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SOLVABILITY OF p-LAPLACIAN FRACTIONAL HIGHER ORDER TWO-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, we consider p-Laplacian fractional higher order two-point boundary value problems. We establish criteria to determine the values of parameter λ for which the p-Laplacian fractional order eigenvalue problem has at least one positive solution. Later, we establish sufficient conditions for the existence of even number of positive solutions for p-Laplacian fractional order boundary value problem by applying an Avery-Henderson functional fixed point theorem.

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1. INTRODUCTION

The study of fractional order differential equations has emerged as an important area of mathematics. The intention of differential equations is to understand the phenomena of nature by developing mathematical models. Among all, a class of differential equations governed by nonlinear differential operators, which have wide applications and interest developed to study such type of equations. In this theory, the most investigated operator is the classical p-Laplacian, given by $\phi_p(s) = |s|^{p-2}s$, $p > 1$. These problems have a wide range of applications in physics and related sciences such as turbulent filtration in porous media, biophysics, plasma physics, and chemical reaction design. For more details on applications of p-Laplacian operator, we refer [7].

Recently, much interest has been created in establishing positive solutions and multiple positive solutions for two-point, multi-point boundary value problems (BVPs) associated with ordinary and fractional order differential equations. For positive solutions of BVPs for ordinary differential equations discussed by authors, we mention few of them, Erbe and Wang [9], Eloe and Henderson [8], Davis, Henderson, Prasad

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and Yin [5], Henderson and Ntouyas [11] and for fractional order differential equations, we mention a few papers, see Bai and Lü [3], Kauffman and Mboumi [12], Su and Zhang [20], Prasad and Krushna [18, 19]. Yang and Yan [21] investigated the existence of positive solutions for the third-order Sturm-Liouville BVPs with p-Laplacian operator by using the fixed point index method. Chai [4] established the existence and multiplicity of positive solutions for a class of fractional order BVPs with p-Laplacian operator by means of the fixed point theorem on cones.

Inspired by the papers mentioned above, in this paper, we are concerned with p-Laplacian fractional order two-point boundary value problem,

$$D^{\beta}_{0^+}\Big(\phi_p\Big(D^{\alpha}_{0^+}x(t)\Big)\Big)+\lambda f(t,x(t))=0,\quad t\in(0,1), \tag{1.1}$$

$$x^{(j)}(0)=0,\quad 0\le j\le n-4,\quad x^{(a)}(1)=0, D^{\alpha}_{0^+}x(0)=0,\quad D^{\alpha}_{0^+}x(1)=0, \tag{1.2}$$

and the p-Laplacian fractional order boundary value problem,

$$D^{\beta}_{0^+}\Big(\phi_p\Big(D^{\alpha}_{0^+}x(t)\Big)\Big)=f(t,x(t)),\quad t\in(0,1), \tag{1.3}$$

with the same boundary conditions, where $\lambda>0$, $\phi_p(s)=|s|^{p-2}s$, $p>1$, $\phi_p^{-1}=\phi_q$, $1/p+1/q=1$, $n-3<\alpha\le n-2$, $n\ge 5$, $1\le a\le \alpha-1$ is a fixed integer, $1<\beta\le 2$ and $D^{\alpha}_{0^+}, D^{\beta}_{0^+}$ are the standard Riemann-Liouville fractional order derivatives.

We assume the following conditions hold throughout this paper:

$(P1)$ $f:[0,1]\times R^+\to R^+$ is continuous,
$(P2)$ each of f_0, f^0, f_∞ and f^∞ by

$$f_0=\lim_{x\to 0^+}\min_{t\in[0,1]}\frac{f(t,x)}{\phi_p(x)},\quad f^0=\lim_{x\to 0^+}\max_{t\in[0,1]}\frac{f(t,x)}{\phi_p(x)},$$
$$f_\infty=\lim_{x\to \infty}\min_{t\in[0,1]}\frac{f(t,x)}{\phi_p(x)}\quad\text{and}\quad f^\infty=\lim_{x\to \infty}\max_{t\in[0,1]}\frac{f(t,x)}{\phi_p(x)},$$

exist as positive real numbers.

The rest of the paper is organized as follows. In Section 2, we construct the Green functions for the homogeneous BVPs corresponding to (1.1)–(1.2) and estimate the bounds for the Green functions. In Section 3, we establish criteria to determine the values of parameter λ for which the p-Laplacian fractional order BVP (1.1)–(1.2) has at least one positive solution by using the Guo-Krasnosel'skii fixed point theorem. In Section 4, we establish sufficient conditions for the existence of even number of positive solutions for p-Laplacian fractional order BVP (1.3), (1.2) by using an Avery-Henderson functional fixed point theorem. In Section 5, as an application, we demonstrate our results with examples.

2. GREEN FUNCTIONS AND BOUNDS

In this section, we construct the Green functions for the homogeneous fractional order BVPs and estimate the bounds for the Green functions, which are needed to establish the main results.

Lemma 2.1. *Let $h(t) \in C[0,1]$. Then the fractional order BVP,*

$$D_{0+}^{\alpha} x(t) + h(t) = 0, \quad t \in (0,1), \tag{2.1}$$

$$x^{(j)}(0) = 0, \quad 0 \le j \le n-4, \quad x^{(a)}(1) = 0, \tag{2.2}$$

has a unique solution,

$$x(t) = \int_0^1 G(t,s)h(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1-a}, & 0 \le t \le s \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1-a} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1. \end{cases} \tag{2.3}$$

Proof. Assume that $x(t)$ is a solution of fractional order BVP (2.1)–(2.2). Now, (2.1) can be written as

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = -I_{0+}^{\alpha} h(t)$$

$$x(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + \cdots + c_{n-2} t^{\alpha-n+2}.$$

From $x^{(j)}(0) = 0$, $0 \le j \le n-4$, we obtain $c_{n-2} = c_{n-3} = c_{n-4} = \cdots = c_2 = 0$. Then

$$x(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + c_1 t^{\alpha-1},$$

$$x^{(a)}(t) = c_1 \prod_{i=1}^{a} (\alpha - i) t^{\alpha-1-a} - \prod_{i=1}^{a} (\alpha - i) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1-a} h(s)ds.$$

