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POSITIVE SOLUTIONS OF A NONLINEAR THIRD ORDER THREE POINT BOUNDARY VALUE PROBLEM

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ABSTRACT. We consider a third order three point boundary value problem. Some new upper estimates for positive solutions of the problem are obtained. New sufficient conditions for the existence and nonexistence of positive solutions of the problem are established.

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1. INTRODUCTION

Multi-point boundary value problems have important applications in physical sciences, and they received a lot of attention in the last decade. We refer the reader to [1, 2, 3, 4, 6, 7, 8, 9] for some work in this direction. In this paper, we consider the third order three point boundary value problem

$$u'''(t) + g(t)f(u(t)) = 0, \quad 0 \le t \le 1, \tag{1.1}$$

$$u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta). \tag{1.2}$$

Our interest here is in obtaining positive solutions to this boundary value problem, that is, solutions $u(t)$ such that $u(t) > 0$ for $t \in (0, 1)$. Throughout the paper we assume the following two conditions hold.

(H1) α and η are constants such that $0 < \eta < 1$ and $1 < \alpha < 1/\eta$;

(H2) $f : [0, \infty) \to [0, \infty)$ and $g : [0, 1] \to [0, \infty)$ are continuous functions, and $g(t) \not\equiv 0$ on $[0, 1]$.

Define the function $G : [0, 1] \times [0, 1] \to [0, \infty)$ by

$$G(t,s) = \begin{cases} \dfrac{(2ts - s^2)(1 - \alpha\eta) + t^2 s(\alpha - 1)}{2(1 - \alpha\eta)}, & s \le \min\{\eta, t\}, \\ \dfrac{t^2(1 - \alpha\eta) + t^2 s(\alpha - 1)}{2(1 - \alpha\eta)}, & t \le s \le \eta, \\ \dfrac{(2ts - s^2)(1 - \alpha\eta) + t^2(\alpha\eta - s)}{2(1 - \alpha\eta)}, & \eta \le s \le t, \\ \dfrac{t^2(1 - s)}{2(1 - \alpha\eta)}, & \max\{\eta, t\} \le s. \end{cases}$$

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According to [4], $G(t,s)$ is the Green function for the problem (1.1)–(1.2), and the problem (1.1)–(1.2) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)g(s)f(u(s))\,ds, \quad 0 \le t \le 1.$$

In 2008, Guo, Sun, and Zhao [4] considered the problem (1.1)–(1.2). They obtained the following estimates on the Green function $G(t,s)$.

Theorem 1.1. *If (H1) holds, then*

$$0 \le G(t,s) \le q(s), \quad \forall (t,s) \in [0,1] \times [0,1]$$

and

$$\gamma q(s) \le G(t,s), \quad \forall (t,s) \in [\eta/\alpha, \eta] \times [0,1],$$

where $0 < \gamma = \eta^2(2\alpha^2(1+\alpha))^{-1} \cdot \min\{\alpha - 1, 1\} < 1$, *and*

$$q(s) = \frac{1+\alpha}{1-\alpha\eta} s(1-s), \quad s \in [0,1]. \tag{1.3}$$

In 2009, Graef, Kong, and Yang [3] considered a higher order boundary value problem, which includes the problem (1.1)–(1.2) as a special case. They obtained the following theorem.

Theorem 1.2. *Suppose that (H1) holds. If* $u \in C^3[0,1]$ *satisfies* (1.2) *and*

$$u'''(t) \le 0, \quad 0 \le t \le 1, \tag{1.4}$$

then $u(t) \ge 0$ *on* $[0,1]$, $u'(t) \ge 0$ *on* $[0,1]$, *and*

$$t^2 u(1) \le u(t) \le u(1), \quad 0 \le t \le 1. \tag{1.5}$$

It was shown in [3] that Theorem 1.2 improves Theorem 1.1 significantly. One of the purposes of this paper is to prove some new upper estimates for positive solutions of the problem (1.1)–(1.2) and further improve Theorem 1.2. Shaper estimates are always desired because they can help to establish better existence and nonexistence results for positive solutions.

