-Wiener measurable functional of* x *on* $C_0(Q)$ *and*

$$\int_{C_0(Q)} f(x(\cdot,\beta))(dx) \overset{*}{=} \int_{C_0[0,S]} f(\sqrt{\beta}\, w)\mathfrak{w}(dw), \tag{4.2}$$

where $\mathfrak{w}$ *denotes the ordinary Wiener measure and the equality* ($\overset{*}{=}$) *is in the sense that if one of the integrals exists then the other exists and the equality holds.*

Proof. Take $\phi : [0,S] \to Q$ to be $\phi(\tau) = (\tau, \beta)$ and note that $a(s,t) = 0$ and $b(s,t) = st$. Applying Theorem 4.1 to any tame functional $F(x) = f(x(s_1,\beta),\ldots,x(s_n,\beta))$ we obtain

$$\begin{aligned}
&\int_{C_0(Q)} F(x)\mathfrak{m}(dx) \\
&\quad = \left(\prod_{j=1}^{n} 2\pi(\beta s_j - \beta s_{j-1})\right)^{-\frac{1}{2}} \int_{\mathbb{R}^n} f(u_1,\ldots,u_n)\exp\left(-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{\beta s_j - \beta s_{j-1}}\right) \mathrm{d}\mathbf{u} \\
&\quad = \left(\prod_{j=1}^{n} 2\pi(s_j - s_{j-1})\right)^{-\frac{1}{2}} \int_{\mathbb{R}^n} f(\sqrt{\beta}\, w_1,\ldots,\sqrt{\beta}\, w_n) \\
&\qquad\qquad \exp\left(-\frac{1}{2}\sum_{j=1}^{n}\frac{(w_j-w_{j-1})^2}{s_j - s_{j-1}}\right) \mathrm{d}\mathbf{w} \\
&\quad = \int_{C_0[0,S]} f(\sqrt{\beta}\, y(s_1),\ldots,\sqrt{\beta}\, y(s_n))\mathfrak{w}(dw).
\end{aligned}$$

The theorem holds in the general case by the same argument used to finish the proof of Theorem 4.1. □

5. n-line Theorem

We can use Theorem 4.1 to extend the n-line theorem of Cameron and Storvick from [2].

Theorem 5.1. *Let* $0 < \beta_1 < \cdots < \beta_n \le T$ *and let* $F(x) = f(x(\cdot,\beta_1),\ldots,x(\cdot,\beta_n))$ *be* μ*-measurable. Put* $a_1(s) = a(s,\beta_1)$ *and* $a_k(s) = a(s,\beta_k) - a(s,\beta_{k-1})$ *and put* $b_1(s) = b(s,\beta_1)$ *and* $b_k(s) = b(s,\beta_k) - b(s,\beta_{k-1})$ *for* $k = 2,\ldots,n$. *Let* $\mathfrak{m}_1,\ldots,\mathfrak{m}_n$ *be Gaussian measures on* $C_0[0,S]$*, each subordinate to the corresponding* a_k *and* b_k*. Then*

$$\begin{aligned}
&\int_{C_{a,b}(Q)} F(x)\mathfrak{m}(dx) \\
&\quad \overset{*}{=} \int_{C_{a_n,b_n}[0,S]} \cdots \int_{C_{a_1,b_1}[0,S]} f(y_1, y_1+y_2,\cdots, y_1+y_2+\cdots+y_n)\mathfrak{m}_1(dy_1)\cdots\mathfrak{m}_n(dy_n),
\end{aligned}$$

where the equality ($\overset{*}{=}$) *is in the sense that if one of the integrals exists then the other exists and the equality holds.*