From $x^{(a)}(1) = 0$, we have

$$c_1 \prod_{i=1}^{a} (\alpha - i) - \prod_{i=1}^{a} (\alpha - i) \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1-a} h(s)ds = 0.$$

Therefore, $c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1-a} h(s)ds$. Thus, the unique solution of (2.1)–(2.2) is

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1-a} h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds. \end{aligned}$$

Here, $G(t,s)$ is the Green's function for the homogeneous BVP corresponding to (2.1)–(2.2). Hence the result given as in (2.3). ∎

Lemma 2.2. *Let $y(t) \in C[0,1]$. Then the fractional order differential equation,*

$$D_{0^+}^{\beta}\Big(\phi_p\Big(D_{0^+}^{\alpha}x(t)\Big)\Big) = y(t), \quad t \in (0,1), \tag{2.4}$$

satisfying the boundary condition (1.2), *has a unique solution,*

$$x(t) = \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)y(\tau)d\tau\Big)ds,$$

where

$$H(t,s) = \begin{cases} \frac{[t(1-s)]^{\beta-1}}{\Gamma(\beta)}, & 0 \le t \le s \le 1, \\ \frac{[t(1-s)]^{\beta-1}-(t-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \le s \le t \le 1. \end{cases} \tag{2.5}$$

Proof. We construct an equivalent integral equation for (2.4), and it is given by

$$\phi_p\Big(D_{0^+}^{\alpha}x(t)\Big) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}y(\tau)d\tau + c_1t^{\beta-1} + c_2t^{\beta-2}.$$

From $D_{0^+}^{\alpha}x(0) = 0, D_{0^+}^{\alpha}x(1) = 0$, one can determine

$$c_2 = 0 \text{ and } c_1 = \frac{-1}{\Gamma(\beta)}\int_0^1 (1-\tau)^{\beta-1}y(\tau)d\tau.$$

Then,

$$\begin{aligned} \phi_p\Big(D_{0^+}^{\alpha}x(t)\Big) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}y(\tau)d\tau - \frac{1}{\Gamma(\beta)}\int_0^1 [t(1-\tau)]^{\beta-1}y(\tau)d\tau \\ &= -\int_0^1 H(t,\tau)y(\tau)d\tau. \end{aligned}$$

Therefore, $\phi_p^{-1}\Big(\phi_p\Big(D_{0^+}^{\alpha}x(t)\Big)\Big) = -\phi_p^{-1}\Big(\int_0^1 H(t,\tau)y(\tau)d\tau\Big)$. Consequently,

$$D_{0^+}^{\alpha}x(t) + \phi_q\Big(\int_0^1 H(t,\tau)y(\tau)d\tau\Big) = 0.$$

Hence, $x(t) = \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)y(\tau)d\tau\Big)ds$ is the solution to fractional order BVP (2.4), (1.2). □

Lemma 2.3. *The Green's function $G(t,s)$ satisfies the following inequalities*

(i) $G(t,s) \ge 0$, *for all* $(t,s) \in [0,1] \times [0,1]$,
(ii) $G(t,s) \le G(1,s)$, *for all* $(t,s) \in [0,1] \times [0,1]$,
(iii) $G(t,s) \ge \eta G(1,s)$, *for all* $(t,s) \in I \times [0,1]$,

where $\eta = \frac{1}{4^{\alpha-1}}$ *and* $I = [\frac{1}{4}, \frac{3}{4}]$.

Proof. The Green's function $G(t,s)$ is given in (2.3). We prove the inequality (i). For $0 \le t \le s \le 1$,

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\Big[t^{\alpha-1}(1-s)^{\alpha-1-a}\Big] \ge 0.$$

For $0 \leq s \leq t \leq 1$,

$$\begin{aligned} G(t,s) &= \frac{1}{\Gamma(\alpha)}\Big[t^{\alpha-1}(1-s)^{\alpha-1-a} - (t-s)^{\alpha-1}\Big] \\ &\geq \frac{1}{\Gamma(\alpha)}\Big[t^{\alpha-1}(1-s)^{\alpha-1-a} - (t-ts)^{\alpha-1}\Big] \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\Big[(1-s)^{\alpha-1-a} - (1-s)^{\alpha-1}\Big] \geq 0. \end{aligned}$$

To prove the inequality (ii). For $0 \leq t \leq s \leq 1$,

$$\frac{G(t,s)}{G(1,s)} = \frac{t^{\alpha-1}(1-s)^{\alpha-1-a}}{(1-s)^{\alpha-1-a}} = t^{\alpha-1} \leq 1.$$

For $0 \leq s \leq t \leq 1$,

$$\frac{G(t,s)}{G(1,s)} = \frac{t^{\alpha-1}(1-s)^{\alpha-1-a} - (t-s)^{\alpha-1}}{(1-s)^{\alpha-1-a} - (1-s)^{\alpha-1}} \leq 1.$$

Now, we can establish the inequality (iii). For $0 \leq t \leq s \leq 1$ and $t \in I$,

$$\frac{G(t,s)}{G(1,s)} = \frac{t^{\alpha-1}(1-s)^{\alpha-1-a}}{(1-s)^{\alpha-1-a}} = t^{\alpha-1} \geq \frac{1}{4^{\alpha-1}} = \eta.$$

For $0 \leq s \leq t \leq 1$ and $t \in I$,

$$\begin{aligned} \frac{G(t,s)}{G(1,s)} &= \frac{t^{\alpha-1}(1-s)^{\alpha-1-a} - (t-s)^{\alpha-1}}{(1-s)^{\alpha-1-a} - (1-s)^{\alpha-1}} \\ &\geq \frac{t^{\alpha-1}(1-s)^{\alpha-1-a} - (t-ts)^{\alpha-1}}{(1-s)^{\alpha-1-a} - (1-s)^{\alpha-1}} \\ &= t^{\alpha-1} \geq \frac{1}{4^{\alpha-1}} = \eta, \end{aligned}$$

where $\eta = \frac{1}{4^{\alpha-1}}$. □

Lemma 2.4. *For $t, s \in [0,1]$, the Green's function $H(t,s)$ satisfies the following inequalities*

(i) $H(t,s) \geq 0$,
(ii) $H(t,s) \leq H(s,s)$.

