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CRITERIA FOR THE OSCILLATION OF SECOND ORDER NONLINEAR DYNAMIC INCLUSIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

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ABSTRACT. In this paper we investigate some new criteria for the oscillation of second order nonlinear inclusions with distributed arguments on time scales. We establish the case of strongly superlinear and the case strongly sublinear subject to various conditions.

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1. INTRODUCTION

In this paper we consider the second order nonlinear inclusions with distributed arguments

$$\left(r(t)\, x^{\Delta}(t)\right)^{\Delta} \in \int_a^b q(t,\tau) F\left(t, x^{\sigma}\left(g(t,\tau)\right)\right) \Delta\tau, \qquad \text{for a.e. } t \geq t_0 \in \mathbb{T}, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T} \subseteq \mathbb{R}$ with $\sup \mathbb{T} = \infty$ and $0 < a < b$. Whenever, we write $t \geq t_1$, we mean $t \in [t_1, \infty) \cap \mathbb{T} = [t_1, \infty)_{\mathbb{T}}$. We assume that:

(i) $r : \mathbb{T} \to \mathbb{R}^+ = (0, \infty)$ is a single real-valued, rd-continuous function and

$$\int^{\infty} \frac{\Delta s}{r(s)} < \infty; \tag{1.2}$$

(ii) $q : \mathbb{T} \times [a, b] \to \mathbb{R}^+$ is a rd-continuous function;

(iii) $g : \mathbb{T} \times [a, b] \to \mathbb{T}$ is a decreasing with respect to second variable and $g(t, \tau) \to \infty$ as $t \to \infty$, $\tau \in [a, b]$;

(iv) $F : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multifunction ($2^{\mathbb{R}}$ denotes the family of nonempty subsets of $\mathbb{R}$).

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We note that the usual standard notation in inclusion theory is used here, e.g.

$$|F(t,u)| := \sup\{|v| : v \in F(t,u)\}$$

and

$$F(t,u) > 0 \qquad \text{means} \qquad w > 0 \text{ for each } w \in F(t,u).$$

In this paper by a solution to inclusion (1.1), we mean a function $x \in C_{rd}$ with $rx^{\Delta} \in C_{rd}$ and $\left(rx^{\Delta}\right)^{\Delta} \in L^{1}_{loc}[t_0,\infty)_{\mathbb{T}}$, where C_{rd} is the space of right-dense continuous functions. We assume throughout that inclusion (1.1) possesses such solutions. We recall that a solution of inclusion (1.1) is said to be nonoscillatory if there exists a $t_1 \in \mathbb{T}$ such that $x(t)\,x^{\sigma}(t) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, where the forward jump operator $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, otherwise, it is said to be oscillatory. Inclusion (1.1) is said to be oscillatory if all its solutions are oscillatory.

Recently there has been an increasing interest in the study of theory of inclusions and inparticular the oscillation of differential inclusion

$$(r(t)\,x'(t))' \in F(t,x(t)), \qquad \text{for a.e. } t \geq t_0, \tag{1.3}$$

where

$$\int^{\infty} \frac{ds}{r(s)} = \infty. \tag{1.4}$$

In [1, 2, 3, 4, 5, 14], Agarwal *et al* initiated such a study. In this paper, we proceed further in this direction to establish new criteria for the oscillation of inclusion (1.1) with distributed deviating arguments. For the oscillation of second order nonlinear dynamic equations, we refer to [10, 9, 16, 17, 18, 19, 20, 21, 11, 13, 12, 6, 22, 23, 24] and the references cited therein. The obtained results are new for the continuous case i.e., $\mathbb{T} = \mathbb{R}$ as well as the discrete case i.e., $\mathbb{T} = \mathbb{Z}$.

2. MAIN RESULTS

We shall employ the following two lemmas.