Proof. Let $0 = s_0 < s_1 < \ldots < s_m \le S$ and $t_k = \beta_k$ for $k = 1, \ldots, n$ and let $p_{j,k} < q_{j,k}$ for all $j = 1, \ldots m$ and $k = 1, \ldots, n$. Define

$$I_j = \{x \in C_{a,b}(Q) : p_{j,k} < x(s_j, \beta_k) \le q_{j,k} \text{ for } k = 1, \ldots, n\},$$
$$E_j = \{(u_{j,1}, \ldots, u_{j,n}) \in \mathbb{R}^n : p_{j,k} < u_{j,k} \le q_{j,k} \text{ for } k = 1, \ldots, n\},$$
$$J_j = \{(y_1, \ldots, y_n) \in \times_{k=1}^n C_{a_k,b_k}[0, S] : p_{j,k} < \sum_{l=1}^{k} y_l(s_j) \le q_{j,k} \text{ for } k = 1, \ldots, n\}.$$

Notice that measurability of E_j in $\mathbb{R}^n$ assures the measurability of I_j and J_j in their respective spaces. Moreover, for a cylinder set $I(p_{1,1}, \ldots, p_{m,n}, q_{1,1}, \ldots, q_{m,n}) \subseteq C_{a,b}(Q)$ determined solely by the values of $x(\cdot,\cdot)$ at the points (s_j, β_k) for $j = 1, \ldots, m$ and $k = 1, \ldots, n$, we have

$$\begin{aligned} &I(p_{1,1}, \ldots, p_{m,n}, q_{1,1}, \ldots, q_{m,n}) \\ &\quad = \{x \in C_{a,b}(Q) : p_{j,k} < x(s_j, \beta_k) \le q_{j,k} \text{ for } j = 1, \ldots, m; k = 1, \ldots, n\} \\ &\quad = \bigcap_{j=1}^{m} I_j. \end{aligned}$$

Begin by considering the case in which

$$F(x) = \chi_I(x) = \prod_{j=1}^{m} \chi_{I_j}(x) = \prod_{j=1}^{m} \chi_{E_j}(x(\cdot, \beta_1), \ldots, x(\cdot, \beta_n)).$$

By Theorem 2.4,

$$\begin{aligned} \int_{C_{a,b}(Q)} F(x)\mathfrak{m}(dx) &= \int_{C_{a,b}(Q)} \prod_{j=1}^{m} \chi_{E_j}(x(s_j, \beta_1), \ldots, x(s_j, \beta_n))\mathfrak{m}(dx) \\ &= \left(\prod_{k=1}^{n}\prod_{j=1}^{m} 2\pi\Delta_k\Delta_j b(s, \beta)\right)^{-\frac{1}{2}} \int_{\mathbb{R}^{mn}} \prod_{j=1}^{m} \chi_{E_j}(u_{j,1}, \ldots, u_{j,n}) \qquad (5.1) \\ &\qquad \prod_{k=1}^{n} \exp\left(-\frac{1}{2}\sum_{j=1}^{m} \frac{(\Delta_k\Delta_j(u - a(s,t)))^2}{\Delta_k\Delta_j b(s,t)}\right) \mathrm{d}u_{1,1}\cdots \mathrm{d}u_{m,n}. \end{aligned}$$

Note that

$$\begin{aligned} \Delta_k\Delta_j(u - a(s,t)) &= u_{j,k} - u_{j,k-1} - a(s_j, \beta_k) + a(s_j, \beta_{k-1}) \\ &\qquad - u_{j-1,k} + u_{j-1,k-1} + a(s_{j-1}, \beta_k) - a(s_{j-1}, \beta_{k-1}) \qquad (5.2) \\ &= [u_{j,k} - u_{j,k-1}] - [a(s_j, \beta_k) - a(s_j, \beta_{k-1})] \qquad (5.3) \\ &\qquad - [u_{j-1,k} - u_{j-1,k-1}] + [a(s_{j-1}, \beta_k) - a(s_{j-1}, \beta_{k-1})], \end{aligned}$$

and also that

$$\begin{aligned} \Delta_k\Delta_j b(s,t) &= b(s_j, \beta_k) - b(s_{j-1}, \beta_k) - b(s_j, \beta_{k-1}) + b(s_{j-1}, \beta_{k-1}) \\ &= [b(s_j, \beta_k) - b(s_j, \beta_{k-1})] - [b(s_j, \beta_k) - b(s_{j-1}, \beta_{k-1})]. \qquad (5.4) \end{aligned}$$

