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PULSATILE CONSTANT AND CHARACTERISATION OF FIRST ORDER NEUTRAL IMPULSIVE DIFFERENTIAL EQUATIONS

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ABSTRACT. In this work, we study the oscillation and nonoscillation properties of first order neutral differential equations with constant coefficients and constant delays by using pulsatile constant. Also, an attempt is made to extend the constant coefficient results to variable coefficient equations.

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1. INTRODUCTION

Oscillation and nonoscillation properties of linear impulsive differential equations with a single delay were first investigated by Gopalsamy and Zhang [5]. Later, papers devoted to oscillatory behaviour of solutions of linear impulsive differential equations with one or more delays were carried out by Bainov et al. [1, 2], Berezansky and Braverman [3], Chen et al. [4], Shen [10], Shen and Wang [9], Zhang et al. [12] and to mention a few. Indeed, the theory of impulsive differential equations with deviating arguments, due to the theoretical and practical difficulties, has been developed slowly. So, in this work an attempt is made to study the oscillation and nonoscillation properties of the neutral impulsive differential equations of the form:

$$(y(t) - ry(t-\tau))' + qy(t-\sigma) = 0, \quad t \neq \tau_k,\, k \in \mathbb{N} \tag{1.1}$$

(E)

$$\Delta(y(\tau_k) - ry(\tau_k - \tau)) + py(\tau_k - \sigma) = 0, \quad k \in \mathbb{N}, \tag{1.2}$$

where $\tau > 0$, $\sigma \geq 0$ are real constants, $r \in \mathbb{R}\backslash\{0\}$, $p, q \in \mathbb{R}$, and τ_k, $k \in \mathbb{N}$ with $\tau_1 < \tau_2 < \cdots < \tau_k < \cdots$ and $\lim_{k\to\infty} \tau_k = +\infty$ are fixed moments of impulsive effect

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with the property $\max\{\tau_{k+1}-\tau_k\} < +\infty$, $k \in \mathbb{N}$. For (E), Δ is the difference operator defined by

$$\Delta(y(\tau_k) - ry(\tau_k - \tau)) = y(\tau_k + 0) - ry(\tau_k - \tau + 0) - y(\tau_k - 0) + ry(\tau_k - \tau - 0);$$
$$y(\tau_k - 0) = y(\tau_k) \text{ and } y(\tau_k - \tau - 0) = y(\tau_k - \tau), \quad k \in \mathbb{N}.$$

The objective of this work is to study (E) and establish conditions for oscillation and nonoscillation of solutions of (E) subject to its associated characteristic equation. We may expect the possible solutions of (E) as

$$y(t) = e^{-\lambda t} A^{i(t_0,t)}, \quad t_0 \geq \rho = \max\{\tau, \sigma\}, \tag{1.3}$$

where $i(t_0, t) = k =$ number of impulses τ_k, $k \in \mathbb{N}$, and $A \neq 0$ is a real number which is called as the pulsatile constant. A close observation reveals that $y(t) = C_1 e^{-\lambda t}$ is a possible solution of (1.1) when (E) is without impulses and $y(n) = C_2 A^n$ is a possible solution of (1.2) when $i(t_0, t) = n$ and the impulses are the discrete values only ($\because$ in case (1.2), $\lambda = 0$). Therefore, (1.3) seems to be the possible choice of solution of (E).

