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SYSTEMS-DISCONJUGACY OF A FOURTH-ORDER DIFFERENTIAL EQUATION WITH A MIDDLE TERM

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ABSTRACT. Systems-conjugate points have been introduced and studied by John Barrett [3] in relation with the self-adjoint fourth order differential equation

$$(r(x)y'')'' - (q(x)y')' = p(x)y,$$

where $r(x) > 0$, $p(x) > 0$ and $q \equiv 0$. In this paper we extend some of his results to more general cases, when $q(x)$ is free of any sign restrictions.

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1. Introduction

This paper shall be concerned with the fourth-order differential equation

$$(r(x)y'')'' - (q(x)y')' = p(x)y, \tag{1.1}$$

where $r(x) > 0$, $p(x) > 0$ and $q(x)$ are continuous functions on $[a, \infty)$, $a \geq 0$.

Definition 1.1. The systems-conjugate point of a, which is denoted by $\hat{\eta}_1(a)$, is defined as the smallest number $b \in (a, \infty)$ for which the two point boundary conditions

$$y(a) = y_1(a) = y(b) = y_1(b) = 0 \tag{1.2}$$

($y_1(x) = r(x)y''$) are satisfied by a nontrivial solution of equation (1.1).

Similarly, the systems-focal point of a, which is denoted by $\hat{\mu}_1(a)$, is defined as the smallest number $b \in (a, \infty)$ for which the two point boundary conditions

$$y(a) = y_1(a) = y'(b) = Ty(b) = 0 \tag{1.3}$$

($Ty(x) = (p(x)y'')' - q(x)y'$) are satisfied by a nontrivial solution of equation (1.1).

The notation $y_1(x)$ and $Ty(x)$ will be used throughout the paper.

The systems-conjugate point and systems-focal point were first defined and studied by Barrett [3, 4] with respect to equation (1.1), for $r(x) > 0$, $p(x) > 0$ and $q \equiv 0$. In his work, he showed that $\hat{\eta}_1(a)$ exists, if and only if $\hat{\mu}_1(a)$ exists, and

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$a < \hat{\mu}_1(a) < \hat{\eta}_1(a)$, without further conditions on $r(x)$ and $p(x)$. Later on, using a Morse system-formulation [11], Atkinson [1, Chap. 10.6] extended a part of Barrett's result to the case $q(x) \geq 0$ (i.e., if $\hat{\eta}_1$ exists then $\hat{\mu}_1(a)$ exists and $a < \hat{\mu}_1(a) < \hat{\eta}_1(a)$). Cheng [6] also studied the existence and the relation between $\hat{\mu}_1(a)$ and $\hat{\eta}_1(a)$ for a system of two second-order differential equations; in particular, he gave a physical interpretation of the numbers $\hat{\eta}_1$ and $\hat{\mu}_1$. At the end of this work, he applied his results to equation (1.1) for $q(x) \leq 0$ and the additional condition $p - q''/2 + q^2/4r > 0$. Note that the systems-focal point studied in [6] do not coincide with that defined above for (1.1) only for $q \equiv const$.

The main goal of the present paper is to establish Barrett's result related to equation (1.1) with some relaxation of the sign of $q(x)$. Furthermore, in Sections 3 and 4 we establish a comparison theorem for $\hat{\mu}_1(a)$, and we show, without further restrictions on r, p and q, that if $\hat{\mu}_1(a)$ exists then it is realized by a positive increasing solution. These results are analogous to those obtained by Barrett [5] for the focal point $\mu_1(a)$ related to equation (1.1) and the boundary conditions $y(a) = y'(a) = y_1(b) = Ty(b) = 0$. However, here we use a different approach, which is essentially based on the Leighton-Nehari transformation [10] and the properties of the Rayleigh quotients. Finally, in Section 5 we establish two criteria for the existence of $\hat{\eta}_1(a)$. Similar results were given in [3] and [6] for $q(x) \equiv 0$ and $q(x) \leq 0$, respectively.

2. Relation between $\hat{\eta}_1$ and $\hat{\mu}_1$

The main result of this section is the following

Theorem 2.1. *1) If the first systems-conjugate point $\hat{\eta}_1(a)$ exits and*

$$I(w, a, b) = \int_a^b [r(w')^2 + qw^2] > 0 \tag{2.1}$$

for each $b > a$ and each nontrivial admissible function $w \in W_2^1[a, b]$ (where $W_2^1[a, b]$ is the Sobolev function space having a generalized first derivative in $L_2[a, b]$), then the first systems-focal point $\hat{\mu}_1(a)$ exists and

$$a < \hat{\mu}_1(a) < \hat{\eta}_1(a). \tag{2.2}$$

2) If the number $\hat{\mu}_1(a)$ exists and $\int^\infty q(t) = -\infty$, then $\hat{\eta}_1(a)$ exits. If in addition the condition (2.1) is satisfied, then (2.2) holds.

