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# ON THE QUALITATIVE BEHAVIORS OF SOLUTIONS TO A KIND OF NONLINEAR THIRD ORDER DIFFERENTIAL EQUATION WITH DELAY

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**ABSTRACT.** By defining an appropriate Lyapunov functional, we establish some new sufficient conditions to the asymptotically stability and boundedness of the solutions for a kind of non-autonomous differential equations of third order. Our results improve and extend some known results in the literature.

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## 1. INTRODUCTION

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, for example, in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows. In particular, stability, boundedness and asymptotic behavior of solutions of nonlinear third order differential equations have in the past and also recently been researched, because of the absence of its complete solution. It still one of the most burning problems of control theory, dynamical systems, time varying non linear systems and etc. In many works, the authors dealt with the problems by using Lyapunov's functions, and interesting results have been obtained we mention only a sampling of such papers [1–23] and other references therein.

In 1974, Hara [6] investigated the asymptotic behavior of solutions of the differential equation without delay of the form

$$x''' + a(t)x'' + b(t)x' + c(t)f(x) = e(t), \tag{1.1}$$

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and showed that all solutions of the equation (1.1) are uniformly bounded and satisfy $x(t) \to 0,\ x'(t) \to 0$ and $x''(t) \to 0$ as $t \to \infty$.

In 2005, Sadek [10], considered the following nonlinear differentiable of third order, with a constant deviating argument $r$ ensure the stability of system

$$x''' + a(t)x'' + b(t)x' + c(t)f(x(t-r)) = 0, \tag{1.2}$$

the following result was proved

**Theorem 1.1** ([10]). *Suppose that $a(t)$, $b(t)$, and $c(t)$ are continuously differentiable on $[0, +\infty[$ and the following conditions are satisfied:*

- S1) $0 < a_0 \le a(t) \le A$, $0 < b_0 \le b(t) \le B$ *and* $0 < c_0 \le c(t) \le C$; *for all* $t \in [0, +\infty[$.
- S2) $0 < f_0 \le \frac{f(x)}{x}$ *with* $x \neq 0$ *and* $f(0) = 0$, $|f'(x)| \le f_1 \le 1$, *for all* $x$.
- S3) $a_0 b_0 - C > 0$.
- S4) $b'(t) + \mu a'(t) - \frac{1}{\mu} c'(t) < \frac{a_0 b_0 - C}{2}$, *such that* $\mu = \frac{a_0 b_0 + C}{2 b_0}$.
- S5) $\int_0^\infty |c'(t)| < \infty, c'(t) \to 0$ *as* $t \to \infty$.

*Then the zero solution of* (1.2) *is uniformly asymptotically stable, provided that*

$$r < \min\left\{ \frac{2c_0 f_0}{f_1 C}, \frac{a_0 b_0 - C}{(1 + a_0) b_0 f_1 C}, \frac{a_0 b_0 - C + 4a_0 C(1 - f_1)}{2 f_1 C(1 + 2\mu + 2a_0^2 + a_0 + (a_0 b_0 - C)C)} \right\}.$$

We shall be concerned here, with asymptotic stability of zero solution and boundedness of all solutions of

$$(q(t)(p(t)x'(t))')' + a(t)x''(t) + b(t)x'(t) + c(t)f(x(t-r)) = 0 \tag{1.3}$$

and

$$(q(t)(p(t)x'(t))')' + a(t)x''(t) + b(t)x'(t) + c(t)f(x(t-r)) = e(t), \tag{1.4}$$

where $q(t)$, $p(t)$, $a(t)$, $b(t)$, $c(t)$, $e(t)$, and $f(x)$ are real valued functions continuous in their respective arguments and $f(0) = 0$. The derivatives $a'(t)$, $b'(t)$, $c'(t)$, $p(t)$, $p'(t)$, $q(t)$, $f'(x)$, exist and are continuous in their respective arguments. The equation discussed by Sadek [10] is a special case of equation (1.3) when $p(t) = q(t) = 1$.

