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GENERALIZATION OF GRONWALL'S INEQUALITY AND ITS APPLICATIONS IN FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we briefly review the recent development of research on Gronwall's inequality. Then obtain a result for the following nonlinear integral inequality:

$$w(u(t)) \le K + \sum_{i=1}^{n} \int_{\alpha_i} (t_0)^{\alpha_i(t)} f_i(s) \prod_{j=1}^{m} H_{ij}(u(s)) G_{ij} \left(\max_{s-h\le\xi\le s} u(\xi) \right) ds.$$

As an application, we study the abstract functional differential equation, $\frac{du}{dt} = f(t, u_t)$ with Lyapunov's second method. Then, we obtain an estimate of solutions of functional differential equations, $u' = F(t, u_t)$ with conditions like:

i) $W_1(|u(t)|_{\mathbf{X}}) \le V(t, u_t) \le W_2(\|u_t\|_{\mathbf{CX}})$

ii) $V'_{(3.1)}(t, u_t) \le 0$.

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1. INTRODUCTION

In 1919, Gronwall [7] gave the following lemma when he studied a system of differential equations with a parameter.

Theorem 1.1 (Gronwall's Original Inequality)**.** *Let α, a, b and h be nonnegative constants, and $u : [\alpha, \alpha + h] \to [0, \infty)$ be continuous. If*

$$0 \le u(t) \le \int_{\alpha}^{t} [bu(s) + a]ds, \quad \alpha \le t \le \alpha + h,$$

then

$$0 \le u(t) \le ahe^{bh}, \quad \alpha \le t \le \alpha + h.$$

His lemma stayed basically quietly until Bellman generalized it in 1943, which is now commonly known as Gronwall's Inequality, or Gronwall-Bellman's Inequality. This version of Gronwall's inequality can be found in many references, for example [1, 5, 12].

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Theorem 1.2 (Gronwall's Inequality)**.** *Let* α, β *and* c *be nonnegative constants, and* $u, f : [\alpha, \beta] \to [0, \infty)$ *continuous. If*

$$u(t) \leq c + \int_{\alpha}^{t} f(s)u(s)ds, \quad \alpha \leq t \leq \beta,$$

then

$$u(t) \leq ce^{\int_{\alpha}^{t} f(s)ds}, \quad \alpha \leq t \leq \beta.$$

In 1958, Bellman [3] generalized Theorem 1.2, his own result, by allowing that c is a nonnegative and nondecreasing function, which is stated as following, and can be founded in many references, for example [8, 12].

Theorem 1.3 (Gronwall-Bellman's Inequality)**.** *If* $u(t)$ *and* $\alpha(t)$ *are real valued continuous functions on* $[a, b]$*,* $\alpha(t)$ *is nondecreasing, and* $\beta(t) \geq 0$ *is integrable on* $[a, b]$ *with*

$$u(t) \leq \alpha(t) + \int_{a}^{t} \beta(s)u(s)ds \quad \text{for } a \leq t \leq b,$$

then

$$u(t) \leq \alpha(t)e^{\int_{a}^{t} \beta(r)dr}, \quad \text{for } a \leq t \leq b.$$

Today, Gronwall's inequality has been found very useful in research of boundedness and stability of differential equations; research of inequalities of the Gronwall type has exploded; and generalization of Gronwall's inequality has gone into many different directions. For example, the integral in Gronwall's inequality is generalized to iterated integrals or a sum of integrals; the function $u(t)$ is generalized to a format of $W(u(t))$ and from a one-variable function to a two-variable function with double integrals; the upper limit of the integral is generalized to a function, instead of a simple variable t; and the integral inequality is generalized to a difference inequality. Applications of this type of inequalities are also expanded from ordinary differential equations to functional differential equations, fractional differential equations and difference equations. In 1998, Pachpatte published a book [12] summarizing the development of inequalities of the Gronwall type up to 1998. Here are some examples of new generalizations of the Gronwall type in the past a few years.