In [11], Tripathy has considered

$$\begin{aligned} &(y(t) - r(t)y(t-\tau))' + q(t)f(y(t-\sigma)) = 0,\ t \neq \tau_k, \quad k \in \mathbb{N}, \\ &\Delta(y(\tau_k) - r_k y(\tau_k - \tau)) + q_k f(y(\tau_k - \sigma)) = 0, \quad k \in \mathbb{N} \end{aligned} \tag{1.4}$$

and studied the oscillatory character of the solutions of the system. For all ranges of $r(t)$, he has established the oscillation criteria for the impulsive system (1.4) which is highly nonlinear. It is observed that the study of (1.4) is easier than the study of (E) subject to its characteristic equations. With our effort, we encounter the linearized oscillation for the highly nonlinear impulsive system

$$\begin{aligned} &(y(t) - r(t)g(y(t-\tau)))' + q(t)f(y(t-\sigma)) = 0, \quad t \neq \tau_k,\ k \in \mathbb{N}, \\ &\Delta(y(\tau_k) - r_k g(y(\tau_k - \tau))) + p(\tau_k)f(y(\tau_k - \sigma)) = 0, \quad k \in \mathbb{N}. \end{aligned}$$

For more details about the theory of impulsive differential equations we refer the monographs [7] and [8] to the readers.

Definition 1.1. A function $y : [-\rho, +\infty) \to \mathbb{R}$ is said to be a solution of (E) with initial function $\o \in C([-\rho, 0], \mathbb{R})$, $y(t) = \o(t)$ for $t \in [-\rho, 0]$, $y \in PC(\mathbb{R}_+, \mathbb{R})$, $z(t) = y(t) + p(t)y(t-\tau)$ and $r(t)z'(t)$ are continuously differentiable for $t \in \mathbb{R}_+$, and $y(t)$ satisfies (E) for all sufficiently large $t \geq 0$, where $\rho = \max\{\tau, \sigma\}$ and $PC(\mathbb{R}_+, \mathbb{R})$ is the set of all functions $U : \mathbb{R}_+ \to \mathbb{R}$ which are continuous for $t \in \mathbb{R}_+$, $t \neq \tau_k$, $k \in \mathbb{N}$, continuous from the left- side for $t \in \mathbb{R}_+$, and have discontinuity of the first kind at the points $\tau_k \in \mathbb{R}_+$, $k \in \mathbb{N}$.

Definition 1.2. A nontrivial solution $y(t)$ of (E) is said to be nonoscillatory, if there exists a point $t_0 \geq 0$ such that $y(t)$ has a constant sign for $t \geq t_0$. Otherwise, the solution $y(t)$ is said to be oscillatory.

Definition 1.3. A solution $y(t)$ of (E) is said to be regular, if it is defined on some interval $[T_y, +\infty) \subset [t_0, +\infty)$ and

$$\sup\{|y(t)| : t \geq T_y\} > 0$$

for every $T_y \geq T$. A regular solution $y(t)$ of (E) is said to be eventually positive (eventually negative), if there exists $t_1 > 0$ such that $y(t) > 0$ $(y(t) < 0)$, for $t \geq t_1$.

2. MAIN RESULTS

In this section, we study the oscillatory and nonoscillatory behaviour of solutions of (E) through its associated characteristic equation provided (1.3) holds.

Theorem 2.1. *Let $\tau > \sigma > 0$ and $p \neq 0 \neq q$. Then (E) admits an oscillatory solution in the exponential impulsive form (1.3) if and only if the algebraic equation*

$$-\lambda\left(1 - \frac{p}{q}\lambda\right)^{L} + r\lambda e^{\lambda\tau}\left(1 - \frac{p}{q}\lambda\right)^{L-s} + qe^{\lambda\sigma} = 0 \tag{2.1}$$

has at least one real root λ with $\lambda > \frac{q}{p}$, for $pq > 0$ and $\lambda < \frac{q}{p}$, for $pq < 0$, where $i(t-\sigma, t) = L > 0$ is a constant and $i(t-\tau, t) = s =$ number of impulses between $t - \tau$ and t.