Take $b_1(\cdot) = b(\cdot,\beta_1)$, $b_k(\cdot) = b(\cdot,\beta_k) - b(\cdot,\beta_{k-1})$, $a_1(\cdot) = a(\cdot,\beta_1)$, and $a(\cdot) = b(\cdot,\beta_k) - b(\cdot,\beta_{k-1})$ for $k = 2,\dots,n$ as in the statement of the theorem. Let

$$v_{j,k} = u_{j,k} - u_{j,k-1}, \tag{5.5}$$

and observe that $\mathrm{d}v_{j,k} = \mathrm{d}u_{j,k}$ under this change of variables, and that

$$u_{j,k} = v_{j,k} + u_{j,k-1} = v_{j,k} + v_{j,k-1} + \cdots + v_{j,1} \tag{5.6}$$

for $1 \le k \le n$. Substitute (5.2), (5.4), (5.5), and (5.6) in (5.1) to obtain

$$\begin{aligned}
&\prod_{k=1}^{n}\left(\prod_{j=1}^{m} 2\pi\Delta_j b_k(s)\right)^{-\frac{1}{2}} \int_{\mathbb{R}^{mn}} \prod_{j=1}^{m} \chi_{E_j}(v_{j,1}, v_{j,1}+v_{j,2},\dots,v_{j,1}+\cdots+v_{j,n}) \\
&\qquad \prod_{k=1}^{n} \exp\left(-\frac{1}{2}\sum_{j=1}^{m} \frac{(\Delta_j(v_j - a_j(s)) - v_{j-1} + a_{j-1}(s))^2}{\Delta_j b_k(s)}\right) \mathrm{d}v_{1,1}\cdots\mathrm{d}v_{m,n} \\
&= \int_{C_{a_1,b_1}[0,S]} \cdots \int_{C_{a_n,b_n}[0,S]} \prod_{j=1}^{m} \chi_{E_j}(y_1(s_j),\dots,y_1(s_j)+\cdots+y_n(s_j))\mathfrak{m}_n(dy_n)\cdots\mathfrak{m}_1(dy_1) \\
&= \int_{C_{a_1,b_1}[0,S]} \cdots \int_{C_{a_n,b_n}[0,S]} \prod_{j=1}^{m} \chi_{E_j}(y_1(\cdot),\dots,y_1(\cdot)+\cdots+y_n(\cdot))\mathfrak{m}_n(dy_n)\cdots\mathfrak{m}_1(dy_1).
\end{aligned}$$

Therefore the theorem is true for characteristic functions of cylinder sets that are dependent only on the value of $x(\cdot,\cdot)$ at the points $\{(s_j,\beta_k)$ for $j = 1,\dots,m;\ k = 1,\dots,n\}$. In the usual manner we can prove the theorem for characteristic functions of measurable sets depending only on the values of $x(\cdot,\beta_k)$ for $k = 1,\dots n$. The proof is then completed in the same fashion as the proof of Theorem 4.1. □

6. Examples

Example 6.1. This first example demonstrates the use of Theorem 5.1. Let $a(s,t) = b(s,t) = st$ on $[0,S]\times[0,2T]$ and put $F(x) = \int_0^S x(s,T)x(s,2T)\mathrm{d}s$. We find the value of $\int_{C_{a,b}(Q)} F(x)\mu(dx)$. Note that $a_1(s) = b_1(s) = sT$ and $a_2(s) = b_2(s) = 2sT - sT = sT$, and thus

$$\begin{aligned}
\int_{C_{a,b}(Q)} F(x)\mathfrak{m}(dx) &= \int_{C_{a_2,b_2}[0,S]}\int_{C_{a_1,b_1}[0,S]}\int_0^S y_1(s)(y_1(s)+y_2(s))\mathrm{d}s\,\mathfrak{m}_1(dy_1)\mathfrak{m}_2(dy_2) \\
&= \int_0^S \int_{C_{a_2,b_2}[0,S]}\int_{C_{a_1,b_1}[0,S]} \left(y_1^2(s) + y_1(s)y_2(s)\right)\mathfrak{m}_1(dy_1)\mathfrak{m}_2(dy_2)\mathrm{d}s \\
&= \int_0^S \int_{C_{a_2,b_2}[0,S]} \left(sT + s^2T^2 + sTy_2(s)\right)\mathfrak{m}_2(dy_2)\mathrm{d}s \\
&= \int_0^S \left(sT + s^2T^2 + s^2T^2\right)\mathrm{d}s
\end{aligned}$$