Proof. The Green's function $H(t,s)$ is given in (2.5). We prove the inequality (i). For $0 \leq t \leq s \leq 1$,

$$H(t,s) = \frac{1}{\Gamma(\beta)}[t(1-s)]^{\beta-1} \geq 0.$$

For $0 \le s \le t \le 1$,

$$\begin{aligned} H(t,s) &= \frac{1}{\Gamma(\beta)}\Big[t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}\Big] \\ &= \frac{1}{\Gamma(\beta)}\Big[t^{\beta-1}(1-s)^{\beta-1} - t^{\beta-1}\Big(1-\frac{s}{t}\Big)^{\beta-1}\Big] \\ &\ge \frac{1}{\Gamma(\beta)}\Big[t^{\beta-1}(1-s)^{\beta-1} - t^{\beta-1}(1-s)^{\beta-1}\Big] \\ &\ge 0. \end{aligned}$$

To prove the inequality (ii). For $0 \le t \le s \le 1$,

$$\frac{\partial H(t,s)}{\partial t} = \frac{1}{\Gamma(\beta)}[(\beta-1)t^{\beta-2}(1-s)^{\beta-1}] > 0.$$

Therefore, $H(t,s)$ is increasing in t, for $s \in [0,1)$, which implies $H(t,s) \le H(s,s)$. For $0 \le s \le t \le 1$,

$$\begin{aligned} \frac{\partial H(t,s)}{\partial t} &= \frac{1}{\Gamma(\beta)}\Big[(\beta-1)t^{\beta-2}(1-s)^{\beta-1} - (\beta-1)(t-s)^{\beta-2}\Big] \\ &= \frac{1}{\Gamma(\beta)}\Big[(\beta-1)t^{\beta-2}(1-s)^{\beta-1} - (\beta-1)t^{\beta-2}\Big(1-\frac{s}{t}\Big)^{\beta-2}\Big] \\ &\le \frac{1}{\Gamma(\beta)}\Big[(\beta-1)t^{\beta-2}(1-s)^{\beta-1} - (\beta-1)t^{\beta-2}(1-s)^{\beta-2}\Big] \\ &= \frac{1}{\Gamma(\beta)}\Big[(\beta-1)t^{\beta-2}(1-s)^{\beta-2}(1-s-1)\Big] \\ &< 0. \end{aligned}$$

Therefore, $H(t,s)$ is decreasing in t, for $s \in (0,1]$, and hence $H(t,s) \le H(s,s)$. $\square$

Lemma 2.5. *Let $\xi \in (\frac{1}{4}, \frac{3}{4})$. Then the Green's function $H(t,s)$ satisfies the inequality,*

$$\min_{t \in I} H(t,s) \ge \vartheta(s) H(s,s), \quad \textit{for } 0 < s < 1, \tag{2.6}$$

where

$$\vartheta(s) = \begin{cases} \frac{[\frac{3}{4}(1-s)]^{\beta-1} - (\frac{3}{4}-s)^{\beta-1}}{[s(1-s)]^{\beta-1}}, & s \in (0,\xi], \\ \frac{1}{(4s)^{\beta-1}}, & s \in [\xi, 1). \end{cases}$$

Proof. For $s \in (0,1)$, $H(t,s)$ is increasing in t when $t \le s$ and decreasing in t when $s \le t$. Define

$$h_1(t,s) = \frac{[t(1-s)]^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)} \quad \text{and} \quad h_2(t,s) = \frac{[t(1-s)]^{\beta-1}}{\Gamma(\beta)}.$$

Then

$$
\begin{aligned}
\min_{t\in I} H(t,s) &= \begin{cases} h_1(\frac{3}{4},s), & s\in(0,\frac{1}{4}],\\ \min\{h_1(\frac{3}{4},s),h_2(\frac{1}{4},s)\}, & s\in[\frac{1}{4},\frac{3}{4}],\\ h_2(\frac{1}{4},s), & s\in[\frac{3}{4},1),\end{cases}\\
&= \begin{cases} h_1(\frac{3}{4},s), & s\in(0,\xi],\\ h_2(\frac{1}{4},s), & s\in[\xi,1),\end{cases}\\
&\geq \begin{cases} \frac{(\frac{3}{4})^{\beta-1}(1-s)^{\beta-1}-(\frac{3}{4}-s)^{\beta-1}}{s^{\beta-1}(1-s)^{\beta-1}}H(s,s), & s\in(0,\xi],\\ \frac{1}{(4s)^{\beta-1}}H(s,s), & s\in[\xi,1),\end{cases}\\
&= \vartheta(s)H(s,s).
\end{aligned}
$$

□

3. EXISTENCE OF EIGENVALUE INTERVALS

In this section, we establish criteria to determine the values of parameter λ for which the p-Laplacian fractional order BVP (1.1)–(1.2) has at least one positive solution by using the Guo-Krasnosel'skii fixed point theorem.

Let $X=\{u : u\in C[0,1]\}$ be the Banach space equipped with the norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|.$$

Define a cone $P\subset X$ by

$$P=\Big\{u\in X \mid u(t)\geq 0,\ t\in[0,1] \text{ and } \min_{t\in I} u(t)\geq \eta\|u\|\Big\}.$$

Let $T : P\to X$ be the operator defined by

$$Tx(t)=\lambda\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds. \tag{3.1}$$

Lemma 3.1. *The operator T defined by* (3.1) *is self map.*

Proof. Let $x\in P$. Clearly, $Tx\geq 0$, for all $t\in[0,1]$, and

$$
\begin{aligned}
Tx(t) &= \lambda\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds\\
&\leq \lambda\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds,
\end{aligned}
$$

so that

$$\|Tx\|\leq \lambda\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds.$$

Next, for $x \in P$,

$$\begin{aligned}
\min_{t\in I} Tx(t) &= \min_{t\in I} \lambda \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds \\
&\geq \lambda\eta \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds \\
&\geq \eta\|Tx\|.
\end{aligned}$$

Therefore,

$$\min_{t\in I} Tx(t) \geq \eta\|Tx\|.$$

□

Lemma 3.2. *The operator T is completely continuous, where T is defined in* (3.1).