Throughout we let

$$F_0 = \limsup_{x \to 0^+} (f(x)/x), \quad f_0 = \liminf_{x \to 0^+} (f(x)/x),$$
$$F_\infty = \limsup_{x \to +\infty} (f(x)/x), \quad f_\infty = \liminf_{x \to +\infty} (f(x)/x).$$

Also, we define the constants

$$A = \int_0^1 G(1,s)g(s)s^2\,ds \quad \text{and} \quad B = \int_0^1 G(1,s)g(s)\,ds.$$

The following two theorems, which give some sufficient conditions for the existence and nonexistence of positive solutions for the problem (1.1)–(1.2), were proved in [3].

Theorem 1.3. *Suppose that (H1) and (H2) hold. If either* $BF_0 < 1 < Af_\infty$ *or* $BF_\infty < 1 < Af_0$*, then the problem* (1.1)–(1.2) *has at least one positive solution.*

Theorem 1.4. *Suppose that (H1) and (H2) hold. If either*

$$Bf(x) < x \text{ for all } x > 0$$

or

$$Af(x) > x \text{ for all } x > 0,$$

then the problem (1.1)–(1.2) *has no positive solutions.*

Our second goal is to improve these existence and nonexistence results. To prove some of our existence results in this paper, we will use the following fixed point theorem known as the Krasnosel'skii fixed point theorem [5].

Theorem 1.5. *Let* $(X, \|\cdot\|)$ *be a Banach space over the reals, and let* $P \subset X$ *be a cone in* X*. Assume that* Ω_1 *and* Ω_2 *are bounded open subsets of* X *with* $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$*, and let*

$$L : P \cap (\overline{\Omega_2} - \Omega_1) \to P$$

be a completely continuous operator such that, either one of the following two conditions hold.

(K1) $\|Lu\| \le \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Lu\| \ge \|u\|$ *for* $u \in P \cap \partial\Omega_2$*,*

(K2) $\|Lu\| \ge \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Lu\| \le \|u\|$ *for* $u \in P \cap \partial\Omega_2$*.*

Then L *has a fixed point in* $P \cap (\overline{\Omega_2} - \Omega_1)$*.*

Throughout we let $X = C[0,1]$ be equipped with the supremum norm

$$\|v\| = \max_{t\in[0,1]} |v(t)|, \quad v \in X.$$

The rest of the paper is organized as follows. In Section 2, we present some new upper estimates for positive solutions of the problem (1.1)–(1.2). In Section 3, we give our existence and nonexistence results for positive solutions of the problem. An example is included at the end of the paper to illustrate our existence and nonexistence results.

2. NEW UPPER ESTIMATES

First, we note that

$$G(1,s) = \begin{cases} \dfrac{(2s - s^2)(1 - \alpha\eta) + s(\alpha - 1)}{2(1 - \alpha\eta)}, & 0 \le s \le \eta, \\ \dfrac{(1 - s)(\alpha\eta + s - s\alpha\eta)}{2(1 - \alpha\eta)}, & \eta < s \le 1. \end{cases}$$

It is easy to see that $G(1,0) = G(1,1) = 0$ and

$$G(1,s) > 0, \quad 0 < s < 1.$$

We define the functions $b_1 : [0, 1] \to [0, \infty)$ and $b_2 : [0, 1] \to [0, \infty)$ as

$$b_1(t) = \min\{t/\eta, 1\}, \quad 0 \le t \le 1,$$

$$b_2(t) = t, \quad 0 \le t \le 1.$$

It is easy to see that

$$b_1(t) \ge b_2(t) \ge t^2, \quad 0 \le t \le 1.$$

The next lemma gives a new upper estimate for the Green function $G(t, s)$.

Lemma 2.1. *If (H1) holds, then*

$$G(t, s) \le b_1(t)G(1, s), \quad 0 \le t, s \le 1. \tag{2.1}$$

Proof. We need only to show that

$$b_1(t)G(1, s) - G(t, s) \ge 0, \quad 0 \le t, s \le 1. \tag{2.2}$$

Since $b_1(t)$, $G(t, s)$, and $G(1, s)$ are all piecewise functions, we need take six cases to prove the inequality (2.2). Our strategy is to decompose $b_1(t)G(1, s) - G(t, s)$ into pieces so that each piece is non-negative.