Lemma 2.1 ([15])**.** *Suppose that $|x|^{\Delta}$ is of one sign on $[t_0,\infty)_{\mathbb{T}}$, $\lambda > 0$, and $\lambda \neq 1$. Then*

$$\frac{|x|^{\Delta}}{(|x|^{\sigma})^{\lambda}} \leq \frac{\left(|x|^{1-\lambda}\right)^{\Delta}}{1-\lambda} \leq \frac{|x|^{\Delta}}{|x|^{\lambda}} \qquad on\ [t_0,\infty)_{\mathbb{T}}. \tag{2.1}$$

We let

$$A(t) := \int_t^{\infty} \frac{\Delta s}{r(s)} \qquad \text{for } t \in [t_0,\infty)_{\mathbb{T}}.$$

Lemma 2.2 ([15])**.** *Assume that condition* (1.2) *holds. Suppose x solves* (1.1) *and is of one sign on $[t_0,\infty)_{\mathbb{T}}$. Then either*

$$|x|^{\Delta} \geq 0 \qquad on\ [t_0,\infty)_{\mathbb{T}}, \tag{2.2}$$

or there exists $t_1 \geq t_0$ *such that*

$$|x|^{\Delta} \leq 0 \qquad on \ [t_1, \infty)_{\mathbb{T}} . \tag{2.3}$$

Moreover, let

$$\bar{c} := \left\{ |x(t_0)| + r(t_0) \left| x^{\Delta}(t_0) \right| |A(t_0)| \right\} \ \text{sgn} \ x(t_0),$$

and

$$\hat{c} := \begin{cases} \dfrac{x(t_0)}{A(t_0)}, & if \ (2.2) \ holds \\ r(t_1) \left| x^{\Delta}(t_1) \right| \ \text{sgn} \ x(t_0), & if \ (2.3) \ holds. \end{cases}$$

Then

$$|x| \leq |\bar{c}| \qquad on \ [t_0, \infty)_{\mathbb{T}} \ \ where \ \bar{c} \ x > 0, \tag{2.4}$$

and

$$|x| \geq |\hat{c}A| \qquad on \ [t_0, \infty)_{\mathbb{T}} \ \ where \ \hat{c} \ Ax > 0. \tag{2.5}$$

The following result is concerned with the oscillatory behaviour of inclusion (1.1) when F is strongly superlinear, i.e., F satisfies condition (2.7) below.

Theorem 2.3. *Let*

$$\begin{cases} F(t, x) < 0, & for \ (t, x) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^+ \\ F(t, x) > 0, & for \ (t, x) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^- \end{cases} \tag{2.6}$$

and assume there exists a constant $\lambda > 1$ *such that the following condition is satisfied: there exists* $f : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ *with*

$$\begin{cases} (a) & xf(t, x) > 0 \ for \ a. \ e. \ t \geq t_0 \ and \ x \neq 0; \\ (b) & |f(t, x)| / |x|^{\lambda} \ \ is \ nondecreasing \ in \ |x| \ \ for \ a. \ e. \ t \geq t_0; \\ (c) & \begin{cases} |F(t, x)| \geq f(t, x), & for \ \ (t, x) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^+; \\ |F(t, x)| \geq -f(t, x), & for \ \ (t, x) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^-. \end{cases} \end{cases} \tag{2.7}$$

If

$$\bar{g}(t) := g(t, a) \leq t, \qquad for \ t \in [t_0, \infty)_{\mathbb{T}}, \tag{2.8}$$

and

$$\int_{t_0}^{\infty} Q(s) \left| f(s, \hat{c}A^{\sigma}(s)) \right| \Delta s = \infty, \tag{2.9}$$

for all nonzero constant $\hat{c}$ *and*

$$Q(t) := \int_a^b q(t, \tau) \, \Delta\tau, \tag{2.10}$$

then inclusion (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of inclusion (1.1) on $[t_0,\infty)_{\mathbb{T}}$. Suppose $x(t)>0$ and $x(g(t,\tau))>0$ for $t\geq t_0$ and $a\leq\tau\leq b$. Let

$$\begin{cases} y(t) := \left(r(t)\,x^{\Delta}(t)\right)^{\Delta} \text{ with } y(t)\in\int_a^b q(t,\tau)F\left(t,x^{\sigma}\left(g(t,\tau)\right)\right)\Delta\tau \\ \text{and} \\ y\in L^1_{loc}\left[t_0,\infty\right)_{\mathbb{T}}. \end{cases} \tag{2.11}$$