In 2007, Pachpatte [13] investigated the following two types of inequalities of the Gronwall type involving double integrals and summations:

$$u(x, y) \leq c + \int_{0}^{x} p(s, y)u(s, y)ds + \int_{0}^{x}\int_{0}^{y} f(s, t)[u(s, t) + \int_{0}^{s}\int_{0}^{t} g(\sigma, \tau)u(\sigma, \tau)d\tau d\sigma + \int_{0}^{a}\int_{0}^{b} h(\sigma, \tau)u(\sigma, \tau)d\tau d\sigma]dtds,$$

and

$$u(n,m) \le c + \sum_{s=0}^{n-1} p(s,m)u(s,m) + \sum_{s=0}^{n-1}\sum_{t=0}^{m-1} f(s,t)[u(s,t)$$
$$+ \sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1} g(\sigma,\tau)u(\sigma,\tau) + \sum_{\sigma=0}^{\alpha}\sum_{\tau=0}^{\beta} h(\sigma,\tau)u(\sigma,\tau)].$$

In 2012, Bohner, Hristova and Stefanova [4] studied the following inequality:

$$\psi(u(t)) \le k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)u^p(s)\omega_i(u(s))ds$$
$$+ \sum_{j=1}^{m} \int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)u^p(s)\tilde{\omega}_j\left(\max_{\xi\in[s-h,s]} u(\xi)\right) ds. \tag{1.1}$$

In 2013, Lin [11] gave the following result of the Gronwall type when he considered fractional differential equations: Let $\beta_i > 0$ be constants; $b_i(t)$ defined on $[0,T)$ be bounded, continuous and monotonically increasing; $i = 1, 2, \ldots, n$, $a(t)$ and $u(t)$ be continuous. If

$$u(t) \le a(t) + b(t)\int_0^t (t-s)^{\beta-1}u(s)ds, \quad t \in [0,T), \tag{1.2}$$

then

$$u(t) \le a(t) + \sum_{k=1}^{\infty}\left(\sum_{i'=1}^{n} \frac{\prod_{i=1}^{k}[b_{i'}(t)\Gamma(\beta_{i'})]}{\Gamma\left(\sum_{i=1}^{k}\beta_{i'}\right)}\int_0^t (t-s)^{\sum_{i=1}^{k}\beta_{i'}-1}a(s)ds\right).$$

Meanwhile, by reversing the direction of Inequality (1.2), Lin also gave the following result: Let $\beta_i > 0$ be constants; $b_i(t)$ defined on $[0,T)$ be bounded, continuous, nonnegative and monotonically increasing; $i = 1, 2, \ldots, n$; $u(t)$ be continuous. If

$$u(t) \ge \sum_{i=1}^{n} b_i(t)\int_0^t (t-s)^{\beta_i-1}u(s)ds, \quad t \in [0,T),$$

then $u(t) \ge 0$, $t \in [0,T)$.

In 2014, Wang, Zheng and Guo [14] studied the following nonlinear delay sum-difference inequality:

$$u(m,n) \le a(m,n) + \sum_{i=1}^{k}\sum_{s=\alpha_i(0)}^{\alpha_i(m-1)}\sum_{s=\beta_i(0)}^{\beta_i(n-1)} f_i(m,n,s,t)w_i(u(s,t)), \quad m,n \in \mathbb{N}_0.$$

2. A NONLINEAR INTEGRAL INEQUALITY

In this section, we generalize Inequality (1.1) to the following inequality and simplify the proof of [4]:

$$w(u(t)) \leq K + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s) \prod_{j=1}^{m} H_{ij}(u(s)) G_{ij}\left(\max_{s-h\leq\xi\leq s} u(\xi)\right) ds, \quad t_0 \leq t < T. \tag{2.1}$$

Notations and conditions for this inequalty are as following: Let $h > 0, K > 0, t_0$, and T are constants, $0 \leq t_0 < T \leq \infty$, $\mathbb{R}_+ = [0, \infty)$.