## 2. PRELIMINARIES

We now give some definitions and important lemma which will play an important role in the proof of our main results. We consider

$$x' = f(t, x_t), \quad x_t(\theta) = x(t + \theta)\ , \quad -r \le \theta \le 0, \quad t \ge 0, \tag{2.1}$$

where $f : I \times C_H \to \mathbb{R}^n$ is a continuous mapping, $f(t, 0) = 0$, $C_H := \{\phi \in (C[-r, 0],\ \mathbb{R}^n) : \|\phi\| \le H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(t, \phi)| < L(H_1)$ when $\|\phi\| < H_1$.

**Definition 2.1** ([2]). An element $\psi \in C$ is in the $\omega - limit$ set of $\phi$, say $\Omega(\phi)$, if $x(t, 0, \phi)$ is defined on $[0, +\infty)$ and there is a sequence $\{t_n\}, t_n \to \infty$, as $n \to \infty$, with $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$ where $x_{t_n}(\phi) = x(t_n + \theta, 0, \phi)$ for $-r \le \theta \le 0$.

**Definition 2.2** ([2]). A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of (2.1), $x(t, 0, \phi)$, is defined on $[0, \infty)$ and $x_t(\phi) \in Q$ for $t \in [0, \infty)$.

**Lemma 2.3** ([1]). *If $\phi \in C_H$ is such that the solution $x_t(\phi)$ of (2.1) with $x_0(\phi) = \phi$ is defined on $[0, \infty)$ and $\|x_t(\phi)\| \le H_1 < H$ for $t \in [0, \infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and*

$$dist(x_t(\phi), \Omega(\phi)) \to 0 \qquad as\ t \to \infty.$$

**Lemma 2.4** ([5]). *Let $V(t, \phi) : I \times C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(t, 0) = 0$, and such that:*

- **(i)** $W_1(|\phi(0)|) \le V(t, \phi) \le W_2(\|\phi\|)$ *where* $W_1(r)$, $W_2(r)$ *are wedges.*
- **(ii)** $V'_{(2.1)}(t, \phi) \le 0$, *for* $\phi \in C_H$.

*Then the zero solution of (2.1) is uniformly stable. If $Z = \{\phi \in C_H : V'_{(2.1)}(t, \phi) = 0\}$, then the zero solution of (2.1) is uniformly asymptotically stable, provided that the largest invariant set in $Z$ is $Q = \{0\}$.*

## 3. ASSUMPTIONS AND MAIN RESULTS

Throughout this paper, we will use the following notations

$$A(t) = \frac{a(t)}{p(t)q(t)}, \text{ and } B(t) = \frac{b(t)p(t) - a(t)p'(t)}{p^2(t)}.$$

We assume that there are positive constants $a_0$, $b_0$, $c_0$, $m$, $\mu$, $\delta_0, \delta_1$, $C$, $L$, $M$, such that the following conditions hold

i) $0 < a_0 \le a(t)$, $0 < b_0 \le b(t)$ and $0 < c_0 \le c(t) \le C$; for all $t \ge 0$.
ii) $0 < m \le q(t) \le p(t) \le M$, $-L \le p'(t) \le 0$, and $-L \le q'(t) \le 0$, for all $t \ge 0$.
iii) $(p(t)c(t))' \le (q(t)c(t))'$ for all $t \ge 0$.
iv) $a_0 b_0 > \frac{M^3 C \delta_1}{m}$.
v) $0 < \delta_0 \le \frac{f(x)}{x}$ with $x \ne 0$ and $f(0) = 0$, $|f'(x)| \le \delta_1$.

Our main results is the following Theorem.

**Theorem 3.1.** *In addition to conditions (i)–(v) being satisfied, suppose that the following conditions hold*

H1) $\frac{MC\delta_1}{b_0} < \mu < \frac{a_0}{M}$, $\mu = \frac{a_0 b_0 + M^2 C \delta_1}{2 b_0 M}$.
H2) $B'(t) + \mu A'(t) - \frac{1}{\rho} c'(t) < \frac{1}{2M}\left(\frac{a_0 b_0}{M^2} - \frac{MC\delta_1}{m}\right)$ *for all* $t \ge 0$, *such that* $\rho = \frac{\mu}{\delta_1}$.
H3) $\int_0^t |[q(s)c(s)]'| ds \le N < \infty$.