Proof. Let $y(t)$ be a regular nontrivial solution of the system (E) such that $y(t) = e^{-\lambda t}A^{i(t_0,t)}$, $t > t_0 > \rho$. Then (1.1) becomes

$$-\lambda e^{-\lambda t}A^{i(t_0,t)} + r\lambda e^{-\lambda t}e^{\lambda\tau}A^{i(t_0,t-\tau)} + qe^{-\lambda(t-\sigma)}A^{i(t_0,t-\sigma)} = 0,$$

that is,

$$qe^{\lambda\sigma} - \lambda A^{i(t_0,t)-i(t_0,t-\sigma)} + r\lambda e^{\lambda\tau}A^{i(t_0,t-\tau)-i(t_0,t-\sigma)} = 0. \tag{2.2}$$

Indeed, $i(t_0, t) - i(t_0, t-\sigma) = i(t-\sigma, t) = L$ and

$$i(t_0, t-\tau) - i(t_0, t-\sigma) = -i(t-\tau, t-\sigma) = -[i(t-\tau, t) - i(t-\sigma, t)] = L - s$$

implies that

$$-\lambda A^{L} + r\lambda e^{\lambda\tau}A^{L-s} + qe^{\lambda\sigma} = 0 \tag{2.3}$$

due to (2.2). Once again we use (1.3) in (1.2) to obtain a relation of the form

$$y(\tau_k + 0) - ry(\tau_k - \tau + 0) - y(\tau_k - 0) + ry(\tau_k - \tau - 0) + py(\tau_k - \sigma) = 0,$$

that is,

$$e^{-\lambda\tau_k}A^{i(t_0,\tau_k+0)} - re^{-\lambda(\tau_k-\tau)}A^{i(t_0,\tau_k-\tau+0)} - e^{-\lambda\tau_k}A^{i(t_0,\tau_k-0)} + re^{-\lambda(\tau_k-\tau)}A^{i(t_0,\tau_k-\tau-0)} + pe^{-\lambda(\tau_k-\sigma)}A^{i(t_0,\tau_k-\sigma)} = 0.$$

We may note that $i(t_0, \tau_k + 0) - i(t_0, \tau_k - 0) = 1$. Hence, the last inequality becomes

$$A^{1+i(t_0,\tau_k-0)} - re^{\lambda\tau}A^{1+i(t_0,\tau_k-\tau-0)} - A^{i(t_0,\tau_k-0)} + re^{\lambda\tau}A^{i(t_0,\tau_k-\tau-0)} + pe^{\lambda\sigma}A^{i(t_0,\tau_k-\sigma)} = 0,$$

that is,

$$(A-1)A^{i(t_0,\tau_k)} - r(A-1)e^{\lambda\tau}A^{i(t_0,\tau_k-\tau)} + pe^{\lambda\sigma}A^{i(t_0,\tau_k-\sigma)} = 0.$$

Therefore,

$$(A-1)A^{i(t_0,\tau_k)-i(t_0,\tau_k-\sigma)} - r(A-1)e^{\lambda\tau}A^{i(t_0,\tau_k-\tau)-i(t_0,\tau_k-\sigma)} + pe^{\lambda\sigma} = 0. \tag{2.4}$$

Using the fact

$$i(t_0,\tau_k) - i(t_0,\tau_k-\sigma) = i(\tau_k-\sigma,\tau_k) = L$$

and

$$i(t_0,\tau_k-\tau) - i(t_0,\tau_k-\sigma) = -i(\tau_k-\tau,\tau_k-\sigma) = -[i(\tau_k-\tau,t) - i(\tau_k-\sigma,t)] = L-s,$$

we obtain from (2.4) that

$$(A-1)A^{L} - r(A-1)e^{\lambda\tau}A^{L-s} + pe^{\lambda\sigma} = 0. \tag{2.5}$$