Before proving this theorem we need some preliminaries. It is known that any solution of equation (1.1) which satisfies the initial condition $y(a) = y_1(a) = 0$ may be expressed as a linear combination of $u(x)$ and $v(x)$ which are the fundamental solutions of (1.1) whose initial conditions are

$$u(a) = u_1(a) = Tu(a) = 0, \quad u'(a) = 1, \tag{2.3}$$

$$v(a) = v'(a) = v_1(a) = 0, \quad Tv(a) = 1. \tag{2.4}$$

We introduce the following subwronskians:

$$r\hat{\sigma}' = uv_1 - vu_1, \quad \hat{\tau}' = u'Tv - v'Tu, \tag{2.5}$$

and

$$\hat{\sigma} = uv' - vu', \quad \hat{\tau} = uTv - vTu, \quad \hat{\rho} = u_1Tv - v_1Tu. \tag{2.6}$$

It is easy to see that $\hat{\eta}_1$ and $\hat{\mu}_1$ are the first zeros on (a, ∞) of the subwronskians $\hat{\sigma}'$ and $\hat{\tau}'$, respectively. The following identities involving the above subwronskians are useful and easily verified. Similar ones have been stated in [5] for (1.1) with Dirichlet boundary conditions at $x = a$ ($y(a) = y'(a) = 0$).

$$r\hat{\sigma}'\hat{\tau}' = \hat{\tau}^2 + \hat{\rho}\hat{\sigma} \tag{2.7}$$

$$\hat{\tau}'' = \frac{\hat{\rho}}{r} - p\hat{\sigma}, \quad (r\hat{\sigma}')' = 2\hat{\tau} + q\hat{\sigma}. \tag{2.8}$$

Note also, the initial conditions

$$\hat{\tau}(a) = 0, \quad \hat{\tau}'(a) = 1, \tag{2.9}$$

$$\hat{\sigma}(a) = \hat{\sigma}'(a) = (r\hat{\sigma}')'(a) = 0, (r\hat{\sigma}')''(a) = 2, \tag{2.10}$$

$$\hat{\rho}(a) = 0, \quad \hat{\rho}'(a) = q(a), \tag{2.11}$$

insure that $\hat{\sigma}$, $\hat{\sigma}'$, $\hat{\tau}$ and $\hat{\tau}'$ are all positive in a right-hand neighborhood of $x = a$.

Throughout our discussion we will use the following transformation given by Leighton-Nehari [10] for removing the middle term $(qy')'$ from equation (1.1). However, this transformation cannot be used in a straightforward way, since as will be seen below, it changes the form of the initial conditions (2.3) and the subwronskians $\hat{\sigma}'$, $\hat{\rho}$.

Let us denote by h a positive solution on the interval $[a, b]$ of the second-order equation

$$(py')' - qy = 0. \tag{2.12}$$

Hence, the following substitution [10, Theorem 12.1]

$$t(x) := \int_0^x h(s)ds \tag{2.13}$$

transform equation (1.1) into

$$\{(rh^3(t)\ddot{y})\}^{\cdot\cdot} = h^{-1}p(t)y, \tag{2.14}$$

where $p(x), h(x), r(x), y(x)$ are taken as functions of t and $^{\cdot} := \frac{d}{dt}$. Therefore, if y is a nontrivial solution of (1.1), then $\tilde{y}(t) \equiv y(x(t))$ is a nontrivial solution of (2.14). Thus, we have the relations:

$$\dot{\tilde{y}} = y'h^{-1}, \quad h^3\ddot{\tilde{y}} = hy'' - y'h', \tag{2.15}$$

$$(\tilde{r}\tilde{h}^3\ddot{\tilde{y}})^{\cdot} = (ry'')' - qy'. \tag{2.16}$$

In what follows, for each of the quantities involving (2.14), the same notations as for (1.1) will be used with the addition of the superscript "$\sim$". Let $\tilde{\sigma}$, $\dot{\tilde{\sigma}}$, $\tilde{\tau}$, $\dot{\tilde{\tau}}$ and $\tilde{\rho}$ denote the subwronskians associated with equation (2.14) and the fundamental solutions $\tilde{u}$, $\tilde{v}$ satisfying the initial conditions

$$\tilde{u}(0) = \tilde{h}(0)\ddot{\tilde{u}}(0) + \dot{\tilde{h}}(0) = (\tilde{r}\tilde{h}^3\ddot{\tilde{u}})^{\cdot}(0) = 0,\ \dot{\tilde{u}}(0) = 1, \tag{2.17}$$

$$\tilde{v}(0) = \dot{\tilde{v}}(0) = \ddot{\tilde{v}}(0) = 0,\ (\tilde{r}\tilde{h}^3\ddot{\tilde{v}})^{\cdot}(0) = 1. \tag{2.18}$$

The relations between these subwronskians and those of equation (1.1) are expressed as follows:

$$\hat{\sigma}(x) = \tilde{h}(t)\tilde{\sigma}(t), \quad \hat{\tau}(x) = \tilde{\tau}(t), \quad \hat{\tau}'(x) = \tilde{h}(t)\dot{\tilde{\tau}}(t), \tag{2.19}$$

$$\hat{\sigma}'(x) = \tilde{h}^2(t)\dot{\tilde{\sigma}}(t) + \tilde{h}\dot{\tilde{h}}\tilde{\sigma}(t), \tag{2.20}$$

Lemma 2.2. *1) If $\hat{\mu}_1(a)$ exists, then $\hat{\rho}(\hat{\mu}_1(a)) < 0$.*
2) Let $\hat{\xi}_1(a), \hat{\xi}_2(a)\dots$ denote the zeros of the subwronskian $\hat{\rho}$ defined by (2.5). If $\hat{\mu}_1(a)$ exists and $\hat{\rho}$ has a first zero $\hat{\xi}_i(a)$ ($i \in \{2, 3, \dots\}$) beyond $\hat{\mu}_1(a)$, then $\hat{\sigma}'$ has a zero $\hat{\eta}_1(a)$ in $(a, \hat{\xi}_i(a)]$.