*Then every solution of* (1.3) *is uniformly asymptotically stable, provided that*

$$r < \min\left\{\frac{m^2 a_1}{M\delta_1 C(1+\mu+\mu m^2)}, \frac{2(a_0 m - \mu M^2)}{CMm^2\delta_1}\right\},$$

*where* $a_1 = \frac{a_0 b_0}{M^2} - \frac{MC\delta_1}{m}$.

**Remark 3.2.** When $p(t) = q(t) = 1$, the conditions (iv), (H1), (H2) and (H3) of Theorem 3.1 are an extension of conditions (S3), (S4) and (S5) of Theorem 1.1 for $M = m = 1$. Remark that in condition (v) of Theorem 3.1 we assume that $|f'(x)| \le \delta_1$, but in assumption (S2) of Theorem 1.1 we have $|f'(x)| \le f_1 \le 1$ which is particularly restrictive.

*Proof.* We write equation (1.3) as the following equivalent system

$$\begin{aligned} x' &= \frac{1}{p(t)}y \\ y' &= \frac{1}{q(t)}z \\ z' &= -A(t)z - B(t)y - c(t)f(x) + c(t)\int_{t-r}^{t}\frac{y(s)}{p(s)}f'(x(s))ds. \end{aligned} \tag{3.1}$$

Consider the Lyapunov functional $W(t, x_t, y_t, z_t) = W$ defined as follows

$$W = \exp\left(-\frac{\theta(t)}{\beta}\right)V(t, x_t, y_t, z_t), \tag{3.2}$$

where

$$\begin{aligned} V(t, x_t, y_t, z_t) &= \mu p(t)c(t)F(x) + q(t)c(t)f(x)y + \frac{1}{2}z^2 + \mu yz \\ &\quad + \frac{q(t)}{2}\big(\mu A(t) + B(t)\big)y^2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds, \end{aligned} \tag{3.3}$$

such that $F(x) = \int_0^x f(u)du$. $\beta$ and $\lambda$ are positive constants which will be determined later and $\theta(t) = \int_0^t |[q(s)c(s)]'|\, ds < \infty$. We can rewrite (3.3) as follows

$$\begin{aligned} V(t, x_t, y_t, z_t) &= \mu p(t)c(t)F(x) + q(t)c(t)f(x)y + \frac{q(t)c(t)}{2\rho}y^2 + \frac{1}{2}\big(z + \mu y\big)^2 \\ &\quad + \frac{q(t)}{2}\left(B(t) + \mu A(t) - \frac{c(t)}{\rho} - \frac{\mu^2}{q(t)}\right)y^2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds. \end{aligned}$$

An easy calculation shows that

$$\begin{aligned} V(t, x_t, y_t, z_t) &= \mu\big(p(t)c(t) - q(t)c(t)\big)F(x) + \mu q(t)c(t)F(x) + q(t)c(t)f(x)y \\ &\quad + \frac{q(t)c(t)}{2\rho}y^2 + \frac{q(t)}{2}\left(B(t) + \mu A(t) - \frac{c(t)}{\rho} - \frac{\mu^2}{q(t)}\right)y^2 \\ &\quad + \frac{1}{2}\big(z + \mu y\big)^2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds. \end{aligned}$$

Since $p(t) \geq q(t)$ and the fact that the integral $\int_{t+s}^{t} y^2(\xi)d\xi$ is nonnegative

$$\begin{aligned} V(t, x_t, y_t, z_t) &\geq \mu q(t)c(t)\Psi(t) + \frac{q(t)}{2}\left(B(t) + \mu A(t) - \frac{c(t)}{\rho} - \frac{\mu^2}{q(t)}\right) y^2 \\ &+\frac{1}{2}\big(z + \mu y\big)^2, \end{aligned}$$

where

$$\Psi(t) = F(x) + \frac{1}{\mu}f(x)y + \frac{1}{2\mu\rho}y^2.$$

From (v) we get

$$\begin{aligned} \Psi(t) &= F(x) + \frac{1}{2\mu\rho}(y + \rho f(x))^2 - \frac{\rho}{2\mu}f^2(x) \\ &\geq \int_0^x \left(1 - \frac{\rho}{\mu}f'(u)\right) f(u)du \\ &\geq \frac{\delta_0}{2}\left(1 - \frac{\rho\delta_1}{\mu}\right) x^2. \end{aligned}$$