If we choose $A = 1-\frac{p}{q}\lambda$, then it is easy to verify that (2.5) reduces to (2.3). Consequently, (2.3) is same as the algebraic equation (2.1). Moreover, (2.1) is the required characteristic equation for (E). Ultimately, if $y(t)$ is an oscillatory solution of (E) with the pulsatile constant $A = 1-\frac{p}{q}\lambda < 0$, where $\lambda > \frac{q}{p}$ for $pq > 0$ and $\lambda < \frac{q}{p}$ for $pq < 0$, then λ satisfies the characteristic equation (2.1). Conversely, consider the characteristic equation (2.1) and assume that $\lambda = \lambda^*$ is the real root of (2.1) with $\lambda^* > \frac{q}{p}$, for $pq > 0$ and $\lambda^* < \frac{q}{p}$, for $pq < 0$. Then (E) admits an oscillatory solution $y(t) = e^{-\lambda^* t}A^{i(t_0,t)}$ with the pulsatile constant $A = 1-\frac{p}{q}\lambda^* < 0$. This completes the proof of the theorem. □

Theorem 2.2. *Let all the assumptions of Theorem* 2.1 *hold. Then* (E) *admits an eventually positive solution in the form of* (1.3) *if and only if* (2.1) *has at least one real root* λ *with* $\lambda < \frac{q}{p}$, *for* $pq > 0$ *and* $\lambda > \frac{q}{p}$, *for* $pq < 0$.

Proof. The proof of the theorem follows from the proof of Theorem 2.1 and hence the details are omitted. □

Corollary 2.3. *Let* $p, q, r \in \mathbb{R}\backslash\{0\}$, *and* $\sigma, \tau \in \mathbb{R}_+$ *such that* $\sigma = \tau \neq 0$ *or* $\sigma = 0 \neq \tau$ *hold. Then the conclusion of the Theorems* 2.1 *and* 2.2 *are hold true.*

Corollary 2.4. *In Theorem* 2.1, *let* $p = q \neq 0$. *Then* (E) *admit an oscillatory solution in the exponential impulsive form* (1.3) *if and only if* $\lambda > 1$ *and eventually positive solution if and only if* $\lambda < 1$.

Remark 2.5. Following to Corollary 2.4, we may note that $\lambda = 1$ if and only if $A = 0$, that is, (E) has the trivial solution.

Theorem 2.6. *Let* $\tau > \sigma > 0$ *and* $p = q = 0$. *Then*

i) *for* $r \in (-\infty, 0)$ *and* s *odd or* $r \in (0,\infty)$ *and* s *even,* (E) *admits an oscillatory solution if and only if* $\lambda^* \in (1,\infty)$ *is a root of the characteristic equation of* (E);

ii) *for* $r \in (0,\infty)$, (E) *admits an eventually positive solution if and only if* $\lambda^* \in (-\infty, 1)$ *is a root of the characteristic equation of* (E).