Proof. Let $x \in P$, in view of the positivity and continuity of $G(t,s), H(t,s)$ and f, we have $T : P \to P$ is continuous. Firstly, we prove that T is uniformly bounded. Now, let $\{x_k\}$ be a bounded sequence in P, say $\|x_k\| \leq M$ for all k. Since f is continuous, there exists $N > 0$ such that $|f(t,x(t))| \leq N$ for all $x \in [0,\infty)$ with $0 \leq x \leq M$ then, for each $t \in [0,1]$ and for each k,

$$\begin{aligned}
|Tx_k(t)| &= \Big|\lambda \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x_k)d\tau\Big)ds\Big| \\
&\leq \lambda \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)Nd\tau\Big)ds \\
&\leq \lambda N^{q-1} \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds < +\infty.
\end{aligned}$$

That is, for each $t \in [0,1]$, $\{Tx_k\}$ is a bounded sequence of real numbers. By choosing successive subsequences, for each t, there exists a subsequence $\{Tx_{k_j}\}$ which converges uniformly for $t \in [0,1]$. Hence, T is uniformly bounded. To prove T is equicontinuous. Let $x \in P$ and $\epsilon > 0$ be given. By the continuity of $G(t,s)$, for $t,s \in [0,1]$, there exists a $\delta > 0$ such that

$$|G(t_2,s) - G(t_1,s)| < \frac{\epsilon}{\lambda N^{q-1}\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)}$$

whenever $|t_2 - t_1| < \delta,$ for any $t_1, t_2 \in [0,1]$.

$$\begin{aligned}
&|Tx(t_2) - Tx(t_1)| \\
&\quad= \Big|\int_0^1 \lambda[G(t_2,s) - G(t_1,s)]\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds\Big| \\
&\quad\leq \int_0^1 \lambda|G(t_2,s) - G(t_1,s)|\phi_q\Big(\int_0^1 H(s,\tau)Nd\tau\Big)ds \\
&\quad\leq \lambda N^{q-1} \int_0^1 |G(t_2,s) - G(t_1,s)|\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds < \epsilon.
\end{aligned}$$

Therefore, T is equicontinuous. Hence T is completely continuous. □

To establish the existence of at least one positive solution for p-Laplacian fractional order BVP (1.1)–(1.2) by employing Guo-Krasnosel'skii fixed point theorem.

Theorem 3.3 ([10, 14]). *Let X be a Banach Space, $P \subseteq X$ be a cone and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is completely continuous operator such that either*

(i) $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$, *or*
(ii) $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$ *holds.*

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Let

$$L_1 = \frac{1}{\Big[\eta^2 \int_{s\in I} G(1,s)\phi_q\Big(\int_{\tau\in I} \vartheta(\tau)H(\tau,\tau)d\tau\Big)ds\Big](f_\infty)^{q-1}}$$

and

$$L_2 = \frac{1}{\Big[\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds\Big](f^0)^{q-1}}.$$

Theorem 3.4. *Assume that the conditions $(P1)$–$(P2)$ are satisfied. Then, for each λ satisfying*

$$L_1 < \lambda < L_2, \tag{3.2}$$

there exists at least one positive solution of p-Laplacian fractional order BVP (1.1)–(1.2) that lies in P.

Proof. Let λ be given as in (3.2). Now, let $\epsilon > 0$ be chosen such that

$$\frac{1}{\Big[\eta^2 \int_{s\in I} G(1,s)\phi_q\Big(\int_{\tau\in I} \vartheta(\tau)H(\tau,\tau)d\tau\Big)ds\Big](f_\infty - \epsilon)^{q-1}} \leq \lambda$$

and

$$\lambda \leq \frac{1}{\Big[\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds\Big](f^0 + \epsilon)^{q-1}}.$$

We seek fixed point of the completely continuous operator $T : P \to P$ defined by (3.1). Now, from the definitions of f^0, there exists an $H^1 > 0$ such that

$$\max_{t\in[0,1]} \frac{f(t,x)}{\phi_p(x)} \leq (f^0 + \epsilon), \quad \text{for } 0 < x \leq H^1.$$

It follows that $f(t,x) \leq (f^0+\epsilon)\phi_p(x)$, for $0 < x \leq H^1$. Let us choose $x \in P$ with $\|x\| = H^1$. Then, we have from Lemma 3.1,

$$\begin{aligned}
Tx(t) &= \lambda \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x)d\tau\Big)ds \\
&\leq \lambda \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x)d\tau\Big)ds \\
&\leq \lambda \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)(f^0+\epsilon)\phi_p(x)d\tau\Big)ds \\
&\leq \lambda \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)(f^0+\epsilon)^{q-1}x ds \\
&\leq \lambda \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)(f^0+\epsilon)^{q-1}\|x\| ds \\
&\leq \|x\|, \ t \in [0,1].
\end{aligned}$$

Therefore, $\|Tx\| \leq \|x\|$. Hence, if we set

$$\Omega_1 = \{u \in X : \|u\| < H^1\}.$$

Then,

$$\|Tx\| \leq \|x\|, \text{ for } x \in P \cap \partial\Omega_1. \tag{3.3}$$

By the definition of f_∞, there exists $\overline{H}^2 > 0$ such that

$$\min_{t\in[0,1]} \frac{f(t,x)}{\phi_p(x)} \geq (f_\infty - \epsilon), \text{ for } x \geq \overline{H}^2.$$