Let

$$U(t) = B(t) - \frac{c(t)}{\rho} + \mu(A(t) - \frac{\mu}{q(t)}),$$

using condition (H1) we have $U(t) > 0$ and according to the conditions (i) and (ii) we obtain

$$V(t, x_t, y_t, z_t) \geq \frac{\mu m c_0\delta_0}{2}\left(1 - \frac{\rho\delta_1}{\mu}\right) x^2 + \frac{m}{2}U(t)y^2 + \frac{1}{2}\big(z + \mu y\big)^2 \geq 0.$$

Hence there exists some positive constant $k_1$ such that

$$V(t, x_t, y_t, z_t) \geq k_1(x^2 + y^2 + z^2). \tag{3.4}$$

Therefore, we can find a continuous function $W_1(|\phi(0)|)$ with

$$W_1(|\phi(0)|) \geq 0 \quad \text{and} \quad W_1(|\phi(0)|) \leq W(t, \phi).$$

The existence of a continuous function $W_2(\|\phi\|)$ which satisfies the inequality $W(t, \phi) \leq W_2(\|\phi\|)$, is easily verified.

Let $\dot{V}_{(3.1)}(t, x_t, y_t, z_t) = \dot{V}_{(3.1)}$ denote the time derivative of the Lyapunov functional $V(t, x_t, y_t, z_t)$, along the trajectories of the system (3.1). An easy computation shows that

$$\begin{aligned} \dot{V}_{(3.1)} = \mu(p(t)c(t))'F(x) &+ \frac{1}{2\rho}(q(t)c(t))'y^2 + (q(t)c(t))'f(x)y + \left(\frac{\mu}{q(t)} - A(t)\right) z^2 \\ &+ \left(\frac{q(t)}{p(t)}c(t)f'(x) - \mu B(t) + \lambda r\right) y^2 + \frac{1}{2}\left((qB)'(t) + \mu(qA)'(t) - \frac{1}{\rho}(qc)'(t)\right) y^2 \\ &+ c(t)(\mu y + z)\int_{t-r}^{t} \frac{y(s)}{p(s)}f'(x(s))ds - \lambda\int_{t-r}^{t} y^2(\xi)d\xi, \end{aligned}$$

from which we deduce

$$\begin{aligned}\dot{V}_{(3.1)} &= \mu\big((p(t)c(t))' - (q(t)c(t))'\big)F(x) + \mu(q(t)c(t))'F(x) + \left(\frac{\mu}{q(t)} - A(t)\right)z^2 \\ &\quad + (q(t)c(t))'f(x)y + \frac{1}{2\rho}(q(t)c(t))'y^2 + \left(\frac{q(t)}{p(t)}c(t)f'(x) - \mu B(t) + \lambda r\right)y^2 \\ &\quad + \frac{1}{2}\left((qB)'(t) + \mu(qA)'(t) - \frac{1}{\rho}(qc)'(t)\right)y^2 \\ &\quad + c(t)(\mu y + z)\int_{t-r}^{t}\frac{y(s)}{p(s)}f'(x(s))ds - \lambda\int_{t-r}^{t}y^2(\xi)d\xi.\end{aligned}$$

But

$$\begin{aligned}(qB)'(t) + \mu(qA)'(t) - \frac{1}{\rho}(qc)'(t) &= q(t)\left[B'(t) + \mu A'(t) - \frac{1}{\rho}c'(t)\right] \\ &\quad + q'(t)\left[B(t) + \mu A(t) - \frac{1}{\rho}c(t)\right] \\ &\leq q(t)\left[\mu A'(t) + B'(t) - \frac{1}{\rho}c'(t)\right],\end{aligned}$$

since $\mu A(t) + B(t) - \frac{1}{\rho}c(t) > \mu A(t) + B(t) - \frac{\mu^2}{q(t)} - \frac{c(t)}{\rho} > 0$, and by (iii) we obtain