Proof. 1) If $\hat{\mu}_1(a)$ exists, then $\hat{\sigma} > 0$ on $(a, \hat{\mu}_1(a)]$. In fact, suppose that $\hat{\sigma}$ has a zero $s_0 \in (a, \hat{\mu}_1(a))$ which is the closest to a. From the initial conditions (2.9)-(2.10), we have $\hat{\tau}' > 0$ and $\hat{\sigma} > 0$ in a right-hand neighborhood of $x = a$, and hence, $\hat{\tau}'(s_0) > 0$ and $\hat{\sigma}'(s_0) < 0$. On the other hand, by (2.7), $\hat{\tau}'\hat{\sigma}'(s_0) \geq 0$, which is a contradiction. If $s_0 = \hat{\mu}_1(a)$, then again by (2.7), $\hat{\tau}(s_0) = 0$. Thus, from Rolle's theorem and the initial conditions (2.9), there exists a zero of $\hat{\tau}'$ less than $\hat{\mu}_1(a)$, which is a contradiction. Since $\hat{\sigma}(\hat{\mu}_1(a)) > 0$ then by (2.7), we have $\hat{\rho}\hat{\sigma}(\hat{\mu}_1(a)) \leq 0$. If $\hat{\rho}(\hat{\mu}_1(a)) = 0$, then $\hat{\tau}(\hat{\mu}_1(a)) = 0$, and as before, this is not possible.

2) Suppose that $\hat{\rho}$ has a first zero $\hat{\xi}_i$ beyond $\hat{\mu}_1(a)$ (i.e., the first in $(\hat{\mu}_1(a), \infty)$). By (2.7), we have $\hat{\tau}'\hat{\sigma}'(\hat{\xi}_i(a)) \geq 0$. If $\hat{\tau}'(\hat{\xi}_i(a)) < 0$, then $\hat{\sigma}'(\hat{\xi}_i(a)) \geq 0$, and hence, from the initial conditions (2.10), $\hat{\sigma}'$ has a zero $\hat{\eta}_1(a)$ in $(a, \hat{\xi}_i(a)]$. If $\hat{\tau}'(\hat{\xi}_i(a)) \geq 0$, then $\hat{\mu}_2(a)$ exists and $a < \hat{\mu}_2 \leq \hat{\xi}_i(a)$. According to Lemma 2.3, $\hat{\sigma}$ has a zero in the interval $(\hat{\mu}_1(a), \hat{\mu}_2(a)]$. Thus, by Rolle's theorem, $\hat{\eta}_1(a)$ exists and $a < \hat{\eta}_1(a) \leq \hat{\xi}_i(a)$. The lemma is proved. □

Lemma 2.3. *If $\hat{\mu}_1(a)$ and $\hat{\mu}_2(a)$ (the second zero of τ') both exist, then $\hat{\sigma}$ has a zero in the interval $(\hat{\mu}_1(a), \hat{\mu}_2(a)]$.*

Proof. By Lemma 2.2 and its proof, we have $\hat{\sigma}(\hat{\mu}_1(a)) > 0$ and $\hat{\rho}(\hat{\mu}_1(a)) < 0$. Thus, $\hat{\tau}''(\hat{\mu}_1(a)) = (\frac{\hat{\rho}}{r} - p\hat{\sigma})(\hat{\mu}_1(a)) < 0$, which implies the simplicity of $\hat{\mu}_1(a)$, and hence, $\hat{\mu}_1(a) < \hat{\mu}_2(a)$. Suppose $\hat{\sigma} > 0$ on $(\hat{\mu}_1(a), \hat{\mu}_2(a)]$. Since $\hat{\tau}(\hat{\mu}_1(a)) > 0$, then by using the identity (2.7), we obtain

$$\left(\frac{\hat{\tau}'}{\hat{\sigma}}\right)' = -p - \frac{1}{r}\left(\frac{\hat{\tau}}{\hat{\sigma}}\right)^2 < 0.$$

Integration of this expression yields

$$\int_{\hat{\mu}_1}^{\hat{\mu}_2} p + \frac{1}{r}\left(\frac{\hat{\tau}}{\hat{\sigma}}\right)^2 dx = 0,$$

which is a contradiction, and so $\hat{\sigma}$ vanishes in $(\hat{\mu}_1(a), \hat{\mu}_2(a)]$. □