$$= \frac{1}{2}S^2T + \frac{2}{3}S^3T^2,$$

where Fubini's theorem justifies the change in order of integration. In this example, we can easily verify our result without using Theorem 5.1, for

$$\begin{aligned}\int_{C_{a,b}(Q)} F(x)\mathfrak{m}(dx) &= \int_{C_{a,b}(Q)} \int_0^S x(s,T)x(s,2T)\mathrm{d}s\mathfrak{m}(dx)\\ &= \int_0^S \int_{C_{a,b}(Q)} x(s,T)x(s,2T)\mathfrak{m}(dx)\mathrm{d}s\\ &= \int_0^S \left(sT + 2s^2T^2\right)\mathrm{d}s\\ &= \frac{1}{2}S^2T + \frac{2}{3}S^3T^2.\end{aligned}$$

Example 6.2. We next demonstrate the use of Theorem 4.1. Let $Q = [0,S]^2$, $a(s,t) = st$, $b(s,t) = s^2t^2$, and $F(x) = \exp\left(\int_0^S x(s,s)\mathrm{d}s\right)$. Note that $\phi : [0,S] \to Q$ defined by $\phi(s) = (s,s)$ is increasing. Then

$$\int_{C_{a,b}(Q)} F(x)\mathfrak{m}(dx) = \int_{C_{a_1,b_1}[0,S]} \exp\left(\int_0^S y(s)\mathrm{d}s\right)\mathfrak{m}_\phi(dy), \tag{6.1}$$

where $a_1(s) = a(\phi(s)) - a(\phi(0)) = a(s,s) - a(0,0) = s^2$ and $b_1(s) = b(\phi(s)) - b(\phi(0)) = s^4$. Integrating by parts we obtain that

$$\int_0^S y(s)\mathrm{d}s = Sy(S) - \int_0^S s\mathrm{d}y(s) = \langle S, y\rangle - \langle s, y\rangle = \langle S - s, y\rangle,$$

for $\mathfrak{m}$ a.e. $y \in C_{a_\phi,b_\phi}[0,S]$. In this case we let $\langle f, y\rangle$ denote the single-variable PWZ stochasitc integral of the function $f \in L^2_{a_\phi,b_\phi}[0,S]$. Compute the values $A = \int_0^S (S - s)\mathrm{d}a_\phi(s) = \frac{1}{3}S^3$ and $B = \int_0^S (S-s)^2\mathrm{d}b_\phi(s) = \frac{1}{15}S^6$ and make use of a theorem from [20] to compute the right-hand side of (6.1) to obtain

$$\begin{aligned}&\int_{C_{a_1,b_1}[0,S]} \exp\left(\int_0^S y(s)\mathrm{d}s\right)\mathfrak{m}_\phi(dy)\\ &\quad= \frac{1}{\sqrt{2\pi B}}\int_{-\infty}^{\infty} \exp(u)\exp\left(-\frac{(u-A)^2}{2B}\right)\mathrm{d}u\\ &\quad= \frac{1}{\sqrt{2\pi B}}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2B}[u^2 - 2Au - 2Bu + A^2]\right)\mathrm{d}u\\ &\quad= \exp\left(-\frac{A^2}{2B}\right)\exp\left(\frac{(A+B)^2}{2B}\right)\int_{-\infty}^{\infty} \exp\left(-\frac{[u-(A+B)]^2}{2B}\right)\mathrm{d}u\\ &\quad= \exp\left(\frac{2AB + B^2}{2B}\right)\\ &\quad= \exp\left(\frac{1}{3}S^3 + \frac{1}{30}S^6\right).\end{aligned}$$

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