From (2.6), we have

$$\left(r(t)\,x^{\Delta}(t)\right)^{\Delta}\leq 0,\qquad \text{for a.e. } t\geq t_0.$$

By Lemma 2.2, either (2.2) or (2.3) holds. From (2.7), inclusion (1.1) becomes

$$\left(r(t)\,x^{\Delta}(t)\right)^{\Delta}+\int_a^b q(t,\tau)f\left(t,x^{\sigma}\left(g(t,\tau)\right)\right)\Delta\tau\leq 0,\qquad \text{for a.e. } t\geq t_0. \tag{2.12}$$

In the case of (2.2), we use condition (iii) and the fact that x is increasing on $[t_0,\infty)_{\mathbb{T}}$, we find for sufficiently large $t_1\in[t_0,\infty)_{\mathbb{T}}$

$$x^{\sigma}\left(g(t,\tau)\right)\geq x(t_0),\qquad \text{for } t\geq t_1 \text{ and } \tau\in[a,b].$$

Using (2.7), see that

$$\frac{f\left(t,x^{\sigma}\left(g(t,\tau)\right)\right)}{\left(x^{\sigma}\left(g(t,\tau)\right)\right)^{\lambda}}\geq\frac{f\left(t,x(t_0)\right)}{\left(x(t_0)\right)^{\lambda}},$$

which implies

$$f\left(t,x^{\sigma}\left(g(t,\tau)\right)\right)\geq\frac{f\left(t,x(t_0)\right)}{\left(x(t_0)\right)^{\lambda}}\left(x^{\sigma}\left(g(t,\tau)\right)\right)^{\lambda}\geq f\left(t,x(t_0)\right),\quad \text{for } t\geq t_1 \text{ and } \tau\in[a,b].$$

Then from (2.12), we have

$$\left(r(t)\,x^{\Delta}(t)\right)^{\Delta}+f\left(t,x(t_0)\right)\int_a^b q(t,\tau)\Delta\tau\leq 0,$$

or

$$\left(r(t)\,x^{\Delta}(t)\right)^{\Delta}+Q(t)\,f\left(t,x(t_0)\right)\leq 0,\qquad \text{for } t\geq t_1. \tag{2.13}$$

Integrate (2.13) from t_1 to t, we see that

$$0\leq r(t)\,x^{\Delta}(t)\leq r(t_1)\,x^{\Delta}(t_1)-\int_{t_1}^{t}Q(s)\,f\left(s,x(t_0)\right)\,\Delta s,$$

or

$$\int_{t_1}^{t}Q(s)\,f\left(s,x(t_0)\right)\,\Delta s\leq r(t_1)\,x^{\Delta}(t_1)<\infty,$$

which yields

$$\int_{t_1}^{\infty}Q(s)\,f\left(s,\hat{c}A^{\sigma}(s)\right)\Delta s<\infty,$$

a contradiction to condition (2.9).