$$\begin{aligned}\dot{V}_{(3.1)} &\leq \mu(qc)'(t)F(x) + (qc)'(t)f(x)y + \frac{1}{2\rho}(qc)'(t)y^2 + \left(\frac{\mu}{q(t)} - A(t)\right)z^2 \\ &\quad + \frac{1}{2}q(t)\left(B'(t) + \mu A'(t) - \frac{1}{\rho}c'(t)\right)y^2 + \left(\frac{q(t)}{p(t)}c(t)f'(x) - \mu B(t) + \lambda r\right)y^2 \\ &\quad + c(t)(\mu y + z)\int_{t-r}^{t}\frac{y(s)}{p(s)}f'(x(s))ds - \lambda\int_{t-r}^{t}y^2(\xi)d\xi,\end{aligned}$$

we rearrange

$$\begin{aligned}\dot{V}_{(3.1)} &\leq \mu(qc)'(t)\Psi(t) + \frac{1}{2}q(t)\left(B'(t) + \mu A'(t) - \frac{1}{\rho}c'(t)\right)y^2 \\ &\quad + \left(\frac{q(t)}{p(t)}c(t)f'(x) - \mu B(t) + \lambda r\right)y^2 + \left(\frac{\mu}{q(t)} - A(t)\right)z^2 \\ &\quad + c(t)(\mu y + z)\int_{t-r}^{t}\frac{y(s)}{p(s)}f'(x(s))ds - \lambda\int_{t-r}^{t}y^2(\xi)d\xi.\end{aligned}$$

Using the Schwartz inequality $|uv| \leq \frac{1}{2}(u^2 + v^2)$ and since $|f'(x)| \leq \delta_1$, we obtain

$$\mu c(t)y\int_{t-r}^{t}\frac{y(s)}{p(s)}f'(x(s))ds \leq \mu\frac{\delta_1 Cr}{2}y^2 + \mu\frac{\delta_1 C}{2m^2}\int_{t-r}^{t}y^2(\xi)d\xi,$$

and

$$c(t)z\int_{t-r}^{t}\frac{y(s)}{p(s)}f'(x(s))ds \leq \frac{\delta_1 Cr}{2}z^2 + \frac{\delta_1 C}{2m^2}\int_{t-r}^{t}y^2(\xi)d\xi.$$

Hence

$$\begin{aligned}\dot{V}_{(3.1)} \;\leq\;& \mu(q(t)c(t))'\Psi(t) + \frac{1}{2}q(t)\left(B'(t) + \mu A'(t) - \frac{1}{\rho}c'(t)\right)y^2\\ &+ \left(\frac{q(t)}{p(t)}c(t)f'(x) - \mu B(t) + \lambda r + \frac{\mu C\delta_1 r}{2}\right)y^2\\ &+ \left(\frac{C\delta_1 r}{2} + \frac{\mu}{q(t)} - A(t)\right)z^2\\ &+ \left(\frac{\mu C\delta_1}{2m^2} + \frac{C\delta_1}{2m^2} - \lambda\right)\int_{t-r}^{t} y^2(\xi)d\xi.\end{aligned}$$

If we take $\lambda = \frac{\mu C\delta_1}{2m^2} + \frac{C\delta_1}{2m^2}$, the last inequality becomes

$$\begin{aligned}\dot{V}_{(3.1)} \;\leq\;& \mu(q(t)c(t))'\Psi(t) + \frac{1}{2}q(t)\left(B'(t) + \mu A'(t) - \frac{1}{\rho}c'(t)\right)y^2\\ &+ \left(\frac{q(t)}{p(t)}c(t)f'(x) - \mu B(t) + \frac{C\delta_1}{2m^2}(\mu + 1 + \mu m^2)r\right)y^2\\ &+ \left(\frac{C\delta_1 r}{2} + \frac{\mu}{q(t)} - A(t)\right)z^2.\end{aligned} \tag{3.5}$$

Using (3.2) we obtain

$$\dot{W}_{(3.1)} = \exp\left(-\frac{\theta(t)}{\beta}\right)\left[\dot{V}_{(3.1)}(t, x_t, y_t, z_t) - \frac{|(q(t)c(t))'|}{\beta}V(t, x_t, y_t, z_t)\right].$$