(A1) $\alpha_i \in C^1([t_0, T), \mathbb{R}_+)$ are nondecreasing and $\alpha_i(t) \leq t, t \in [t_0, T)$, $i = 1, 2, \ldots, n$;
(A2) $f_i \in C([0, T), \mathbb{R}_+)$ for $i = 1, 2, \ldots, n$;
(A3) $H_{ij}, G_{ij} \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing, and $H_{ij}(x) > 0$, $G_{ij}(x) > 0$ if $x > 0$;
(A4) $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ is increasing, $w(0) = 0$, and $\lim_{t\to\infty} w(t) = \infty$;
(A5) $u \in C([-h, T), \mathbb{R}_+)$.

Theorem 2.1. *Let $u(t)$ satisfy* (2.1) *with conditions* (A1) *through* (A5)*. Then*

$$u(t) \leq w^{-1}\left[K + W^{-1}\left(\int_{\alpha(t_0)}^{\alpha(t)} \sum_{i=1}^{n} f_i(s) ds\right)\right], \quad t_0 \leq t < T$$

where

$$W(r) = \int_0^r \frac{1}{H^m(w^{-1}(K+s)) G^m(w^{-1}(K+s))} ds, \quad 0 \leq r < \infty$$

$$H(r) = \max_{0\leq i\leq n, 0\leq j\leq m}\{H_{ij}(r)\}, \quad G(r) = \max_{0\leq i\leq n, 0\leq j\leq m}\{G_{ij}(r)\}.$$

Proof. Define

$$Z(t) = K + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s) \prod_{j=1}^{m} H_{ij}(u(s)) G_{ij}\left(\max_{s-h\leq\xi\leq s} u(\xi)\right) ds, \quad t_0 \leq t < T.$$

Then $Z(t)$ is nondecreasing, and $w(u(t)) \leq Z(t)$, $t_0 \leq t < T$.

By (A4), w^{-1} exists and has the same properties as those of w. Therefore,

$$u(t) \leq w^{-1}(Z(t)), \quad t_0 \leq t < T.$$

In addition,

$$\max_{s-h\leq\xi\leq s} u(\xi) \leq \max_{s-h\leq\xi\leq s} w^{-1}(Z(\xi)) = w^{-1}(Z(s)), \quad t_0 \leq s < T.$$

Therefore,

$$\begin{aligned} Z(t) &\le K + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s) \prod_{j=1}^{m} H(u(s)) G\left(\max_{s-h\le\xi\le s} u(\xi) \right) ds \\ &\le K + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s) \prod_{j=1}^{m} H(w^{-1}(Z(s))) G(w^{-1}(Z(s))) ds \\ &= K + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s) H^m(w^{-1}(Z(s))) G^m(w^{-1}(Z(s))) ds. \end{aligned}$$

Let

$$R(t) = \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s) H^m(w^{-1}(Z(s))) G^m(w^{-1}(Z(s))) ds.$$

Clearly, $Z(t) \le K + R(t)$. Then

$$\begin{aligned} R'(t) &= \sum_{i=1}^{n} f_i(\alpha_i(t)) H^m(w^{-1}(Z(\alpha_i(t)))) G^m(w^{-1}(Z(\alpha_i(t)))) {\alpha_i}'(t) \\ &\le \sum_{i=1}^{n} f_i(\alpha_i(t)) H^m(w^{-1}(Z(t))) G^m(w^{-1}(Z(t))) {\alpha_i}'(t) \\ &\le \sum_{i=1}^{n} f_i(\alpha_i(t)) H^m(w^{-1}(K+R(t))) G^m(w^{-1}(K+R(t))) {\alpha_i}'(t) \\ &= H^m(w^{-1}(K+R(t))) G^m(w^{-1}(K+R(t))) \sum_{i=1}^{n} f_i(\alpha_i(t)) {\alpha_i}'(t). \end{aligned}$$

$$\frac{R'(t)}{H^m(w^{-1}(K+R(t))) G^m(w^{-1}(K+R(t)))} \le \sum_{i=1}^{n} f_i(\alpha_i(t)) {\alpha_i}'(t).$$