It follows that $f(t,x) \geq (f_\infty - \epsilon)\phi_p(x)$, for $x \geq \overline{H}^2$. If we set

$$H^2 = \max\left\{2H^1, \frac{\overline{H}^2}{\eta}\right\},$$

and define $\Omega_2 = \{u \in X : \|u\| < H^2\}$. If $x \in P \cap \partial\Omega_2$, so that $\|x\| = H^2$, then

$$\min_{t\in I} x(t) \geq \eta\|x\| \geq \overline{H}^2.$$

Then, we have from Lemma 2.5,

$$\begin{aligned}
Tx(t) &= \lambda \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x)d\tau\Big)ds \\
&\geq \lambda \int_{s\in I} \eta G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x)d\tau\Big)ds \\
&\geq \lambda \int_{s\in I} \eta G(1,s)\phi_q\Big(\int_{\tau\in I} \vartheta(\tau)H(\tau,\tau)(f_\infty-\epsilon)\phi_p(x)d\tau\Big)ds \\
&\geq \lambda \int_{s\in I} \eta G(1,s)\phi_q\Big(\int_{\tau\in I} \vartheta(\tau)H(\tau,\tau)d\tau\Big)(f_\infty-\epsilon)^{q-1}x ds \\
&\geq \lambda \int_{s\in I} \eta G(1,s)\phi_q\Big(\int_{\tau\in I} \vartheta(\tau)H(\tau,\tau)d\tau\Big)(f_\infty-\epsilon)^{q-1}\eta\|x\| ds \\
&\geq \lambda\eta^2 \int_{s\in I} G(1,s)\phi_q\Big(\int_{\tau\in I} \vartheta(\tau)H(\tau,\tau)d\tau\Big)ds(f_\infty-\epsilon)^{q-1}\|x\| \\
&\geq \|x\|.
\end{aligned}$$

Thus,

$$\|Tx\| \geq \|x\|, \quad \text{for } x \in P\cap\partial\Omega_2. \tag{3.4}$$

An application of Theorem 3.3 to (3.3) and (3.4) yields a fixed point of T that lies in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$. This fixed point is the solution of the fractional order BVP (1.1)–(1.2). □

Let

$$L_3 = \frac{1}{\Big[\eta^2 \int_{s\in I} G(1,s)\phi_q\Big(\int_{\tau\in I} \vartheta(\tau)H(\tau,\tau)d\tau\Big)ds\Big](f_0)^{q-1}}$$

and

$$L_4 = \frac{1}{\Big[\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds\Big](f^\infty)^{q-1}}.$$

Theorem 3.5. *Assume that the conditions* $(P1)$–$(P2)$ *are satisfied. Then, for each* λ *satisfying*

$$L_3 < \lambda < L_4, \tag{3.5}$$

there exists at least one positive solution of p-Laplacian fractional order BVP (1.1)–(1.2) *that lies in* P.

Proof. Let λ be given as in (3.5) and let $\epsilon > 0$ be chosen such that

$$\frac{1}{\Big[\eta^2 \int_{s\in I} G(1,s)\phi_q\Big(\int_{\tau\in I} \vartheta(\tau)H(\tau,\tau)d\tau\Big)ds\Big](f_0-\epsilon)^{q-1}} \leq \lambda$$

and

$$\lambda \leq \frac{1}{\Big[\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds\Big](f^\infty+\epsilon)^{q-1}}.$$

We seek fixed point of the completely continuous operator $T : P \to P$ defined by (3.1). Now, from the definition of f_0, there exists $J^1 > 0$ such that

$$\min_{t\in[0,1]} \frac{f(t,x)}{\phi_p(x)} \geq (f_0 - \epsilon), \quad \text{for } 0 < x \leq J^1.$$

It follows that $f(t,x) \geq (f_0 - \epsilon)\phi_p(x)$, for $0 < x \leq J^1$. In this case, define

$$\Omega_1 = \{u \in X : \|u\| < J^1\}.$$

Then, for $x \in P \cap \partial\Omega_1$, we have $f(\tau,x) \geq (f_0 - \epsilon)\phi_p(x)$, $\tau \in I$, and moreover, $x(t) \geq \eta\|x\|$, $t \in I$ and we have

$$\begin{aligned}
Tx(t) &= \lambda \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x)d\tau\Big)ds \\
&\geq \lambda \int_{s\in I} \eta G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x)d\tau\Big)ds \\
&\geq \lambda \int_{s\in I} \eta G(1,s)\phi_q\Big(\int_{\tau\in I} \vartheta(\tau)H(\tau,\tau)(f_0-\epsilon)\phi_p(x)d\tau\Big)ds \\
&\geq \lambda \int_{s\in I} \eta G(1,s)\phi_q\Big(\int_{\tau\in I} \vartheta(\tau)H(\tau,\tau)d\tau\Big)(f_0-\epsilon)^{q-1}x ds \\
&\geq \lambda \int_{s\in I} \eta G(1,s)\phi_q\Big(\int_{\tau\in I} \vartheta(\tau)H(\tau,\tau)d\tau\Big)(f_0-\epsilon)^{q-1}\eta\|x\| ds \\
&\geq \lambda\eta^2 \int_{s\in I} G(1,s)\phi_q\Big(\int_{\tau\in I} \vartheta(\tau)H(\tau,\tau)d\tau\Big)ds(f_0-\epsilon)^{q-1}\|x\| \\
&\geq \|x\|.
\end{aligned}$$

Thus,

$$\|Tx\| \geq \|x\|, \quad \text{for } x \in P \cap \partial\Omega_1. \tag{3.6}$$

By the definition of f^∞, there exists $\overline{J}^2 > 0$ such that

$$\max_{t\in[0,1]} \frac{f(t,x)}{\phi_p(x)} \leq (f^\infty + \epsilon), \quad \text{for } x \geq \overline{J}^2.$$

It follows that $f(t,x) \leq (f^\infty + \epsilon)\phi_p(x)$, for $x \geq \overline{J}^2$. There are two subcases.

Case (i): f is bounded. Suppose $L > 0$ is such that $\max_{t\in[0,1]} f(t,x) \leq L$, for all $0 < x < \infty$.