In the case of (2.3), we use condition (iii) and the fact that x is decreasing on $[t_0,\infty)_{\mathbb{T}}$, we get

$$x^{\sigma}\left(g(t,\tau)\right)\geq x^{\sigma}\left(g(t,a)\right),\qquad \text{for } t\geq t_0 \text{ and } \tau\in[a,b].$$

Using (2.7), see that

$$\frac{f\left(t,x^{\sigma}\left(g\left(t,\tau\right)\right)\right)}{\left(x^{\sigma}\left(g\left(t,\tau\right)\right)\right)^{\lambda}}\geq\frac{f\left(t,x^{\sigma}\left(g\left(t,a\right)\right)\right)}{\left(x^{\sigma}\left(g\left(t,a\right)\right)\right)^{\lambda}},$$

which implies, for $t\geq t_0$ and $\tau\in[a,b]$

$$f\left(t,x^{\sigma}\left(g\left(t,\tau\right)\right)\right)\geq f\left(t,x^{\sigma}\left(g\left(t,a\right)\right)\right)\left(\frac{x^{\sigma}\left(g\left(t,\tau\right)\right)}{x^{\sigma}\left(g\left(t,a\right)\right)}\right)^{\lambda}\geq f\left(t,x^{\sigma}\left(g\left(t,a\right)\right)\right). \quad (2.14)$$

Combining (2.12) and (2.14) we get

$$\left(r\left(t\right)x^{\Delta}\left(t\right)\right)^{\Delta}+\left(\int_a^b q(t,\tau)\Delta\tau\right)f\left(t,x^{\sigma}\left(g\left(t,a\right)\right)\right)\leq 0, \quad (2.15)$$

or

$$\left(r\left(t\right)x^{\Delta}\left(t\right)\right)^{\Delta}+Q\left(t\right)f\left(t,x^{\sigma}\left(\bar{g}\left(t\right)\right)\right)\leq 0, \qquad \text{for } t\geq t_0. \quad (2.16)$$

In view (2.5) and (2.7), one can easily see that

$$\frac{f\left(t,x^{\sigma}\left(\bar{g}\left(t\right)\right)\right)}{\left(x^{\sigma}\left(\bar{g}\left(t\right)\right)\right)^{\lambda}}\geq\frac{f\left(t,x^{\sigma}\left(t\right)\right)}{\left(x^{\sigma}\left(t\right)\right)^{\lambda}}\geq\frac{f\left(t,\hat{c}\ A^{\sigma}\left(t\right)\right)}{\left(\hat{c}\ A^{\sigma}\left(t\right)\right)^{\lambda}}, \qquad \text{for } t\geq t_0. \quad (2.17)$$

Let $u,v,t\in\mathbb{T}$ with $u,v,t\geq t_0$. Let $s\in\mathbb{T}$ with $s\geq t_0$. Integrate (2.16) from v to s and divide the resulting inequality by $r\left(s\right)$. Now, integrate the resulting equation from u to t, we obtain

$$x\left(t\right)\leq x\left(u\right)+r\left(v\right)x^{\Delta}\left(v\right)\int_u^t\frac{\Delta s}{r\left(s\right)}-\int_u^t\frac{1}{r\left(s\right)}\int_v^s Q\left(\tau\right)f\left(\tau,x^{\sigma}\left(\bar{g}\left(\tau\right)\right)\right)\Delta\tau\Delta s. \quad (2.18)$$

Using (2.18) with $t\geq u\geq t_1=v$, we have

$$\begin{aligned} x\left(u\right) &\geq x\left(t\right)-r\left(t_1\right)x^{\Delta}\left(t_1\right)\int_u^t\frac{\Delta s}{r\left(s\right)}+\int_u^t\frac{1}{r\left(s\right)}\int_{t_1}^s Q\left(\tau\right)f\left(\tau,x^{\sigma}\left(\bar{g}\left(\tau\right)\right)\right)\Delta\tau\Delta s \\ &\geq -r\left(t_1\right)x^{\Delta}\left(t_1\right)\int_u^t\frac{\Delta s}{r\left(s\right)}+\int_u^t\frac{1}{r\left(s\right)}\int_{t_1}^s Q\left(\tau\right)f\left(\tau,x^{\sigma}\left(\bar{g}\left(\tau\right)\right)\right)\Delta\tau\Delta s \\ &\geq -r\left(t_1\right)x^{\Delta}\left(t_1\right)\int_u^t\frac{\Delta s}{r\left(s\right)}+\int_u^t\frac{\Delta s}{r\left(s\right)}\int_{t_1}^u Q\left(\tau\right)f\left(\tau,x^{\sigma}\left(\bar{g}\left(\tau\right)\right)\right)\Delta\tau. \quad (2.19)\end{aligned}$$