That is

$$\frac{d(W(R(t)))}{dt} \le \sum_{i=1}^{n} f_i(\alpha_i(t)) {\alpha_i}'(t).$$

Integrate the above inequality from t_0 to t,

$$W(R(t)) - W(R(t_0)) \le \int_{t_0}^{t} \sum_{i=1}^{n} f_i(\alpha_i(s)) {\alpha_i}'(s) ds = \int_{\alpha(t_0)}^{\alpha(t)} \sum_{i=1}^{n} f_i(s) ds.$$

$$W(R(t)) \le \int_{\alpha(t_0)}^{\alpha(t)} \sum_{i=1}^{n} f_i(s) ds$$

$$R(t) \le W^{-1}\left(\int_{\alpha(t_0)}^{\alpha(t)} \sum_{i=1}^{n} f_i(s) ds \right)$$

$$Z(t) \le K + W^{-1}\left(\int_{\alpha(t_0)}^{\alpha(t)} \sum_{i=1}^{n} f_i(s) ds \right)$$

$$u(t) \le w^{-1}\left[K + W^{-1}\left(\int_{\alpha(t_0)}^{\alpha(t)} \sum_{i=1}^{n} f_i(s) ds \right) \right]$$

□

Inequality (2.1) generalizes Inequality (1.1). Let us consider a very simple case of Inequality (2.1) for applications.

Corollary 2.2. *Given*

$$w(u(t)) \leq K + \int_{t_0}^{t} f(s)H(u(s))ds, \quad t_0 \leq t < T, \tag{2.2}$$

or

$$w(u(t)) \leq K + \int_{t_0}^{t} f(s)H(\max_{s-h\leq\xi\leq s} u(\xi))ds, \quad t_0 \leq t < T, \tag{2.3}$$

where f, H, w *and* u *satisfy* (A2), (A3), (A4) *and* (A5), *respectively. Then*

$$u(t) \leq w^{-1}\left[K + W^{-1}\left(\int_{t_0}^{t} f(s)ds\right)\right],$$

where $W(r) = \int_0^r \frac{1}{H(w^{-1}(K+s))}ds$, $0 \leq r < \infty$.

3. APPLICATIONS IN CONJUNCTION WITH LYAPUNOV'S SECOND METHOD

Let us consider the general abstract functional differential equations with finite delay

$$\frac{du}{dt} = F(t, u_t). \tag{3.1}$$

First, let us set forth some notation and terminology. Suppose that $(\mathbf{X}, |\cdot|_{\mathbf{X}})$ and $(\mathbf{Y}, |\cdot|_{\mathbf{Y}})$ are Banach spaces and $\mathbf{X} \cap \mathbf{Y} \neq \emptyset$ with $|u|_{\mathbf{Y}} \leq N|u|_{\mathbf{X}}$ for some constant $N > 0$ and all $u \in \mathbf{X} \cap \mathbf{Y}$. Denote $\mathbf{C_X} = \mathbf{C}([-h, 0], \mathbf{X})$, where $h > 0$ is a constant and

$$\|u\|_{\mathbf{CX}} = \sup\{|u(s)|_{\mathbf{X}} : -h \leq s \leq 0\} \quad \text{for } u \in \mathbf{C_X}.$$

Thus $(\mathbf{C_X}, \|\cdot\|_{\mathbf{CX}})$ is also a Banach space. If $u : [t_0 - h, \beta) \to \mathbf{X}$ for some $\beta > t_0$, define $u_t \in \mathbf{C_X}$ by $u_t(s) = u(t + s)$ for $s \in [-h, 0]$, where $t \in [t_0, \beta)$. For a positive constant H, by $\mathbf{C_{XH}}$ we denote the subset of $\mathbf{C_X}$ for which $\|\phi\|_{\mathbf{CX}} < H$ for each $\phi \in \mathbf{C_X}$. $F : \mathbf{R}_+ \times \mathbf{C_X} \to \mathbf{Y}$ is a function. Here $\mathbf{R}_+ = [0, \infty)$. $\frac{du}{dt}$ is the strong derivative [9] of u at t in $(\mathbf{X}, |\cdot|_{\mathbf{X}})$.