Proof of Theorem 2.1*:* 1) Let h be the solution of equation (2.12) which satisfies the initial conditions

$$y'(a) = 0, \quad y(a) = 1. \tag{2.21}$$

If condition (2.1) holds, then all the eigenvalues of the problem determined by equation (2.12) and the boundary conditions $y'(a) = y'(b) = 0$ (for each $b > a$) are positive, and hence, $h(x) > 0$ on $[a, \infty)$. Furthermore, since $I(1, a, a + \varepsilon) = \int_a^{a+\varepsilon} q > 0$ for sufficiently small $\varepsilon > 0$, $q(x) \geq 0$ in a right-neighborhood of $x = a$. Thus, $h'(x) > 0$ on $[a, \infty)$. Therefore, the change of variables $t(x) := \int_0^x h(s)ds$ is valid to transform equation (1.1) into (2.14). Let $\tilde{\eta}_1(0)$ and $\tilde{\mu}_1(0)$ denote, respectively, the first systems-conjugate point and the first systems-focal point associated with equation (2.14); i.e., the first zeros of the subwronskians $\dot{\tilde{\sigma}}$ and $\dot{\tilde{\tau}}$, respectively. As noted before, these subwronskians are obtained from the original ones via the above change of variables, and the relations between them are expressed by (2.19)-(2.20). Note that also the initial conditions

$$\tilde{\tau}(0) = 0, \quad \dot{\tilde{\tau}}(0) = 1, \tag{2.22}$$

$$\tilde{\sigma}(0) = \dot{\tilde{\sigma}}(0) = (\tilde{r}\tilde{h}^3\dot{\tilde{\sigma}})^{\cdot}(0) = 0, (\tilde{r}\tilde{h}^3\dot{\tilde{\sigma}})^{\cdot\cdot}(0) = 2, \tag{2.23}$$

imply that $\tilde{\sigma}$, $\dot{\tilde{\sigma}}$ and $\dot{\tilde{\tau}}$ are positive in a right-hand neighborhood of $t = 0$.

Suppose $\hat{\eta}_1(a)$ exists. By (2.21), together with the relation (2.20), we have $\dot{\tilde{\sigma}}(\int_a^{\hat{\eta}_1(a)} h) < 0$. Hence, $\tilde{\eta}_1(0)$ exists for (2.14). According to [3, Theorem 1.1], which is applied to equation (2.14), it follows that $\tilde{\mu}_1(0)$ exists, and

$$0 < \tilde{\mu}_1(0) < \tilde{\eta}_1(0). \tag{2.24}$$

Therefore, from the last relation of (2.19), $\hat{\mu}_1(a)$ also exists, and (2.2) holds.

2) Assume that $\int^\infty q = -\infty$, and suppose that $\hat{\mu}_1(a)$ exists, but $\hat{\sigma}' > 0$ on (a, ∞). In view of Lemmas 2.2 (second statement) and 2.3, if $\hat{\xi}_i(a)$ (the first zero of $\hat{\rho}$ beyond $\hat{\mu}_1(a)$) or $\hat{\mu}_2(a)$ exists, then $\hat{\eta}_1(a)$ exists. On the other hand, by the first statement of Lemma 2.2, $\hat{\rho}(\hat{\mu}_1(a)) < 0$. Therefore, if $\hat{\xi}_i(a)$ and $\hat{\mu}_2(a)$ do not exist then we have $k(x) = -\frac{\hat{\rho}}{\hat{\tau}'} < 0$ on $(\hat{\mu}_1(a), \infty)$, and

$$k'(x) = p\left(\frac{\hat{\tau}}{\hat{\tau}'}\right)^2 - q + \frac{1}{r}k^2 \geq 0 \quad \text{on } (a, \infty).$$

Integrating this expression, and taking into account the assumption that $\int^\infty q = -\infty$, it follows that $k(x) \to +\infty$ as $x \to +\infty$. This is a contradiction, and so $\hat{\mu}_1(a)$ exists. If in addition, (2.1) holds, then from the first statement of the theorem, we have $a < \hat{\mu}_1(a) < \hat{\eta}_1(a)$. The theorem is proved. □

3. Wirtinger inequality and comparison theorem for $\hat{\eta}_1(a)$

The following theorem establish the relation between the existence of $\hat{\eta}_1(a)$ and the sign of the quadratic form associated with (1.1). This relation is known as a Wirtinger-type inequality [7]. Note that the method of Cole used in [7] and also in [5] (for a Wirtinger inequality related to the focal point $\mu_1(a)$) cannot be applied here.

Theorem 3.1. *If $\hat{\eta}_1(a)$ does not exist for* (1.1)*, then for each $b \in (a, +\infty)$ and each nontrivial admissible function $w(x)$ on $[a, b]$ (i.e., $w(x) \in C^1[a, b]$, w' is absolutely continuous and $w'' \in L_2[a, b]$) for which $w(a) = w'(b) = 0$, we have*

$$I[w, a, b] = \int_a^b r(w'')^2 + q(w')^2 - pw^2 dx > 0.$$

For the proof of this theorem, we need some preliminarily results. We introduce the following equation similar to (1.1), but depends on a parameter $\lambda \in \mathbb{R}$.

$$(r(x)y'')'' - (q(x)y')' = \lambda p(x)y. \tag{3.1}$$

Let $\bar{\eta}_1(a)$ denote the first conjugate point of a with respect to equation (2.12); i.e., the smallest number $b \in (a, \infty)$ for which the boundary conditions $y'(a) = y(b) = 0$ are satisfied by a nontrivial solution.