By taking $\beta = mc_0$, using (3.5), (3.4) and since $(q(t)c(t))' - |(q(t)c(t))'| \leq 0$ we have

$$\begin{aligned}\dot{W}_{(3.1)} \;\leq\;& \exp\left(-\frac{\theta(t)}{mc_0}\right)\left[\frac{1}{2}q(t)\left(B'(t) + \mu A'(t) - \frac{1}{\rho}c'(t)\right)y^2\right.\\ &+ \left(\frac{q(t)}{p(t)}c(t)f'(x) - \mu B(t) + \frac{C\delta_1}{2m^2}(\mu + 1 + \mu m^2)r\right)y^2\\ &\left.+ \left(\frac{C\delta_1 r}{2} + \frac{\mu}{q(t)} - A(t)\right)z^2\right].\end{aligned}$$

From (i), (ii), (v) and (H1)

$$\begin{aligned}\frac{q(t)}{p(t)}c(t)f'(x) - \mu B(t) \;\leq\;& \frac{MC\delta_1}{p(t)} - \left(\frac{a_0 b_0 + M^2 C\delta_1}{2b_0 M}\right)\frac{b_0}{p(t)}\\ \leq\;& -\frac{1}{2}\left(\frac{a_0 b_0}{M^2} - \frac{MC\delta_1}{m}\right) \leq 0,\end{aligned}$$

hence using (H2) we get

$$\begin{aligned}&\frac{1}{2}q(t)(B'(t) + \mu A'(t) - \frac{1}{\rho}c'(t)) + \frac{q(t)}{p(t)}c(t)f'(x) - \mu B(t)\\ &\qquad < -\frac{1}{4}\left(\frac{a_0 b_0}{M^2} - \frac{MC\delta_1}{m}\right) < 0.\end{aligned}$$

Thus,

$$\begin{aligned}\dot{W}_{(3.1)} \quad \leq \quad & \exp\left(-\frac{\theta(t)}{mc_0}\right)\left[\left(-\frac{1}{4}\left(\frac{a_0 b_0}{M^2}-\frac{MC\delta_1}{m}\right)+\frac{C\delta_1}{2m^2}(\mu+1+\mu m^2)r\right)y^2\right.\\ & \left.+\left(\frac{C\delta_1 r}{2}+\frac{\mu}{m}-\frac{a_0}{M^2}\right)z^2\right].\end{aligned}$$

Condition (H3) shows that $e^{-\frac{\theta(t)}{\beta}} \geq e^{-\frac{-N}{\beta}}$. Therefore, if

$$r<\min\left\{\frac{m^2 a_1}{2\delta_1 C(1+\mu+\mu m^2)}, \frac{2(a_0 m-\mu M^2)}{CM^2 m\delta_1}\right\},$$

where $a_1=\frac{a_0 b_0}{M^2}-\frac{MC\delta_1}{m}$, thus

$$\dot{W}_{(3.1)}(t,x_t,y_t,z_t) \leq -\gamma(y^2+z^2), \quad \text{for some } \ \gamma>0.$$

It is clear that the largest invariant set in $Z$ is $Q=\{0\}$ , where

$$Z=\left\{\phi\in C_H : \frac{d}{dt}W(\phi)=0\right\}.$$

That is, the only solution of system (3.1) for which $\dot{W}_{(3.1)}(t,x_t,y_t,z_t)=0$ is the solution $x=y=z=0$. The above discussion guarantees that the trivial solution of equation (1.3) is uniformly asymptotically stable and completes the proof of the Theorem. □

In the case $e(t)\neq 0$ we establish the following result:

**Theorem 3.3.** *In addition to the assumptions of Theorem* 3.1*, If we assume that* $e(t)$ *is continuous in* $\mathbb{R}$ *and*

$$\int_0^t e(s)ds<\infty \quad \textit{for all } t\geq 0,$$

*then all solutions of the perturbed equation* (1.4) *are bounded.*

*Proof.* The remaining of this proof follows the strategy indicated in the proof of Theorem 2 in [9] and hence it omitted. □

**Example.** We consider the following third order non-autonomous delay differential equation

$$\begin{aligned}&\left(\left(\frac{1}{8(1+t^2)}+\frac{31}{8}\right)\left(\left(\frac{1}{4(1+t^2)}+\frac{31}{8}\right)x'(t)\right)'\right)'+\left(20+e^{-t}\right)x''(t) \qquad (3.6)\\ &+\left(36+e^{-2t}\right)x'(t)+\left(\frac{1}{20(1+t)}+\frac{1}{20}\right)\left(\frac{x(t-r)}{200}+\frac{x(t-r)}{200(1+x^2(t-r))}\right)=2e^{3-t}.\end{aligned}$$