Definition 3.1. A function $u(t_0, \phi)$ is said to be a solution of Eq. (3.1) with initial function $\phi \in \mathbf{C_X}$ at $t = t_0 \geq 0$ and having value $u(t, t_0, \phi)$ if there is a $\beta > 0$ such that $u : [t_0 - h, t_0 + \beta) \to \mathbf{X} \cap \mathbf{Y}$ with $u_t \in \mathbf{C_{X\cap Y}}$ for $t_0 \leq t < t_0 + \beta$, $u_{t_0} = \phi$, and $u(t, t_0, \phi)$ satisfies Eq. (3.1) on $(t_0, t_0 + \beta)$ in $(\mathbf{Y}, |\cdot|_{\mathbf{Y}})$.

In this section, a wedge, denoted by W_i, is a continuous and strictly increasing function from $\mathbf{R}_+ \to \mathbf{R}_+$ with $W_i(0) = 0$, which is related to properties of continuous scalar functionals (called Lyapunov functionals) $V : \mathbf{R}_+ \times \mathbf{C}_{\mathbf{XH}} \to \mathbf{R}_+$ which are differentiated along the solutions of Eq. (3.1) by the relation

$$V'_{(3.1)}(t,\phi) = \sup \lim_{\delta \to 0^+} \sup[V(t+\delta, u_{t+\delta}(t,\phi)) - V(t,\phi)]/\delta$$

and $V(t,0) = 0$ for all $t \in \mathbf{R}_+$, where $u(t,\phi)$ is a solution of Eq. (3.1) satisfying $u_t = \phi$ and the first "sup" runs over such solutions, since the solution of Eq. (3.1) may not be unique. Detailed consequences of this derivative are discussed in [6], [8], [9].

Lyapunov's second method has generated numerous results on stability and boundedness. For example, please refer to Burton [6], Hale and Lunel [8], Ladas, Lakshmikantham and Leela [9, 10], Wang [20, 21], and Wu [22]. The following are two typical theorems [6] about uniform stability and uniformly asymptotical stability, which are stated for a system of ordinary functional differential equations:

$$X' = F(t, X_t), \quad X \in \mathbb{R}^n. \tag{3.2}$$

Theorem 3.1. *Let $D > 0$ be a constant, $V(t,\varphi_t)$ be a scalar continuous function in (t,φ_t) and locally Lipschitz in φ_t when $t_0 \le t < \infty$ and $\varphi_t \in C(t)$ with $\|\varphi_t\| < D$. Suppose that $F(t,0) = 0$, $V(t,0) = 0$, and*

i) $W_1(|\varphi(t)|) \le V(t,\varphi_t) < W_2(\|\varphi_t\|)$,
ii) $V'_{(3.2)}(t,\varphi_t) \le 0$.

Then $X = 0$ is uniformly stable.

Theorem 3.2. *Let $D > 0$ be a constant, $V(t,\varphi_t)$ be a scalar continuous function in (t,φ_t) and locally Lipschitz in φ_t when $t_0 \le t < \infty$ and $\varphi_t \in C(t)$ with $\|\varphi_t\| < D$. Suppose that $F(t,0) = 0$, $V(t,0) = 0$, and*

i) $W_1(|\varphi(t)|) \le V(t,\varphi_t) < W_2(|\varphi(t)|) + W_3(\int_{t-h}^{t} |\varphi(s)|^2 ds)$,
ii) $V'_{(3.2)}(t,\varphi_t) \le -W_4(|\varphi(t)|)$.

Then $X = 0$ is uniformly asymptotically stable.

It has been an interest for a long time if we can use conditions similar to (i) and (ii) in the above theorems to obtain an inequality about solutions of Equation (3.2). Wang has published a few papers [15, 16, 17, 18, 19] on this type of research. Here is a typical one. The proof is done by application of Theorem 1.3 (Gronwall-Bellman's Inequality).