Using (2.17) in (2.19), we get

$$\begin{aligned} x\left(u\right) &\geq bA\left(u\right)+A\left(u\right)\int_{t_1}^u Q\left(\tau\right)f\left(\tau,x^{\sigma}\left(\bar{g}\left(\tau\right)\right)\right)\Delta\tau \\ &\geq bA\left(u\right)+A\left(u\right)\int_{t_1}^u Q\left(\tau\right)\frac{f\left(\tau,\hat{c}\ A^{\sigma}\left(\tau\right)\right)}{\left(\hat{c}\ A^{\sigma}\left(\tau\right)\right)^{\lambda}}\left(x^{\sigma}\left(\bar{g}\left(\tau\right)\right)\right)^{\lambda}\Delta\tau \\ &\geq bA\left(u\right)+A\left(u\right)\int_{t_1}^u Q\left(\tau\right)\frac{f\left(\tau,\hat{c}\ A^{\sigma}\left(\tau\right)\right)}{\left(\hat{c}\ A^{\sigma}\left(\tau\right)\right)^{\lambda}}\left(x^{\sigma}\left(\tau\right)\right)^{\lambda}\Delta\tau, \end{aligned}$$

where $b:=-r\left(t_1\right)x^{\Delta}\left(t_1\right)>0$. Let

$$w\left(u\right):=b+\hat{c}^{-\lambda}\int_{t_1}^u Q\left(\tau\right)f\left(\tau,\hat{c}\ A^{\sigma}\left(\tau\right)\right)\left(\frac{x^{\sigma}\left(\tau\right)}{A^{\sigma}\left(\tau\right)}\right)^{\lambda}\Delta\tau.$$

Therefore

$$w(u) \leq \frac{x(u)}{A(u)},$$

and hence

$$w(u) \geq b + \hat{c}^{-\lambda} \int_{t_1}^{u} Q(\tau) f(\tau, \hat{c}\, A^{\sigma}(\tau)) (w^{\sigma}(\tau))^{\lambda} \Delta\tau,$$

or

$$w^{\Delta}(u) \geq \hat{c}^{-\lambda} Q(u) f(\tau, \hat{c}\, A^{\sigma}(u)) (w^{\sigma}(u))^{\lambda}.$$

Using the first inequality of (2.1) in the above inequality, we obtain

$$\hat{c}^{-\lambda} Q(u) f(\tau, \hat{c}\, A^{\sigma}(u)) \leq \frac{w^{\Delta}(u)}{(w^{\sigma}(u))^{\lambda}} \leq \frac{\left(w^{1-\lambda}(u)\right)^{\Delta}}{1-\lambda}.$$

Integrating this inequality from t_1 to $t \geq t_1$, we have

$$\begin{aligned} w^{1-\lambda}(t_1) &\geq w^{1-\lambda}(t) + \frac{\lambda-1}{\hat{c}^{\lambda}} \int_{t_1}^{t} Q(\tau) f(\tau, \hat{c}\, A^{\sigma}(\tau)) \Delta\tau \\ &\geq \frac{\lambda-1}{\hat{c}^{\lambda}} \int_{t_1}^{t} Q(\tau) f(\tau, \hat{c}\, A^{\sigma}(\tau)) \Delta\tau, \end{aligned}$$

which contradicts condition (2.9). A parallel argument holds when $x(t)$ is negative. This completes the proof. □

Next, we present the following result which is concerned with the case when F is strongly sublinear, i.e., F satisfies condition (2.20) below.