Let

$$J^2 = \max\left\{2J^1, L^{q-1}\lambda \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds\right\},$$

and let

$$\Omega_2 = \{u \in X : \|u\| < J^2\}.$$

Then, for $x \in P \cap \partial\Omega_2$ with $\|x\| = J^2$, we have

$$
\begin{aligned}
Tx(t) &= \lambda \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x)d\tau\Big)ds \\
&\leq \lambda \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)Ld\tau\Big)ds \\
&\leq \lambda \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)L^{q-1}ds \\
&\leq \lambda L^{q-1} \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds \\
&\leq J^2 \\
&= \|x\|, \quad t \in [0,1],
\end{aligned}
$$

and so

$$\|Tx\| \leq \|x\|, \quad \text{for } x \in P \cap \partial\Omega_2. \tag{3.7}$$

Case (ii): f is unbounded. Let $J^2 > \max\Big\{2J^1, \overline{J}^2\Big\}$ be such that $f(t,x) \leq f(t,J^2)$, and let

$$\Omega_2 = \{u \in X : \|u\| < J^2\}.$$

Choosing $x \in P \cap \partial\Omega_2$ and $\|x\| = J^2$, we have

$$
\begin{aligned}
Tx(t) &= \lambda \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x)d\tau\Big)ds \\
&\leq \lambda \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x)d\tau\Big)ds \\
&\leq \lambda \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)f(\tau,J^2)d\tau\Big)ds \\
&\leq \lambda \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)(f^\infty+\epsilon)\phi_p(J^2)d\tau\Big)ds \\
&\leq \lambda \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)(f^\infty+\epsilon)^{q-1}J^2ds \\
&\leq \lambda \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)(f^\infty+\epsilon)^{q-1}\|x\|ds. \\
&\leq \|x\|, \quad t \in [0,1].
\end{aligned}
$$

And so

$$\|Tx\| \leq \|x\|, \quad \text{for } x \in P \cap \partial\Omega_2. \tag{3.8}$$

An application of Theorem 3.3 to (3.7) and (3.8) yields a fixed point of T that lies in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$. This fixed point is the solution of the fractional order BVP (1.1)–(1.2). □

4. Even Number of Positive Solutions

In this section, we establish sufficient conditions for the existence of even number of positive solutions for p-Laplacian fractional order BVP (1.3), (1.2) by using Avery-Henderson functional fixed point theorem.

Let $B = \{x : x \in C[0,1]\}$ be the real Banach space equipped with the norm

$$\|x\| = \max_{t\in[0,1]} |x(t)|.$$

Define a cone $P \subset B$ by

$$P = \Big\{x \in B \mid x(t) \geq 0,\ t \in [0,1] \text{ and } \min_{t\in I} x(t) \geq \eta\|x\|\Big\}.$$

Let

$$\mathcal{R} = \frac{1}{\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds}$$

and

$$\mathcal{S} = \frac{1}{\int_{s\in I} \eta G(1,s)\phi_q\Big(\int_{\tau\in I} \vartheta(\tau)H(\tau,\tau)d\tau\Big)ds}.$$

Let $T : P \to B$ be the operator defined by

$$Tx(t) = \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds. \tag{4.1}$$

Lemma 4.1. *The operator T defined in* (4.1) *is a self map on P.*

Proof. Let $x \in P$. Clearly, $Tx(t) \geq 0$, for all $t \in [0,1]$, and

$$\begin{aligned} Tx(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds \\ &\leq \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds \end{aligned}$$

so that

$$\|Tx\| \leq \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds.$$

Next, if $x \in P$, then by the above inequality we have

$$\begin{aligned} \min_{t\in I} Tx(t) &= \min_{t\in I} \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds \\ &\geq \eta \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds \\ &\geq \eta\|Tx\|. \end{aligned}$$

Hence, $Tx \in P$ and so $T : P \to P$. Standard arguments involving the Arzela-Ascoli theorem shows that T is completely continuous. $\square$

Let ψ be a nonnegative continuous functional on a cone P of the real Banach space B. Then for a positive real number c', the sets are defined as

$$P(\psi, c') = \{y \in P : \psi(y) < c'\} \text{ and } P_a = \{y \in P : \|y\| < a\}.$$

In obtaining even number of positive solutions for p-Laplacian fractional order BVP (1.3), (1.2), the following Avery-Henderson functional fixed point theorem is fundamental.

Theorem 4.2 ([2]). *Let P be a cone in the real Banach space B. Suppose α and γ are increasing, nonnegative continuous functionals on P and θ is nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive numbers c' and k, $\gamma(y) \leq \theta(y) \leq \alpha(y)$ and $\|y\| \leq k\gamma(y)$, for all $y \in \overline{P(\gamma, c')}$. Suppose that there exist positive numbers a' and b' with $a' < b' < c'$ such that $\theta(\lambda y) \leq \lambda\theta(y)$, for all $0 \leq \lambda \leq 1$ and $y \in \partial P(\theta, b')$. Further, let $T : \overline{P(\gamma, c')} \to P$ be a completely continuous operator such that*

(B1) $\gamma(Ty) > c'$, *for all* $y \in \partial P(\gamma, c')$,
(B2) $\theta(Ty) < b'$, *for all* $y \in \partial P(\theta, b')$,
(B3) $P(\alpha, a') \neq \emptyset$ *and* $\alpha(Ty) > a'$ *for all* $y \in \partial P(\alpha, a')$.

Then, T has at least two fixed points $y_1, y_2 \in \overline{P(\gamma, c')}$ such that $a' < \alpha(y_1)$ with $\theta(y_1) < b'$ and $b' < \theta(y_2)$ with $\gamma(y_2) < c'$.

Define the nonnegative, increasing, continuous functionals γ, θ and α on the cone P by

$$\gamma(x) = \min_{t \in I} x(t), \theta(x) = \max_{t \in I} x(t) \text{ and } \alpha(x) = \max_{t \in [0,1]} x(t).$$

We observe that for any $x \in P$,

$$\gamma(x) \leq \theta(x) \leq \alpha(x), \tag{4.2}$$

$$\|x\| \leq \frac{1}{\eta} \min_{t \in I} x(t) = \frac{1}{\eta}\gamma(x) \leq \frac{1}{\eta}\theta(x) \leq \frac{1}{\eta}\alpha(x). \tag{4.3}$$

Theorem 4.3. *Suppose there exist $0 < a' < b' < c'$ such that f satisfies the following conditions:*

(A1) $f(t, x(t)) > \phi_p(c'\mathcal{S})$, $t \in I$ *and* $x \in \left[c', \frac{c'}{\eta}\right]$,
(A2) $f(t, x(t)) < \phi_p(b'\mathcal{R})$, $t \in [0, 1]$ *and* $x \in \left[0, \frac{b'}{\eta}\right]$,
(A3) $f(t, x(t)) > \phi_p(a'\mathcal{S})$, $t \in I$ *and* $x \in \left[a', \frac{a'}{\eta}\right]$.