Theorem 3.3 (Wang, [17])**.** *Let $V : \mathbf{R}_+ \times \mathbf{C}_{\mathbf{XH}} \to \mathbf{R}_+$ be continuous and $D : \mathbf{R}_+ \times \mathbf{C}_{\mathbf{XH}} \to \mathbf{R}_+$ be continuous along the solutions of Eq. (3.1), and η, L, and $P : \mathbf{R}_+ \to \mathbf{R}_+$ be integrable. Suppose the following conditions hold:*

i) $W_1(|u(t)|_{\mathbf{X}}) \le V(t,u_t) \le W_2(D(t,u_t)) + \int_{t-h}^{t} L(s) W_1(|u(s)|_{\mathbf{X}}) ds$,
ii) $V'_{(3.1)}(t,u_t) \le -\eta(t) W_2(D(t,u_t)) + P(t)$.

Then the solutions of Eq. (3.1), $u(t) = u(t, t_0, \phi)$, *satisfy the following inequality:*

$$W_1(|u(t)|_{\mathbf{X}}) \leq \left[K + \int_{t_0}^{t} P(s) e^{\int_{t_0}^{s} \eta(r)dr} ds\right] e^{\int_{t_0}^{t}[-\eta(s)+L(s)(e^{\int_s^{s+h}\eta(r)dr}-1)]ds}, \quad t \geq t_0, \tag{3.3}$$

where $K = V(t_0, \phi) + [e^{\int_{t_0}^{t_0+h}\eta(r)dr} - 1] \int_{-h}^{0} L(s+t_0) W_1(|\phi(s)|_{\mathbf{X}}) ds$.

Now, as an application of Theorem 2.1, we prove another inequality with similar conditions to those in Theorem 3.1.

Theorem 3.4. *Let* $V : \mathbf{R}_+ \times \mathbf{C_{XH}} \to \mathbf{R}_+$ *be continuous and* $D : \mathbf{R}_+ \times \mathbf{C_{XH}} \to \mathbf{R}_+$ *be continuous along the solutions of Eq.* (1). *Suppose the following conditions hold:*

i) $W_1(|u(t)|_{\mathbf{X}}) \leq V(t, u_t) \leq W_2(\|u_t\|_{\mathbf{CX}})$
ii) $V'_{(3.1)}(t, u_t) \leq 0$.

Then

$$|u(t)|_{\mathbf{X}} \leq W_1^{-1}\left[V(t_0, u_{t_0}) + W^{-1}(t - t_0)\right], \quad t_0 \leq t,$$

where

$$W(r) = \int_0^r \frac{1}{W_2(W_1^{-1}(V(t_0, u_{t_0})e^{t_0} + s))} ds.$$

Proof. For convenience, let us denote $V(t) = V(t, u_t)$. With Condition (ii),

$$\begin{aligned} V'_{(3.1)}(t) &\leq -W_2(\|u_t\|_{\mathbf{CX}}) + W_2(\|u_t\|_{\mathbf{CX}}) \\ &\leq -V(t) + W_2(\|u_t\|_{\mathbf{CX}}) \end{aligned}$$

Thus, $V'(t) + V(t) \leq W_2(\|u_t\|_{\mathbf{CX}})$. Multiplying e^t, we get

$$V'(t)e^t + V(t)e^t \leq W_2(\|u_t\|_{\mathbf{CX}})e^t.$$

That is $\frac{d(V(t)e^t)}{dt} \leq W_2(\|u_t\|_{\mathbf{CX}})e^t$. Integrating from t_0 to t, we get

$$V(t)e^t - V(t_0)e^{t_0} \leq \int_{t_0}^{t} W_2(\|u_s\|_{\mathbf{CX}})e^s ds \leq e^t \int_{t_0}^{t} W_2(\|u_s\|_{\mathbf{CX}}) ds.$$

Thus,

$$\begin{aligned} V(t) &\leq V(t_0)e^{-t+t_0} + \int_{t_0}^{t} W_2(\|u_s\|_{\mathbf{CX}}) ds \\ &\leq V(t_0) + \int_{t_0}^{t} W_2(\|u_s\|_{\mathbf{CX}}) ds. \end{aligned}$$

With Condition (i),

$$W_1(|u(t)|_{\mathbf{X}}) \leq V(t, u_t) \leq V(t_0) + \int_{t_0}^{t} W_2(\|u_s\|_{\mathbf{CX}}) ds.$$

Apply Corollary 2.2, we get

$$|u(t)|_{\mathbf{X}} \leq W_1^{-1}\left[V(t_0) + W^{-1}(t - t_0)\right].$$

□

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REFERENCES

[1] R. Bellman, The stability of solutions of linear differential equations, *Duke Math. J.* **10** (1943) no. 4, 643–647.