Theorem 2.4. *Let* (2.6) *and* (2.8) *hold and assume that there exists a constant* λ, $0 < \lambda < 1$ *such that the following condition holds: there exists* $f : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ *with*

$$\begin{cases} (a) & xf(t,x) > 0 \text{ for a. e. } t \geq t_0 \text{ and } x \neq 0; \\ (b) & |f(t,x)| \text{ is nondecreasing in } |x| \text{ for a. e. } t \geq t_0; \\ (c) & |f(t,x)| / |x|^{\lambda} \text{ is nonincreasing in } |x| \text{ for a. e. } t \geq t_0; \\ (d) & \begin{cases} |F(t,x)| \geq f(t,x), & \text{for } (t,x) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{R}^{+}; \\ |F(t,x)| \geq -f(t,x), & \text{for } (t,x) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{R}^{-}. \end{cases} \end{cases} \tag{2.20}$$

If

$$\sigma(g(t,a)) \leq t, \qquad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{2.21}$$

and

$$\int_{t_0}^{\infty} \frac{1}{r(s)} \int_{t_0}^{s} Q(u) \left|f(u, \bar{c})\right| \Delta u\, \Delta s = \infty, \tag{2.22}$$

for all nonzero constant $\bar{c}$ *and* Q *is defined by* (2.10), *then inclusion* (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of inclusion (1.1) on $[t_0,\infty)_{\mathbb{T}}$. Suppose $x(t)>0$ and $x(g(t,\tau))>0$ for $t\geq t_0$ and $a\leq\tau\leq b$. By Lemma 2.2, either (2.2) or (2.3) holds.

In the case of (2.2), as shown in the proof of Theorem 2.3, we find for sufficiently large $t_1\in[t_0,\infty)_{\mathbb{T}}$

$$x^{\sigma}(g(t,\tau))\geq x(t_0), \qquad \text{for } t\geq t_1 \text{ and } \tau\in[a,b],$$

and thus by integrating (2.12) twice from t_1 to t and using (b) of (2.20), one can easily find

$$\begin{aligned} x(t) &\leq x(t_1)+r(t_1)x^{\Delta}(t_1)\int_{t_1}^{t}\frac{\Delta s}{r(s)} \\ &\quad -\int_{t_1}^{t}\frac{1}{r(s)}\int_{t_1}^{s}\int_{a}^{b}q(u,\tau)f(u,x^{\sigma}(g(u,\tau)))\Delta\tau\Delta u\Delta s \\ &\leq x(t_1)+r(t_1)x^{\Delta}(t_1)\int_{t_1}^{t}\frac{\Delta s}{r(s)}-\int_{t_1}^{t}\frac{1}{r(s)}\int_{t_1}^{s}Q(u)f(u,x(t_0))\Delta u\Delta s, \end{aligned}$$

a contradiction to condition (2.22). In the case of (2.3), using (iii), (b) of (2.20) and (2.21) in (2.12), we have

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta}+Q(t)f(t,x(t))\leq 0, \qquad \text{for } t\geq t_1\geq t_0. \tag{2.23}$$

Now, using (2.4) and (c) of (2.20), we find

$$\frac{f(t,x(t))}{x^{\lambda}(t)}\geq\frac{f(t,\bar{c})}{\bar{c}^{\lambda}}, \qquad \text{for } t\geq t_2\geq t_1. \tag{2.24}$$