Lemma 3.2. *Let $\lambda_1(b)$ be the first eigenvalue of Problem* (3.1)-(1.3)*, and assume that $\bar{\eta}_1(a)$ exists. If $\lambda_1(b) > 0$, then $b < \bar{\eta}_1(a)$.*

The proof of this lemma is based on the following result on the monotonicity of the eigenvalues of Sturm-Liouville problem. To the best of my knowledge, this property is known only for $q \geq 0$ (e.g., [13]).

Lemma 3.3. *The eigenvalues $\rho_k(b)$ of the second-order boundary problem*

$$-(r(x)y')' + q(x)y = \rho y, \tag{3.2}$$

$$y'(a) = 0, \quad y(b) = 0 \tag{3.3}$$

decrease as b increases.

Proof. Let

$$F(x, \rho) = \frac{y(x, \rho)}{ry'(x, \rho)},$$

where $y(x, \rho)$ is a nontrivial solution of Problem (3.2)-(3.3). Obviously, for fixed ρ, the zeros and poles of $F(x, \rho)$ do not coincide unless $y(x, \rho) \equiv 0$. If $y(b, \rho_k(b)) = 0$, then $F(b, \rho_k(b)) = 0$ and

$$\frac{\partial F(x, \rho_k(b))}{\partial x}_{|x=b} = 1/r(b) > 0. \tag{3.4}$$

On the other hand, for fixed $x = b$, $F(x,\rho)$ is a finite-order meromorphic function of ρ, and

$$\frac{\partial F(b,\rho)}{\partial \rho}_{|\rho=\rho_k(b)} = y^{-2}(b,\rho_k(b)) \int_a^b p(x) y^2(x,\rho_k(b))dx > 0 \tag{3.5}$$

(e.g., see, [1, Chap. 6]). From the implicit-function theorem, together with (3.4)-(3.5), we obtain

$$\rho_k'(b) = -\frac{\frac{\partial F(x,\rho_k(b))}{\partial x}_{|x=b}}{\frac{\partial F(b,\rho)}{\partial \rho}_{|\rho=\rho_k(b)}} < 0,$$

and this completes the proof of the lemma. □

Proof of Lemma 3.2: Suppose $\lambda_1(b) > 0$, but $b \geq \bar{\eta}_1(a)$. In this case, the mini-max principle yields:

$$\lambda_1(b) = \min_{w \in \mathrm{H}} \frac{I(w)}{\int_a^b p(w)^2 dx} > 0,$$

where $I(w) = \int_a^b [r(w'')^2 + q(w')^2]dx$, and H is a set of nontrivial admissible functions w (i.e., $w(x) \in C^1[a,b]$, w' is absolutely continuous and $w'' \in L_2[a,b]$) for which $w(a) = w'(b) = 0$. On the other hand, by Lemma 3.3, $\rho_1(b) \leq 0$, and hence, the corresponding eigenfunction $v(x)$ satisfies the inequality

$$\int_a^b [r(v')^2 + q(v)^2]dx \leq 0.$$

Let $y(x) = \int_a^x v dx$. Then $y(a) = 0$, $y'(b) = 0$ and $\int_a^b [r(y'')^2 + q(y')^2]dx \leq 0$, which is a contradiction. The lemma is proved. □

The conclusion in the second part of the following lemma is similar to that of Greenberg [8] stated for the first eigenvalue of the problem determined by equation (3.1) and the Dirichlet boundary conditions $y(a) = y'(a) = y(b) = y'(b) = 0$.

Lemma 3.4. *The first eigenvalue* $\lambda_1(b)$ *of Problem* (3.1)-(1.3) *is simple. Furthermore, if* $b \to +a$ *then* $\lambda_1(b) \to +\infty$.

Proof. By Lemma 3.2, if $\lambda_1(b) > 0$ then $b < \bar{\eta}_1(a)$. Therefore, the solution h of the initial-value problem (2.12)-(2.21) is positive on the interval $[a,b]$, and hence, it is possible to transform equation (3.1) (with $\lambda = \lambda_1(b)$) into

$$(rh^3(t)\ddot{y})^{\cdot\cdot} = \lambda h^{-1} p(t) y \tag{3.6}$$

(with $\lambda = \lambda_1(b)$), and the boundary conditions (1.3) into

$$\tilde{y}(0) = \ddot{\tilde{v}}(0) = \dot{\tilde{v}}(\tilde{b}) = (\tilde{r}\tilde{h}^3\ddot{\tilde{v}})^{\cdot}(\tilde{b}) = 0, \tag{3.7}$$

where $\tilde{b} = \int_a^b h dx$. Obviously, if $\lambda = \lambda_1(b)$ is a multiple eigenvalue of Problem (3.1)-(1.3), then it is so for Problem (3.6)-(3.7). But, this is not possible since all the eigenvalues of this problem are simple (e.g., see [2]).