[2] R. Bellman, *Stability Theory of Differential Equations*, McGraw-Hill Book Company, Inc., New York, 1953.

[3] R. Bellman, Asymptotic series for the solutions of linear differential-difference equations, *Rendiconti del Circolo Matematico di Palermo* **7** (1958) no. 2, 261–269.

[4] M. Bohner, S. Hristova, K. Stefanova, Nonlinear integral inequalities involving maxima of the unknown scalar functions, *Math. Inequal. Appl.* **15** (2012), no. 4, 811–825.

[5] T. A. Burton, *Volterra Integral and Differential Equations. Second Edition*, in Mathematics in Science and Engineering, 202, Elsevier, New York, 2005.

[6] T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional-Differential Equations*, in Mathematics in Science and Engineering, 178, Academic Press, Orlando, Florida, 1985.

[7] T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, *Ann. of Math.* **20** (1919), no. 4, 292–296.

[8] J. Hale and S. Lunel, *Introduction to Functional-Differential Equations*, in Applied Matheamtics Science, 99, Springer, New York, 1993.

[9] G. E. Ladas and V. Lakshmikantham, *Differential Equations in Abstract Spaces*, in Mathematics in Science and engineering, Vol. 85, Academic Press, New York-London, 1972.

[10] V. Lakshmikantham, and S. Leela, *Differential and Integral Inequalities, Vol. I and II*, Academic Press, New York, 1969.

[11] Shi-you Lin, Generalized Gronwall inequalities and their applications to fractional differential equations, *J. Inequal. Appl.* **2013** (2013), 9 pp.

[12] B. G. Pachpatte, *Inequalities for Differential and Integral Equations*, in Mathematics in Science and Engineering, 197, Academic Press, San Diego, 1998.

[13] B. G. Pachpatte, On a new integral inequality applicable to certain partial integrodifferential equations. *Inequality Theory and Applications.* Vol. 5, 131–140, Nova Sci. Publ., New York, 2007.

[14] H. Wang, K. Zheng, C. Guo, Chun-xiang, Nonlinear discrete inequality in two variables with delay and its application, *Acta Math. Appl. Sin. Engl.* Ser. 30 (2014), no. 2, 389–400.

[15] T. Wang, Lower and upper bounds of solutions of functional differential equations, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Analy.* **20** (2013) no. 1, 131-141.

[16] T. Wang, Inequalities of Solutions of Volterra Integral and Differential Equations, *Electron. J. Qual. Theory Differ. Equ. 2009, Special Edition I,* No. 28, 10 pp.

[17] T. Wang, Exponential stability and inequalities of abstract functional differential equations, *J. Math. Anal. Appl.* **342** (2006), no.2, 982–991.

[18] T. Wang, Inequalities and stability in a linear scalar functional differential equation, *J. Math. Anal. Appl.* **298** (2004), no. 1, 33–44.

[19] T. Wang, Wazewski's inequality in a linear Volterra integrodifferential equation, *Volterra Equations and Applications*, edited by C. Corduneanu and I. W. Sandberg, Vol. 10: Stability and Control: Theory, Methods and Applications, Gordon and Breach, Amsterdam, 2000.

[20] T. Wang, Stability in abstract functional-differential equations I: General theorems, *J. Math. Anal. Appl.* **186** (1994), 534–558.

[21] T. Wang, Stability in abstract functional-differential equations II: Applications, Ĵ. Math. Anal. Appl. **186** (1994), 835–861.

[22] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, in Applied Mathematical Sciences, 119, Springer-Verlag, New York, 1996.