Integrating (2.23) from t_2 to t and using the fact that $x^{\Delta}<0$ on $[t_2,\infty)_{\mathbb{T}}$, we get

$$\begin{aligned} -x^{\Delta}(t) &\geq -\frac{r(t_2)x^{\Delta}(t_2)}{r(t)}+\frac{1}{r(t)}\int_{t_2}^{t}Q(s)f(s,x(s))\Delta s \\ &\geq \frac{(\bar{c})^{-\lambda}}{r(t)}\int_{t_2}^{t}Q(s)f(s,\bar{c})x^{\lambda}(s)\Delta s \\ &\geq \left(\frac{(\bar{c})^{-\lambda}}{r(t)}\int_{t_2}^{t}Q(s)f(s,\bar{c})\Delta s\right)x^{\lambda}(t), \qquad \text{for } t\geq t_2, \end{aligned}$$

or

$$\frac{(\bar{c})^{-\lambda}}{r(t)}\int_{t_2}^{t}Q(s)f(s,\bar{c})\Delta s\leq-\frac{x^{\Delta}(t)}{x^{\lambda}(t)}$$

and by the second inequality of (2.1), we have

$$\frac{(\bar{c})^{-\lambda}}{r(t)}\int_{t_2}^{t}Q(s)f(s,\bar{c})\Delta s\leq-\frac{x^{\Delta}(t)}{x^{\lambda}(t)}\leq-\frac{\left(x^{1-\lambda}(t)\right)^{\Delta}}{1-\lambda}.$$

Integrating this inequality from t_2 to $t \geq t_2$, we obtain

$$\begin{aligned} x^{1-\lambda}(t_2) &\geq x^{1-\lambda}(t) + \frac{1-\lambda}{(\bar{c})^{\lambda}} \int_{t_2}^{t} \frac{1}{r(s)} \int_{t_2}^{s} Q(\tau) f(\tau, \bar{c}) \Delta\tau \Delta s \\ &\geq \frac{1-\lambda}{(\bar{c})^{\lambda}} \int_{t_2}^{t} \frac{1}{r(s)} \int_{t_2}^{s} Q(\tau) f(\tau, \bar{c}) \Delta\tau \Delta s, \end{aligned}$$

which contradicts condition (2.22). This completes the proof. □

Next, we present the following result.

Theorem 2.5. *Let conditions (i)–(iv) and* (2.6) *hold and assume that there exists* $f : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ *with*

$$\begin{cases} (a) & xf(t,x) > 0 \text{ for a. e. } t \geq t_0 \text{ and } x \neq 0; \\ (b) & |f(t,x)| \text{ is nondecreasing in } |x| \text{ for a. e. } t \geq t_0; \\ (c) & \begin{cases} |F(t,x)| \geq f(t,x), & \text{for } (t,x) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{R}^{+}; \\ |F(t,x)| \geq -f(t,x), & \text{for } (t,x) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{R}^{-}. \end{cases} \end{cases} \tag{2.25}$$

If

$$g(t,\tau) \leq t, \qquad \text{for } t \geq t_0 \text{ and } \tau \in [a,b], \tag{2.26}$$

and

$$\int_{t_0}^{\infty} \frac{1}{r(s)} \int_{t_0}^{s} Q(u) \left| f(u, \hat{c} A^{\sigma}(u)) \right| \Delta u \Delta s = \infty, \tag{2.27}$$

for all nonzero constant $\hat{c}$ *and* Q *is defined by* (2.10)*, then inclusion* (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of inclusion (1.1) on $[t_0, \infty)_{\mathbb{T}}$, Say $x(t) > 0$ and $x(g(t,\tau)) > 0$ for $t \geq t_0$ and $a \leq \tau \leq b$. A parallel argument holds when $x(t)$ is negative. By Lemma 2.2, either (2.2) or (2.3) holds.