Let $b_0 > a$. For each $b \leq b_0$, consider the quadratic form

$$I(y) = \int_a^b [r(y'')^2 + q(y')^2]dx,$$

defined on the set of all nontrivial admissible functions y (i.e., $y(x) \in C^1[a,b]$, y' is absolutely continuous and $y'' \in L_2[a,b]$) for which $y(a) = y'(b) = 0$. For such y, we have the following expressions, which follows from the Cauchy-Schwarz inequality:

$$\int_a^b (y)^2 dx \leq (b-a) \int_a^b (y')^2 dx,$$

and

$$\int_a^b (y')^2 dx \leq (b-a) \int_a^b (y'')^2 dx.$$

Therefore,

$$I(y) \geq \frac{r^* \int_a^b (y)^2 dx}{(b-a)^2} + \frac{q^* \int_a^b (y)^2 dx}{(b-a)},$$

where, $f^* = \min_{x \in [a,b_0]} f(x)$. Thus,

$$\frac{I(y)}{\int_a^b p(y)^2 dx} \geq \frac{1}{p_*} \left(\frac{r^*}{(b-a)^2} + \frac{q^*}{(b-a)} \right),$$

where, $p_* = \max_{x \in [a,b_0]} p(x)$. The mini-max principle implies

$$\lambda_1(b) \geq \frac{1}{p_*} \left(\frac{r^*}{(b-a)^2} + \frac{q^*}{(b-a)} \right),$$

and hence, $\lim_{b \to a} \lambda_1(b) = +\infty$. □

Proof of Theorem 3.1: In view of Lemma 3.4, $\lambda_1(b) \to +\infty$ if $b \to +a$ (recall that $\lambda = \lambda_1(b)$ denotes the smallest eigenvalue of Problem (3.1)-(1.3)). Thus, there exists $b > a$ such that $\lambda_1(b) > 1$. Let $\hat{\tau}'(\lambda, x)$ denotes the subwronskian defined by (2.5) related to equation (3.1). It is easily remarked that, for fixed $x = b$, the zeros of the function $\hat{\tau}'(\lambda, x)$ and the eigenvalues of Problem (3.1)-(1.3), together with their multiplicities, coincide. In particular, the simplicity of $\lambda_1(b)$ (see Lemma 3.4) yields

$$\hat{\tau}'(\lambda_1(b), b) = 0, \quad \frac{\partial \hat{\tau}'}{\partial \lambda}(\lambda, b)_{|\lambda = \lambda_1(b)} \neq 0.$$

It then follows from the implicit-function theorem that $\lambda_1(b)$ is a continuous function of $b \in (a, \infty)$. Therefore, as b varies along the interval (a, ∞), $\lambda_1(b)$ cannot pass through the value $\lambda = 1$, since otherwise, we have for some $b > a$, $\hat{\eta}_1(a) = b$ exists for (1.1), and this is in contradiction to the hypothesis of the theorem. Hence, $\lambda_1(b) > 1$ for all $b \in (a, +\infty)$, and so, for every nontrivial admissible function w for which $w(a) = w'(b) = 0$, we obtain

$$\int_a^b r(w'')^2 + q(w')^2 dx > \int_a^b p w^2 dx.$$

The theorem is proved. □

We now establish a comparison theorem for $\hat{\mu}_1(a)$.

Theorem 3.5. *Let $r_0(x) > 0$, $p_0(x)) > 0$ and $q_0(x)$ be continuous functions on $[a, \infty)$, such that*

$$r \le r_0, \qquad p_0 \le p, \qquad q_0 \ge q, \tag{3.8}$$

and there exists the first systems-focal point, say $\hat{\mu}_1^0(a)$, for the equation

$$(r_0(x)y'')'' - (q_0(x)y')' = p_0(x)y. \tag{3.9}$$

Then $\hat{\mu}_1(a)$ exists for the original equation (1.1) *and*

$$a < \hat{\eta}_1(a) \le \hat{\eta}_1^0(a).$$

Proof. Suppose that $\hat{\eta}_1^0(a)$ exists but $\hat{\sigma}' > 0$ on $(a, \hat{\eta}_1^0(a)]$. Let y_0 be the corresponding eigenfunction; then Theorem 3.1 yields

$$I[y_0, a, \hat{\eta}_1^0(a)] = \int_a^{\hat{\eta}_1^0(a)} r(y_0'')^2 + q(y_0')^2 - p(y_0)^2 dx > 0$$

and

$$I^0[y_0, a, \hat{\eta}_1^0(a)] = \int_a^{\hat{\eta}_1^0(a)} r_0(y_0'')^2 + q_0(y_0')^2 - p_0(y_0)^2 dx = 0.$$

Subtracting these two expressions and taking into account (3.8), we obtain

$$0 \le \int_a^{\hat{\eta}_1^0(a)} (r_0 - r)(y_0'')^2 + (q_0 - q)(y_0')^2 + (p - p_0)(y_0)^2 dx < 0.$$

This contradiction shows that there exists $\hat{\eta}_1(a) \le \hat{\eta}_1^0(a)$. □

4. Oscillation of the eigenfunction associated to $\hat{\mu}_1(a)$

Theorem 4.1. *If $\hat{\mu}_1(a)$ exists, then it is realized by an unique eigenfunction $y_{\hat{\mu}_1}$ up to a multiplicative constant. It has the properties*

$$y_{\hat{\mu}_1} > 0, \quad y_{\hat{\mu}_1}' > 0, \quad T(y_{\hat{\mu}_1}) < 0 \quad \text{on} \quad (a, \hat{\mu}_1).$$

Also, if $q \le 0$ on $[a, \hat{\mu}_1)$, then $y_{\hat{\mu}_1}'' < 0$ on $(a, \hat{\mu}_1)$.