Proof. Proceeding as in the proof of Theorem 2.1 we have the impulsive system

$$-\lambda A^{i(t_0,t)} + r\lambda e^{\lambda\tau} A^{i(t_0,t-\tau)} = 0,$$

$$(A-1)A^{i(t_0,\tau_k)} - r(A-1)e^{\lambda\tau} A^{i(t_0,\tau_k-\tau)} = 0$$

which in turn implies that

$$-\lambda A^{k} + r\lambda e^{\lambda\tau} A^{k-s} = 0,$$

$$(A-1)A^{k} - r(A-1)e^{\lambda\tau} A^{k-s} = 0.$$

Consequently, the above system becomes

$$-\lambda A^{s} + r\lambda e^{\lambda\tau} = 0,$$

$$(A-1)A^{s} - r(A-1)e^{\lambda\tau} = 0$$

which is equivalent to say that

$$A = 1-\lambda, \quad -\lambda A^{s} + r\lambda e^{\lambda\tau} = 0$$

and hence

$$-\lambda(1-\lambda)^{s} + r\lambda e^{\lambda\tau} = 0 \tag{2.6}$$

is the resulting characteristic equation for (E). Clearly, $\lambda \neq 1$ for $r \neq 0$ in (2.6). Hence to solve (2.6), it happens that either $\lambda \in (-\infty, 1)$ or $\lambda \in (1,\infty)$. If the former holds, then $1-\lambda > 0$, that is, $A > 0$ and (2.6) holds true when $r \in (0,\infty)$. Therefore, (E) admits an eventually positive solution in the form (1.3) if and only if $\lambda^* \in (-\infty, 1)$ is a root of (2.6). Assume that the latter holds. Then $1-\lambda < 0$, that is, $A < 0$ and (2.6) holds true when $r \in (-\infty, 0)$ with odd s or $r \in (0,\infty)$ with even s. Therefore, (E) admits an oscillatory solution in the form (1.3) if and only if $\lambda^* \in (1,\infty)$ is a root of (2.6). This completes the proof of the theorem. □

Remark 2.7. Indeed, (2.6) doesn't hold if $r \in (-\infty, 0)$ and $\lambda \in (-\infty, 1)$.

Theorem 2.8. *Let* $p, r \in \mathbb{R}\backslash\{0\}$ *be such that* $p + r > 0$. *Assume that* $\tau = \sigma \neq 0$, $q = 0$ *and* $i(t-\tau, t) = 1$. *Then for* $r \in (-1,\infty)$, $r \neq 1$, (E) *admits an eventually positive solution if and only if* $4p \leq (r-1)^2$, *and for* $r \in (-\infty,-1)$, (E) *admits an oscillatory solution if and only if* $4p \leq (r-1)^2$.

Proof. Let $y(t)$ be a regular nontrivial solution of (E) in the form of (1.3). Then proceeding as in Theorem 2.1, we have the system of equations

$$-\lambda A + r\lambda e^{\lambda\tau} = 0,$$

$$(A-1)A + [p - r(A-1)]e^{\lambda\tau} = 0.$$

In the above system of equations, we have $\lambda = 0$. Otherwise, $A = re^{\lambda\tau}$ and hence $p = 0$ which is absurd. Consequently,

$$2A = (1+r) \pm [(1+r)^2 - 4(p+r)]^{\frac{1}{2}}.$$

Because, we are concerned with the non-zero real roots, then $4p \le (r-1)^2$. If $4p = (r-1)^2$, then $2A = r+1 > 0$, for $r \in (-1,\infty)$ and $r \ne 1$, that is, (E) admits a nonoscillatory solution. For $4p < (r-1)^2$, two roots A_1 and A_2 of A are positive when $r \in (-1,\infty)$ with $r \ne 1$ and $r+p > 0$. Similar observation can be made when $r \in (-\infty,-1)$, for $4p \le (r-1)^2$. Hence, the theorem is proved. □

Corollary 2.9. *Let $p, r \in \mathbb{R}\setminus\{0\}$, $\tau = \sigma \ne 0$, $q = 0$, $4p = (r-1)^2$ and $i(t-\tau,t) = 1$. Then for $r \in (-1,\infty)$, $\ne 1$, (E) admits an eventually positive solution if and only if the algebraic equation $A^2 - A(1+r) + p + r = 0$ has a real root $A^* \in (0,\infty) - \{\frac{1}{2}, 1\}$, and for $r \in (-\infty,-1)$, (E) admits an oscillatory solution if and only if the algebraic equation $A^2 - A(1+r) + p + r = 0$ has a real root $A^* \in (-\infty, 0)$.*

Corollary 2.10. *Let $r, p \in (0,\infty)$, $\sigma = 0 = q$ and $i(t-\tau,t) = 1$. Then (E) admit eventually positive solutions if and only if $p \le 1 + r - \sqrt{4r}$ and oscillatory solutions if and only if $p \ge 1 + r + \sqrt{4r}$.*