Case I. If $0 \le s \le t \le \eta \le 1$, then

$$\begin{aligned} & b_1(t)G(1, s) - G(t, s) \\ = \; & \frac{s}{2\eta(1 - \alpha\eta)} \cdot \big(s(\eta - t)(1 - \alpha\eta) + t(\eta(\eta - t)(\alpha - 1) + (1 - \eta)^2(1 + \alpha))\big) \\ \ge \; & 0. \end{aligned}$$

Case II. If $0 \le s \le \eta \le t \le 1$, then

$$b_1(t)G(1, s) - G(t, s) = \frac{s(1 - t)}{2(1 - \alpha\eta)} \cdot (t(\alpha - 1) + (1 - \alpha\eta) + \alpha(1 - \eta)) \ge 0.$$

Case III. If $0 \le t \le s \le \eta \le 1$, then

$$\begin{aligned} & b_1(t)G(1, s) - G(t, s) \\ = \; & \frac{t}{2\eta(1 - \alpha\eta)} \cdot (s(1 - \eta)(1 - \alpha\eta + \alpha - s) + \eta(s - t)(1 - \alpha\eta + s(\alpha - 1))) \\ \ge \; & 0. \end{aligned}$$

Case IV. If $0 \le \eta \le s \le t \le 1$, then

$$\begin{aligned} & b_1(t)G(1, s) - G(t, s) \\ = \; & \frac{(1 - t)}{2(1 - \alpha\eta)} \cdot (s(1 - t) + \alpha\eta(1 - s) + \alpha\eta(t - s)) \\ \ge \; & 0. \end{aligned}$$

Case V. If $0 \le \eta \le t \le s \le 1$, then

$$b_1(t)G(1, s) - G(t, s) = \frac{(1 - s)}{2(1 - \alpha\eta)} \cdot \big(s - t^2 + \alpha\eta(1 - s)\big) \ge 0.$$

Case VI. If $0 \le t \le \eta \le s \le 1$, then

$$b_1(t)G(1,s) - G(t,s) = \frac{t(1-s)}{2\eta(1-\alpha\eta)} \cdot (s(1-\alpha\eta) + \eta(\alpha - t)) \ge 0.$$

We have shown that, in all the above six cases, the inequality $G(t,s) \le b_1(t)G(1,s)$ holds. The proof of the lemma is complete. □

The next lemma gives a sharper upper estimate for $G(t,s)$ under the extra condition that $2\alpha\eta \ge 1$.

Lemma 2.2. *If (H1) holds and* $2\alpha\eta \ge 1$, *then*

$$G(t,s) \le tG(1,s), \quad 0 \le t, s \le 1. \tag{2.3}$$

Proof. We take four cases to prove the inequality (2.3).

Case I. If $0 \le t \le s \le \eta \le 1$, then

$$\begin{aligned} & tG(1,s) - G(t,s) \\ = \;& \frac{t}{2(1-\alpha\eta)} \cdot (s(1-s)\alpha(1-\eta) + (s-t)(s\alpha - s + 1 - \alpha\eta)) \\ \ge \;& 0. \end{aligned}$$

Case II. If $0 \le s \le t \le 1$ and $0 \le s \le \eta \le 1$, then

$$tG(1,s) - G(t,s) = \frac{s(1-t)}{2(1-\alpha\eta)} \cdot (s - s\alpha\eta + t\alpha - t) \ge 0.$$

Case III. If $0 \le t \le s \le 1$ and $0 \le \eta \le s \le 1$, then

$$tG(1,s) - G(t,s) = \frac{t(1-s)}{2(1-\alpha\eta)} \cdot ((1-s)\alpha\eta + (s-t)) \ge 0.$$

Case IV. If $0 \le \eta \le s \le t \le 1$, then

$$\begin{aligned} & tG(1,s) - G(t,s) \\ = \;& \frac{(1-t)}{4(1-\alpha\eta)} \cdot \big((2\alpha\eta - 1)(t - s^2) + (t-s)^2 + t(1-t)\big) \\ \ge \;& 0. \end{aligned}$$

We have shown that, in all the above four cases, the inequality $G(t,s) \le tG(1,s)$ holds. The proof of the lemma is complete. □

We can easily translate Lemma 2.1 to upper estimates on positive solutions of the problem (1.1)–(1.2).