Then the fractional order BVP (1.3), (1.2) *has at least two positive solutions x_1 and x_2 such that*

$$a' < \max_{t \in [0,1]} x_1(t) \text{ with } \max_{t \in I} x_1(t) < b',$$

$$b' < \max_{t \in I} x_2(t) \text{ with } \min_{t \in I} x_2(t) < c'.$$

Proof. We seek two fixed points $x_1, x_2 \in P$ of T defined by (4.1). From Lemma 4.1, (4.2) and (4.3), for each $x \in P$, $\gamma(x) \le \theta(x) \le \alpha(x)$ and $\|x\| \le \frac{1}{\eta}\gamma(x)$. Also, for any $0 \le \lambda^* \le 1$ and $x \in P$, $\theta(\lambda^* x) = \max_{t\in I}(\lambda^* x)(t) = \lambda^* \max_{t\in I} x(t) = \lambda^*\theta(x)$. It is clear that $\theta(0) = 0$. We now show that the remaining conditions of Theorem 4.2 are satisfied.

Firstly, we shall verify that condition $(B1)$ of Theorem 4.2 is satisfied. Since $x \in \partial P(\gamma, c')$, from (4.3) we have that $c' = \min_{t\in I} x(t) \le \|x\| \le \frac{c'}{\eta}$. Then

$$\begin{aligned}
\gamma(Tx) &= \min_{t\in I}\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds\\
&\ge \int_{s\in I}\eta G(1,s)\phi_q\Big(\int_{\tau\in I}\vartheta(\tau)H(\tau,\tau)f(\tau,x(\tau))d\tau\Big)ds\\
&> \int_{s\in I}\eta G(1,s)\phi_q\Big(\int_{\tau\in I}\vartheta(\tau)H(\tau,\tau)\phi_p\Big(c'\mathcal{S}\Big)d\tau\Big)ds\\
&= c'\mathcal{S}\int_{s\in I}\eta G(1,s)\phi_q\Big(\int_{\tau\in I}\vartheta(\tau)H(\tau,\tau)d\tau\Big)ds = c',
\end{aligned}$$

using the condition $(A1)$.

Now we shall show that condition $(B2)$ of Theorem 4.2 is satisfied. Since $x \in \partial P(\theta, b')$, from (3.2) we have that $0 \le x(t) \le \|x\| \le \frac{b'}{\eta}$, for $t \in [0,1]$. Thus

$$\begin{aligned}
\theta(Tx) &= \max_{t\in I}\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds\\
&\le \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)f(\tau,x(\tau))d\tau\Big)ds\\
&< \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)\phi_p\Big(b'\mathcal{R}\Big)d\tau\Big)ds\\
&= b'\mathcal{R}\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds = b',
\end{aligned}$$

by the condition $(A2)$.

Finally, using the condition $(A3)$, we shall show that condition $(B3)$ of Theorem 4.2 is satisfied. Since $0 \in P$ and $a' > 0$, $P(\alpha, a') \ne \phi$. Since $x \in \partial P(\alpha, a')$, $a' = \max_{t\in[0,1]} x(t) \le \|x\| \le \frac{a'}{\eta}$, for $t \in I$. Therefore,

$$\begin{aligned}
\alpha(Tx) &= \max_{t\in[0,1]}\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\Big)ds\\
&\ge \int_{s\in I}\eta G(1,s)\phi_q\Big(\int_{\tau\in I}\vartheta(\tau)H(\tau,\tau)f(\tau,x(\tau))d\tau\Big)ds\\
&> \int_{s\in I}\eta G(1,s)\phi_q\Big(\int_{\tau\in I}\vartheta(\tau)H(\tau,\tau)\phi_p\Big(a'\mathcal{S}\Big)d\tau\Big)ds\\
&= a'\mathcal{S}\int_{s\in I}\eta G(1,s)\phi_q\Big(\int_{\tau\in I}\vartheta(\tau)H(\tau,\tau)d\tau\Big)ds = a'.
\end{aligned}$$

Thus, all the conditions of Theorem 4.2 are satisfied. Therefore, the fractional order BVP (1.3), (1.2) has at least two positive solutions $x_1, x_2 \in \overline{P(\gamma, c')}$. This completes the proof of the theorem. □

Theorem 4.4. *Let m be an arbitrary positive integer. Assume that there exist numbers a_r $(r = 1, 2, \ldots, m+1)$ and b_s $(s = 1, 2, \ldots, m)$ with $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m < a_{m+1}$ such that f satisfies the following conditions:*

(A4) $f(t, x(t)) > \phi_p(a_r \mathcal{S})$, *$t \in I$ and $x \in \left[a_r, \frac{a_r}{\eta}\right]$, $r = 1, 2, \ldots, m+1$,*

(A5) $f(t, x(t)) < \phi_p(b_s \mathcal{R})$, *$t \in [0, 1]$ and $x \in \left[0, \frac{b_s}{\eta}\right]$, $s = 1, 2, \ldots, m$.*

Then the fractional order BVP (1.3), (1.2) *has at least* $2m$ *positive solutions in* $\overline{P}_{a_{m+1}}$.