The following lemma establishes the relation between $\bar{\eta}_1$ (defined in Section 3) and $\hat{\mu}_1$.

Lemma 4.2. *If $\bar{\eta}_1(a)$ exists, then $\hat{\eta}_1(a)$ exists, and*

$$a < \hat{\mu}_1(a) \le \bar{\eta}_1(a), \tag{4.1}$$

with equality if, and only if, $p(x) \equiv 0$ on $[a, \bar{\eta}_1(a)]$.

Proof. It is easily seen that if $p(x) \equiv 0$, then $\hat{\mu}_1(a) = \bar{\eta}_1(a)$. Therefore, the conclusion of the lemma follows from Theorem 3.5. □

For the proof of Theorem 4.1 we need the following two lemmas.

Lemma 4.3 ([LN, Lemma 2.1]). *Let y be a nontrivial solution of the differential equation* (1.1) *for $q \equiv 0$. If y, y', y'' and Ty are nonnegative at $x = a$ (but not all zero), then they are positive for all $x > a$. If $y, -y', y''$ and $-Ty$ are nonnegative at $x = a$ (but not all zero), then they are positive for all $x < a$.*

Lemma 4.4. *Let u and v be two fundamental solutions of* (1.1) *defined by* (2.3) *and* (2.4), *respectively. Then:*

$$u > 0, \quad u' > 0, \quad Tu > 0 \quad on \quad (a, \hat{\mu}_1]. \tag{4.2}$$

$$v > 0, \quad v' > 0, \quad Tv > 0 \quad on \quad (a, \hat{\mu}_1]. \tag{4.3}$$

If, in addition; (2.1) *holds, then $u'' > 0$ and $v'' > 0$ on $(a, \hat{\mu}_1]$.*

Proof. In view of Lemma 4.2, we have $a < \hat{\eta}_1(a) < \bar{\eta}_1(a)$. In this case, from the definition of $\bar{\eta}_1(a)$, the solution h of (2.12) satisfying the initial conditions $h'(a) = 0$, $h(a) = 1$, is positive on $[a, \hat{\eta}_1(a)]$, and hence, it is possible to use the transformation (2.13) to rewrite equation (1.1) in the form (2.14). Note that, in view of (2.15) and (2.16), the initial conditions (2.3) are preserved after this transformation. Therefore, the solution $\tilde{u} \equiv u(x(t))$ of (2.14) satisfies these initial conditions. According to Lemma 4.3, we obtain

$$\tilde{u} > 0, \quad \dot{\tilde{u}} > 0, \quad (\tilde{r}\tilde{h}^3\ddot{\tilde{u}})^{\cdot} > 0, \quad \text{on} \quad (a, \hat{\mu}_1].$$

Again from (2.15)–(2.16), (4.2) follows. As shown in the proof of Theorem 4.1, if (2.1) holds on $(a, \hat{\mu}_1]$, then $h'(x) > 0$ on $(a, \hat{\mu}_1]$. Therefore, from the second relation in (2.15) we get $u'' > 0$ on $(a, \hat{\mu}_1]$. By similar arguments we prove the same results for v. □

Proof of Theorem 4.1: We introduce the ratios

$$\delta_0 = \frac{u}{v}, \quad \delta_1 = \frac{u'}{v'}, \quad \delta_2 = \frac{Tu}{Tv},$$

together with their derivatives

$$\delta_0' = -\frac{\hat{\sigma}}{v^2}, \quad \delta_1' = -\frac{\hat{\tau}}{r(v')^2}, \quad \delta_2' = \frac{p\hat{\tau}}{(Tv)^2}. \tag{4.4}$$

Let

$$y_{\hat{\mu}_1} = u - \delta_1(\hat{\mu}_1)v. \tag{4.5}$$

By Lemma 4.4, $\delta_1(\hat{\mu}_1)$ is well defined. In this case, we have $y'_{\hat{\mu}_1}(\hat{\mu}_1) = 0$ and $Ty_{\hat{\mu}_1}(\hat{\mu}_1) = \hat{\tau}'(\hat{\mu}_1) = 0$. Therefore, $y_{\hat{\mu}_1}$ is an eigenfunction of the boundary problem (1.1)–(1.3) defined on the interval $[a, \hat{\mu}_1]$. From the definition of $\hat{\mu}_1$ and the initial conditions (2.9), it follows that $\hat{\tau} > 0$ on $(a, \hat{\mu}_1]$. Thus, $\delta_1' < 0$ on this interval, and hence, $y'_{\hat{\mu}_1}(x) \neq 0$ on $(a, \hat{\mu}_1(a)$. From the initial condition $y'_{\hat{\mu}_1}(a) = u'(a) = 1$, it follows that $y'_{\hat{\mu}_1}(x) > 0$ and $y_{\hat{\mu}_1} > 0$. On the other hand, since $Ty_{\hat{\mu}_1}(a) = -1$, $T'y_{\hat{\mu}_1}(x) > 0$ on $(a, \hat{\mu}_1(a)]$ and $Ty_{\hat{\mu}_1}(\hat{\mu}_1(a)) = 0$, then $Ty_{\hat{\mu}_1}(x) < 0$ on $[a, \hat{\mu}_1(a))$.