Theorem 2.3. *Suppose that (H1) and (H2) hold. If* $u \in C^3[0,1]$ *satisfies* (1.2) *and* (1.4), *then*

$$t^2 u(1) \le u(t) \le b_1(t)u(1) \quad \textit{for} \;\; 0 \le t \le 1. \tag{2.4}$$

In particular, if $u(t)$ *is a nonnegative solution to the problem* (1.1)–(1.2), *then* $u(t)$ *satisfies* (2.4).

Proof. Suppose $u \in C^3[0,1]$ satisfies (1.2) and (1.4). By Theorem 1.2, we have $u(t) \geq t^2 u(1)$ on $[0,1]$. For $0 \leq t \leq 1$, we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)(-u'''(s))ds \\ &\leq b_1(t) \int_0^1 G(1,s)(-u'''(s))ds \\ &\leq b_1(t)u(1). \end{aligned}$$

Thus we proved (2.4).

If $u(t)$ is a nonnegative solution to the problem (1.1)–(1.2), then $u(t)$ satisfies (1.2) and

$$u'''(t) = -g(t)f(u(t)) \leq 0, \quad 0 \leq t \leq 1.$$

It follows immediately that $u(t)$ satisfies (2.4). The proof is complete. □

In a similar way, we can prove the next theorem by using Lemma 2.2.

Theorem 2.4. *Suppose that (H1) and (H2) hold and $2\alpha\eta \geq 1$. If $u \in C^3[0,1]$ satisfies (1.2) and (1.4), then*

$$t^2 u(1) \leq u(t) \leq b_2(t)u(1) \quad \textit{for} \;\; 0 \leq t \leq 1. \tag{2.5}$$

In particular, if $u(t)$ is a nonnegative solution to the problem (1.1)–(1.2), then $u(t)$ satisfies (2.5).

3. EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS

With the new upper estimates that we proved in Section 2, we are now ready to establish some new existence and nonexistence results. For each $i = 1, 2$, we define the constant

$$B_i = \int_0^1 G(1,s)g(s)b_i(s)\, ds.$$

Also, for each $i = 1, 2$, we define $P_i \subset X$ as

$$P_i = \big\{ v \in X : v(1) \geq 0,\ t^2 v(1) \leq v(t) \leq v(1)b_i(t) \text{ on } [0,1] \big\}.$$

Let

$$Y = \big\{ v \in X : v(t) \geq 0 \text{ on } [0,1] \big\}.$$

Define the operator $T : Y \to X$ by

$$Tu(t) = \int_0^1 G(t,s)g(s)f(u(s))ds, \quad 0 \leq t \leq 1,\ u \in P.$$

By a standard argument we can show that $T : Y \to X$ is a completely continuous operator.

The next lemma is a summary of some basic properties of the sets P_1, P_2, and Y.

Lemma 3.1. *We have*

1. $P_2 \subset P_1 \subset Y$;
2. P_1, P_2, *and* Y *are all positive cones of* X;
3. *If* $u \in P_1$, *then* $\|u\| = u(1)$;
4. *If* $u \in P_1$ *and* $u(1) > 0$, *then* $u(t) > 0$ *for* $0 < t \le 1$;
5. *If* $u \in P_2$, *then* $\|u\| = u(1)$;
6. *If* $u \in P_2$ *and* $u(1) > 0$, *then* $u(t) > 0$ *for* $0 < t \le 1$.

The proof of Lemma 3.1 is straightforward and is therefore left to the reader.

Now, we rephrase Theorems 2.3 and 2.4 into the next two theorems.