The case (2.2) is similar to that of Theorem 2.2 and hence is omitted. For the case (2.3), using (2.5), (2.6), (2.25) and (2.26), we get

$$f(t, x^{\sigma}(g(t,\tau))) \geq f(t, x^{\sigma}(t)) \geq f(t, \hat{c}\, A^{\sigma}(t)), \qquad \text{for } t \geq t_1 \geq t_0 \text{ and } \tau \in [a,b]. \tag{2.28}$$

Integrating (2.12) twice from t_1 to t and using (2.28), we have

$$x(t) \leq x(t_1) + r(t_1) x^{\Delta}(t_1) \int_{t_1}^{t} \frac{\Delta s}{r(s)} - \int_{t_1}^{t} \frac{1}{r(s)} \int_{t_1}^{s} Q(u) f(u, \hat{c}\, A^{\sigma}(u)) \Delta u \Delta s,$$

which contradicts condition (2.27) and completes the proof. □

From the above results we can obtain some oscillation criteria for inclusion (1.1) on different types of time scales. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $x^{\Delta} = x'$ and (1.1) becomes the differential inclusion

$$(r(t) x'(t))' \in \int_{a}^{b} q(t,\tau) F(t, x(g(t,\tau))) d\tau, \qquad \text{for all } t \geq t_0. \tag{2.29}$$

and for the oscillation of (2.29) we have.

Theorem 2.6. *Let conditions (i)–(iv) hold,* (2.6) *hold and* $\int^{\infty} \frac{ds}{r(s)} < \infty$*. Inclusion* (2.29) *is oscillatory of one of the following conditions holds:*

(I) $\lambda > 1$*, conditions* (2.7) *and* (2.8) *hold and for all nonzero constant* $\hat{c}$*,*

$$\int_{t_0}^{\infty} Q(s) \left| f(s, \hat{c}A(s)) \right| ds = \infty.$$

(II) $0 < \lambda < 1$*, conditions* (2.20) *and* (2.21) *hold and for all nonzero constant* $\bar{c}$*,*

$$\int^{\infty} \frac{1}{r(s)} \int_{t_0}^{s} Q(u) \left| f(u, \bar{c}) \right| du\ ds = \infty.$$

(III) *Conditions* (2.25) *and* (2.26) *hold and for all nonzero constant* $\hat{c}$*,*

$$\int^{\infty} \frac{1}{r(s)} \int_{t_0}^{s} Q(u) \left| f(u, \hat{c}A(u)) \right| du\ ds = \infty,$$

where

$$Q(u) := \int_a^b q(u, \tau)\, d\tau.$$

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$ and $x^{\Delta}(t) = \Delta x(t) = x(t+1) - x(t)$ and (1.1) becomes the difference inclusion

$$\Delta(r(t)\Delta x(t)) \in \sum_{\tau=a}^{b-1} q(t, \tau) F(t, x^{\sigma}(g(t, \tau))), \qquad \text{for all } t \geq t_0, \tag{2.30}$$

and for oscillation result for (2.30) we obtain.

Theorem 2.7. *Let conditions (i)–(iv) hold,* (2.6) *hold and* $\sum^{\infty} \frac{1}{r(s)} < \infty$*. Inclusion* (2.30) *is oscillatory of one of the follwoing conditions holds:*

(I) $\lambda > 1$*, conditions* (2.7) *and* (2.8) *hold and for all nonzero constant* $\hat{c}$*,*

$$\sum^{\infty} Q(s) \left| f(s, \hat{c}A(s+1)) \right| = \infty;$$

(II) $0 < \lambda < 1$*, conditions* (2.20) *and* (2.21) *hold and for all nonzero constant* $\bar{c}$*,*

$$\sum^{\infty} \frac{1}{r(s)} \sum_{u=t_0}^{s-1} Q(u) \left| f(u, \bar{c}) \right| = \infty;$$

(III) *Conditions* (2.25) *and* (2.26) *hold and for all nonzero constant* $\hat{c}$*,*

$$\sum^{\infty} \frac{1}{r(s)} \sum_{u=t_0}^{s-1} Q(u) \left| f(u, \hat{c}A(u+1)) \right| = \infty,$$

where

$$Q(u) := \sum_{\tau=a}^{b-1} q(u, \tau).$$

We may employ other types of time scales, e.g., $\mathbb{T} = \mathbb{R}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, $\mathbb{T} = \mathbb{N}_0^2$ and others, see [7]. The details are left to the readers.

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