Proof. We use induction on m. For $m = 1$, we know from the conditions $(A4)$ and $(A5)$ that $T : \overline{P}_{a_2} \to P_{a_2}$, then it follows from Avery-Henderson functional fixed point theorem that the fractional order BVP (1.3), (1.2) has at least two positive solutions in $\overline{P}_{a_2}$. Next, we assume that this conclusion holds for $m = l$. In order to prove that this conclusion holds for $m = l + 1$. We suppose that there exist numbers $a_r(r = 1, 2, \ldots, l + 2)$ and $b_s(s = 1, 2, \ldots, l + 1)$ with $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_{l+1} < b_{l+1} < a_{l+2}$ such that f satisfies the following conditions:

$$f(t, x(t)) > \phi_p\Big(a_r \mathcal{S}\Big), \quad t \in I \text{ and } x \in \left[a_r, \frac{a_r}{\eta}\right], \quad r = 1, 2, \ldots, l + 2, \tag{4.4}$$

$$f(t, x(t)) < \phi_p\Big(b_s \mathcal{R}\Big), \quad t \in [0, 1] \text{ and } x \in \left[0, \frac{b_s}{\eta}\right], \quad s = 1, 2, \ldots, l + 1. \tag{4.5}$$

By assumption, the fractional order BVP (1.3), (1.2) has at least $2l$ positive solutions $x_i(i = 1, 2, \dot{;} 2l)$ in $\overline{P}_{a_{l+1}}$. At the same time, it follows from Theorem 4.3, (4.4) and (4.5) that the fractional order BVP (1.3), (1.2) has at least two positive solutions x_1, x_2 in $\overline{P}_{a_{l+2}}$ such that $a_{l+1} < \alpha(x_1)$ with $\theta(x_1) < b_{l+1}$ and $b_{l+1} < \theta(x_2)$ with $\gamma(x_2) < a_{l+2}$. Obviously x_1 and x_2 are distinct from x_i $(i = 1, 2, \ldots, 2l)$ in $\overline{P}_{a_{l+1}}$. Therefore, the fractional order BVP (1.3), (1.2) has at least $2l + 2$ positive solutions in $\overline{P}_{a_{l+2}}$, which shows that this conclusion also holds for $m = l + 1$. This completes the proof of theorem. □

5. EXAMPLES

In this section, as an application, we demonstrate our results with examples.

Example 5.1 Consider the p-Laplacian fractional order BVP,

$$D_{0^+}^{1.7}\Big(\phi_p\Big(D_{0^+}^{2.5} x(t)\Big)\Big) + \lambda f(t, x) = 0, \quad t \in (0, 1), \tag{5.1}$$

$$x(0) = 0, \quad x'(0) = 0, \quad x''(1) = 0, \quad D_{0^+}^{2.5} x(0) = 0, \quad D_{0^+}^{2.5} x(1) = 0, \tag{5.2}$$

where

$$f(t, x) = x(5360000 - 5359970 e^{-2x}).$$

Then, the Green functions $G(t,s)$ and $H(t,s)$ are given by

$$G(t,s)=\begin{cases}\frac{t^{1.5}(1-s)^{-0.5}}{\Gamma(2.5)}, & t\leq s,\\ \frac{t^{1.5}(1-s)^{-0.5}-(t-s)^{1.5}}{\Gamma(2.5)}, & s\leq t,\end{cases}$$

$$H(t,s)=\begin{cases}\frac{[t(1-s)]^{0.7}}{\Gamma(1.7)}, & t\leq s,\\ \frac{[t(1-s)]^{0.7}-(t-s)^{0.7}}{\Gamma(1.7)}, & s\leq t.\end{cases}$$

Clearly, the Green's functions $G(t,s),H(t,s)$ are positive. Let $p=2$. By direct calculations, $\eta=0.125$, $f^0=30$ and $f_\infty=5360000$. Employing Theorem 3.3, we get an eigenvalue interval is $0.00821042253<\lambda<0.085261021$, for which the p-Laplacian fractional order BVP (5.1)–(5.2) has at least one positive solution.

Example 5.2 Consider the p-Laplacian fractional order BVP,

$$D_{0^+}^{1.9}\Big(\phi_p\Big(D_{0^+}^{2.8}x(t)\Big)\Big)=f(t,x),\quad t\in(0,1),\tag{5.3}$$

$$x(0)=x'(0)=x''(1)=0,\quad D_{0^+}^{2.8}x(0)=0=D_{0^+}^{2.8}x(1),\tag{5.4}$$

where

$$f(t,x)=\frac{95(x+3)^3}{x^2+5}-e^{\frac{15223}{10000}x}.$$

Then the Green functions $G(t,s)$ and $H(t,s)$ are given by

$$G(t,s)=\begin{cases}\frac{t^{1.8}(1-s)^{-0.2}}{\Gamma(2.8)}, & t\leq s,\\ \frac{t^{1.8}(1-s)^{-0.2}-(t-s)^{1.8}}{\Gamma(2.8)}, & s\leq t,\end{cases}$$

$$H(t,s)=\begin{cases}\frac{[t(1-s)]^{0.9}}{\Gamma(1.9)}, & t\leq s,\\ \frac{[t(1-s)]^{0.9}-(t-s)^{0.9}}{\Gamma(1.9)}, & s\leq t.\end{cases}$$

Clearly, the Green functions $G(t,s)$, $H(t,s)$ are positive, f is continuous and increasing on $[0,\infty)$. Let $p=2$. By direct calculations, $\eta=0.082469$, $\mathcal{R}=7$ and $\mathcal{S}=424$. Choosing $a'=0.0002$, $b'=0.4$ and $c'=3$, then $0<a'<b'<c'$ and f satisfies

(i) $f(t,x)>1272=\phi_p(c'\mathcal{S})$, $t\in\left[\frac{1}{4},\frac{3}{4}\right]$ and $x\in[3,36.377305]$,

(ii) $f(t,x)<2.8=\phi_p(b'\mathcal{R})$, $t\in[0,1]$ and $x\in[0,4.850307]$,

(iii) $f(t,x)>0.0848=\phi_p(a'\mathcal{S})$, $t\in\left[\frac{1}{4},\frac{3}{4}\right]$ and $x\in[0.0002,0.002425]$.

Then all the conditions of Theorem 4.3 are satisfied. Thus by Theorem 4.3, the p-Laplacian fractional order BVP (5.3)–(5.4) has at least two positive solutions x_1 and x_2 satisfying

$$0.0002<\max_{t\in[0,1]}x_1(t)\quad\text{with}\quad\max_{t\in[\frac{1}{4},\frac{3}{4}]}x_1(t)<0.4,$$

$$0.4<\max_{t\in I}x_2(t)\quad\text{with}\quad\min_{t\in[\frac{1}{4},\frac{3}{4}]}x_2(t)<3.$$

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