Theorem 3.2. *Suppose that (H1) and (H2) hold.*

1. *If* $u \in C^3[0,1]$ *satisfies* (1.2) *and* (1.4), *then* $u \in P_1$;
2. *If* $u(t)$ *is a nonnegative solution to the problem* (1.1)–(1.2), *then* $u \in P_1$;
3. $T(P_1) \subset P_1$.

Theorem 3.3. *Suppose that (H1) and (H2) hold, and* $2\alpha\eta \ge 1$.

1. *If* $u \in C^3[0,1]$ *satisfies* (1.2) *and* (1.4), *then* $u \in P_2$;
2. *If* $u(t)$ *is a nonnegative solution to the problem* (1.1)–(1.2), *then* $u \in P_2$;
3. $T(P_2) \subset P_2$.

To find a positive solution to the problem (1.1)–(1.2), we need only to find a fixed point u of T such that $u \in Y$ and $u(1) > 0$. We now give our first existence result.

Theorem 3.4. *Suppose that (H1) and (H2) hold. If either* $B_1F_0 < 1 < Af_\infty$ *or* $B_1F_\infty < 1 < Af_0$, *then the problem* (1.1)–(1.2) *has at least one positive solution.*

Proof. We shall prove the existence of at least one positive solution under the condition $B_1F_0 < 1 < Af_\infty$ only. Choose $\varepsilon > 0$ such that $(F_0 + \varepsilon)B_1 \le 1$. There exists $H_1 > 0$ such that

$$f(x) \le (F_0 + \varepsilon)x \quad \text{for } 0 < x \le H_1.$$

For each $u \in P$ with $\|u\| = H_1$, we have

$$\begin{aligned}
(Tu)(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\
&\le (F_0 + \varepsilon)\int_0^1 G(1,s)g(s)u(s)\,ds \\
&\le (F_0 + \varepsilon)\|u\|\int_0^1 G(1,s)g(s)b_1(s)ds \\
&\le (F_0 + \varepsilon)\|u\|B_1 \le \|u\|,
\end{aligned}$$

which means $\|Tu\| \leq \|u\|$. If we let $\Omega_1 = \{u \in X : \|u\| < H_1\}$, then

$$\|Tu\| \leq \|u\| \quad \text{for } u \in P_1 \cap \partial\Omega_1.$$

Next, we construct Ω_2. Since $1 < Af_\infty$, we can choose $c \in (0, 1/4)$ and $\delta > 0$ such that

$$(f_\infty - \delta) \int_c^1 G_n(1,s)g(s)s^2\, ds > 1.$$

There exists $H_3 > 0$ such that

$$f(x) \geq (f_\infty - \delta)x \quad \text{for } x \geq H_3.$$

Let $H_2 = \max\{H_3 c^{-2}, 2H_1\}$. Now if $u \in P$ with $\|u\| = H_2$, then for $c \leq t \leq 1$, we have

$$u(t) \geq t^2\|u\| \geq c^2 H_2 \geq H_3,$$

and

$$\begin{aligned}(Tu)(1) &\geq \int_c^1 G(1,s)g(s)f(u(s))ds \\ &\geq (f_\infty - \delta) \int_c^1 G(1,s)g(s)u(s)ds \\ &\geq (f_\infty - \delta)\|u\| \int_c^1 G(1,s)g(s)s^2\, ds \geq \|u\|,\end{aligned}$$

which means $\|Tu\| \geq \|u\|$. So, if we let $\Omega_2 = \{u \in X \mid \|u\| < H_2\}$, then $\overline{\Omega_1} \subset \Omega_2$ and

$$\|Tu\| \geq \|u\| \quad \text{for } u \in P_1 \cap \partial\Omega_2.$$

Since the condition (K1) of Theorem 1.5 is satisfied, there exists a fixed point of T in P_1, and this completes the proof of the theorem. □

The proof of the next existence result is quite similar to that of Theorem 3.4 and is therefore left to the reader.

Theorem 3.5. *Suppose that (H1) and (H2) hold and $2\alpha\eta \geq 1$. If either $B_2F_0 < 1 < Af_\infty$ or $B_2F_\infty < 1 < Af_0$, then the problem* (1.1)–(1.2) *has at least one positive solution.*

The next theorem provides a sufficient condition for the nonexistence of positive solutions.