We have

$$\frac{31}{8} < q(t) = \frac{1}{8(1+t^2)} + \frac{31}{8} \leq p(t) = \frac{1}{4(1+t^2)} + \frac{31}{8} \leq 4,$$
$$-\frac{1}{4} \leq p'(t) < 0,\ -\frac{1}{8} \leq q'(t) < 0, \text{ for all } t \in [1, +\infty[.$$
$$\frac{1}{200} \leq \frac{f(x)}{x} = \frac{1}{200} + \frac{1}{200(1+x^2)} \text{ with } x \neq 0, \text{ and } |f'(x)| \leq \frac{1}{10^2} = \delta_1.$$

It follows easily that

$$\frac{1}{20} \leq c(t) = \frac{1}{20(1+t)} + \frac{1}{20} \leq \frac{1}{10},$$
$$36 \leq b(t) = 36 + e^{-2t} \leq 37;\ \textit{and}\ 20 \leq a(t) = 20 + e^{-t} \leq 21,$$
$$-2 \leq b'(t) \leq 0;\quad -1 \leq a'(t) \leq 0,$$
$$-\frac{1}{20} \leq c'(t) \leq 0; \text{ for all } t \in [1, +\infty[.$$

Since $p(t) - q(t) = \dfrac{1}{8(1+t^2)}$, it is easy to see that

$$(p(t)c(t))' - (q(t)c(t))' = (p(t) - q(t))c'(t) + (p'(t) - q'(t))c(t) \leq 0,$$

then $(p(t)c(t))' \leq (q(t)c(t))'$ for all $t \in [1, +\infty[$. Now, we verify conditions (H1) and (H2), we have

$$\begin{aligned}
B'(t) &= \frac{b'(t)p^2(t) - [b(t) + a'(t)]p(t)p'(t) - a(t)p(t)p''(t) + 2a(t)p'^2(t)}{p^3(t)} \\
&\leq \frac{-b(t)p(t)p'(t) + 2a(t)p'^2(t)}{p^3(t)} \leq 0,68; \\
A'(t) &= \frac{a'(t)p(t)q(t) - a(t)(p'(t)q(t) + p(t)q'(t))}{(p(t)q(t))^2} \\
&\leq \frac{-a(t)(p'(t)q(t) + p(t)q'(t))}{(p(t)q(t))^2} \leq 0.14.
\end{aligned}$$

We have $\mu = \frac{a_0 b_0 + M^2 C \delta_1}{2 b_0 M} = 2.5$, we obtain $\frac{1}{\rho} = \frac{\delta_1}{\mu} = 0.004$. We deduce that

$$\mu A'(t) + B'(t) - \frac{1}{\rho} c'(t) < 1, 1 < \frac{1}{2M}\left(\frac{a_0 b_0}{M^2} - \frac{MC\delta_1}{m}\right) = 5,62.$$

Finally we have $\int_1^\infty 2e^{3-s} ds < \infty$. Hence All the assumptions (i) through (v), are satisfied, (H1), (H2) and (H3) also hold, we can conclude using Theorem 3.3 that every solution of (3.6) is bounded.

**Remark 3.4.** An easy computation of derivatives shows that equation (1.3) can be rewritten as

$$x''' + \alpha(t)x'' + \beta(t)x' + \gamma(t)f(x(t-r) = 0, \tag{3.7}$$

where

$$\alpha(t) = \frac{p(t)q'(t) + 2q(t)p'(t) + a(t)}{p(t)q(t)}, \quad \beta(t) = \frac{q'(t)p'(t) + q(t)p''(t) + b(t)}{p(t)q(t)},$$

and $\gamma(t) = \frac{c(t)}{p(t)q(t)}$. If we apply Sadek's Theorem [10] to show that every solution x(t) of (3.7) is uniform-bounded and satisfies $x(t) \to 0$, $x'(t) \to 0$ and $x''(t) \to 0$ as $t \to \infty$, then the differentiability of $\alpha$ and $\beta$ is needed, which implies the use of the second derivative of $q$ and the third derivative of $p$. However in our Theorem this latter conditions are not required since we just need to deal with $p'$, $p''$ and $q'$.

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