The relations (2.15) and (2.16) yield

$$\tilde{y}_{\hat{\mu}_1} > 0, \quad \dot{\tilde{y}}_{\hat{\mu}_1} > 0, \quad (\tilde{r}\tilde{h}^3\ddot{\tilde{y}}_{\hat{\mu}_1})^{\cdot} < 0$$

on $(0, \tilde{\mu}_1(0))$, where $\tilde{\mu}_1(0) = \int_a^{\hat{\mu}_1} h$. From this and $\ddot{\tilde{y}}_{\hat{\mu}_1}(0) = 0$, it follows that $\ddot{\tilde{y}}_{\hat{\mu}_1}(t) < 0$ on $(0, \tilde{\mu}_1(0))$. It is easily seen that, if $q \leq 0$ on $[a, \hat{\mu}_1(a))$, then $h' < 0$ on $(a, \hat{\mu}_1(a))$. Therefore, from the second relation of (2.15), we obtain $y''_{\hat{\mu}_1}(x) < 0$ on $(a, \hat{\mu}_1(a))$. The theorem is proved. □

5. Sufficient conditions for the existence of $\hat{\eta}_1$

We say equation (1.1) is systems-conjugate in (a, ∞) if $\hat{\eta}_1$ exists; otherwise (1.1) is said to be systems-disconjugate. In this section, a number of conjugacy and disconjugacy criteria for (1.1) will be established.

Theorem 5.1. *If $\int^\infty q(t) = -\infty$ and $\int^\infty p(t) = +\infty$ then equation (1.1) is systems-conjugate.*

Proof. If the subwronskian $\hat{\sigma}$ has a zero in (a, ∞), then by Rolle's theorem, $\hat{\eta}_1$ exists. Assume that $\hat{\sigma} > 0$ on (a, ∞) and let $k(x) = \frac{\hat{\tau}'}{\hat{\sigma}}$. By using the identity (2.7), we obtain

$$k'(x) = -P - \frac{k^2}{r} < 0 \qquad \text{on } (a, \infty).$$

Integrating this expression, and taking into account the assumption $\int^\infty p = +\infty$, it follows that $k(x) \to -\infty$ as $x \to +\infty$, and hence, $\hat{\mu}_1(a)$ exists. Therefore, in view of Theorem 2.1 and the assumption $\int^\infty q = -\infty$, $\hat{\eta}_1(a)$ exists, which implies that (1.1) is systems-conjugate. □

Theorem 5.2. *If $\int^\infty \frac{1}{r}(t) = +\infty$ and $\int^\infty q(t) = -\infty$ then equation (1.1) is systems-conjugate.*

For the proof of this theorem we need the following result.

Theorem 5.3 ([9, 14])**.** *If the conditions*

$$\int_a^\infty r^{-1}(x)dx = \infty, \qquad \int_a^\infty q(s)ds = -\infty$$

hold, then the second-order equation (2.12) is oscillatory on (a, ∞); i.e., each of its solution has infinitely many zeros in this interval.

Proof. It is easy to see that the zeros of the subwronskian $\hat{\tau}'$ related to (1.1) for $p \equiv 0$ coincide with those of the solution h of the second-order initial value problem (2.12)–(2.21). In view of Theorem 5.3, h has infinitely many zeros in (a, ∞). Therefore, the first-systems focal point $\hat{\mu}_1(a)$ exists for (1.1) with $p \equiv 0$. By Theorem 3.5, $\hat{\mu}_1(a)$ exists for $p > 0$, and hence the assumption $\int^\infty q = -\infty$ and Theorem 2.1 yield the existence of the first-systems conjugate point $\hat{\eta}_1(a)$. The theorem is proved. □

By combining Theorem 3.1 with the second statement of Theorem 2.1 we obtain the following criterion giving the relation between the systems-disconjugacy of (1.1) and the sign of the associated quadratic functional.

Theorem 5.4. *If* $\int^{\infty} q = -\infty$*, then equation* (1.1) *is systems-disconjugate if, and only if,*

$$I[w,a,b] = \int_a^b r(w'')^2 + q(w')^2 - pw^2 dx > 0$$

for each $b \in (a, +\infty)$ *and each nontrivial admissible function* $w(x)$ *on* $[a,b]$ *(i.e.,* $w(x) \in C^1[a,b]$*,* w' *is absolutely continuous and* $w'' \in L_2[a,b]$*) for which* $w(a) = w'(b) = 0$*.*

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