Theorem 3.6. *Suppose that (H1) and (H2) hold. If $B_1 f(x) < x$ for all $x > 0$, then the problem* (1.1)–(1.2) *has no positive solutions.*

Proof. Assume to the contrary that $u(t)$ is a positive solution of the problem (1.1)–(1.2). Then $u \in P_1$, $u(t) > 0$ for $0 < t \leq 1$, and

$$\begin{aligned} u(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &< B^{-1}\int_0^1 G(1,s)g(s)u(s)\,ds \\ &\leq B_1^{-1}u(1)\int_0^1 G(1,s)g(s)b_1(s)\,ds = u(1), \end{aligned}$$

which is a contradiction. The proof is complete. □

In a similar way, we can prove the following non-existence result.

Theorem 3.7. *Suppose that (H1) and (H2) hold and $2\alpha\eta \geq 1$. If $B_2 f(x) < x$ for all $x > 0$, then the problem* (1.1)–(1.2) *has no positive solutions.*

We conclude the paper with an example.

Example 3.8. Consider the boundary value problem

$$u'''(t) = (1+2t)\cdot\frac{\lambda u(t)(1+9u(t))}{1+u(t)}, \quad 0 < t < 1, \tag{3.1}$$

$$u(0) = u'(0) = 0, \quad u'(1) = (6/5)\cdot u'(5/7). \tag{3.2}$$

Here $\lambda > 0$ is a parameter. This problem is a special case of the problem (1.1)–(1.2) in which $\alpha = 6/5$, $\eta = 5/7$, $g(t) = 1+2t$, and

$$f(u) = \frac{\lambda u(1+9u)}{1+u}, \quad u \geq 0.$$

It is easy to see that $F_0 = f_0 = \lambda$, $F_\infty = f_\infty = 9\lambda$, and $\lambda u \leq f(u) \leq 9\lambda u$ for $u \geq 0$. Also, we note that $2\alpha\eta = 12/7 \geq 1$. For the problem (3.1)–(3.2), calculations show that

$$A = \frac{396625}{705894}, \quad B = \frac{43555}{14406}, \quad B_1 = \frac{53855}{19208}, \quad B_2 = \frac{232625}{201684}.$$

First, we compare the three existence results, Theorems 1.3, 3.4, and 3.5. By Theorem 1.3, we have that if

$$1.9775 \approx \frac{1}{9A} < \lambda < \frac{1}{B} \approx 3.3075,$$

then problem (3.1)–(3.2) has at least one positive solution. By Theorem 3.4, we have that if

$$1.9775 \approx \frac{1}{9A} < \lambda < \frac{1}{B_1} \approx 3.5666,$$

then problem (3.1)–(3.2) has at least one positive solution. By Theorem 3.5, we have that if

$$1.9775 \approx \frac{1}{9A} < \lambda < \frac{1}{B_2} \approx 8.6699,$$

then problem (3.1)–(3.2) has at least one positive solution. It is clear that Theorems 3.4 and 3.5 are better than Theorem 1.3 in this example.

Next, we compare the three nonexistence results, Theorems 1.4, 3.6, and 3.7. By Theorem 1.4, we see that if either

$$\lambda < \frac{1}{9B} \approx 0.3675 \quad \text{or} \quad \lambda > \frac{1}{A} \approx 17.7975,$$

then (3.1)–(3.2) has no positive solutions. By Theorem 3.6, we see that if either

$$\lambda < \frac{1}{9B_1} \approx 0.39629 \quad \text{or} \quad \lambda > \frac{1}{A} \approx 17.7975,$$

then the problem (3.1)–(3.2) has no positive solutions. By Theorem 3.7, we see that if either

$$\lambda < \frac{1}{9B_2} \approx 0.9633 \quad \text{or} \quad \lambda > \frac{1}{A} \approx 17.7975,$$

then the problem (3.1)–(3.2) has no positive solutions. It is clear that Theorems 3.6 and 3.7 are better than Theorem 1.4 in this example.

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