Remark 2.11. If we denote

$$F(\lambda) = -\lambda\left(1 - \frac{p}{q}\lambda\right)^{L} + r\lambda e^{\lambda\tau}\left(1 - \frac{p}{q}\lambda\right)^{L-s} + qe^{\lambda\sigma},$$

then it is easy to verify that $F(0) = q > 0$,

$$F\left(\frac{q}{p}\right) \longrightarrow +\infty, \text{ for } r > 0, \ \ p > 0, \ \ q > 0$$

and

$$F\left(-\frac{q}{p}\right) = \frac{q}{p}2^{L}\left[1 - r2^{-s}e^{-\frac{q}{p}\tau}\right] + qe^{-\frac{q}{p}\sigma} > 0,$$

for $r \in (0,1)$, $p > 0$ and $q > 0$. Keeping in view of Theorem 2.1, hence we have proved the following result:

Theorem 2.12. *Let $p, q > 0$, $r \in (0,1)$ and $\tau > \sigma > 0$. Then every solution of (E) which is of the form (1.3) oscillates if and only if (2.1) has no real roots $\lambda^* \in \left[-\frac{q}{p}, \frac{q}{p}\right]$.*

Example 2.13. Consider the system of equations

$$\left(y(t) - ry\left(t - \frac{1}{3}\right)\right)' + qy\left(t - \frac{1}{6}\right) = 0, \quad t \ne \tau_k,\ t > \frac{1}{3},\ k \in \mathbb{N}$$
$$\Delta\left(y(\tau_k) - ry\left(\tau_k - \frac{1}{3}\right)\right) + py\left(\tau_k - \frac{1}{6}\right) = 0, \quad k \in \mathbb{N}, \tag{2.7}$$

where $r = 0.025014234$, $q = 0.02$, $p = 0.01$. If we choose $L = 3$ and $s = 4$, then from the characteristic equation of (2.7), it follows that $A = -0.5$, $\lambda = 3$ and

$$y(t) = e^{-3t}(-0.5)^{i(\frac{1}{3},t)}$$

is an oscillatory solution of (2.7). Hence, by Theorem 2.1, (2.7) admits an oscillatory solution.

Example 2.14. Consider the system of equations

$$\begin{aligned} &(y(t) - ry(t-4))' + qy(t-2) = 0, \quad t \neq \tau_k,\ t > 4,\ k \in \mathbb{N} \\ &\Delta(y(\tau_k) - ry(\tau_k - 4)) + py(\tau_k - 2) = 0, \quad k \in \mathbb{N}, \end{aligned} \tag{2.8}$$

where $r = 0.05801223$, $q = 0.021$, $p = 0.012$. If we choose $L = 1$ and $s = 2$, then from the characteristic equation of (2.8), it follows that $A = 0.714285714$, $\lambda = \frac{1}{2}$ and

$$y(t) = e^{-\frac{t}{2}}(0.714285714)^{i(4,t)}$$

is an eventually positive solution of (2.8). Hence, by Theorem 2.2, (2.8) admits an eventually positive solution.

3. LINEARIZED OSCILLATION CRITERIA

Consider the nonlinear neutral impulsive delay differential equation of the form:

$$\begin{aligned} &(y(t) - r(t)g(y(t-\tau)))' + q(t)f(y(t-\sigma)) = 0, \quad t \neq \tau_k,\ k \in \mathbb{N} \\ &\Delta(y(\tau_k) - r(\tau_k)g(y(\tau_k - \tau))) + p(\tau_k)f(y(\tau_k - \sigma)) = 0, \quad k \in \mathbb{N}, \end{aligned} \tag{3.1}$$

where $r, \tau, \sigma \in \mathbb{R}_+$, $g, f \in C(\mathbb{R}, \mathbb{R})$ and $p, q \in C(\mathbb{R}_+, \mathbb{R}_+)$. We introduce the following assumptions for the system (3.1):

(A_1) $\lim_{t\to\infty} r(t) = r_0$, $r_0 \in [0, 1)$; $\lim_{t\to\infty} q(t) = q_0 \in \mathbb{R}_+$, $\liminf_{t\to\infty} p(t) = p_0 \in \mathbb{R}_+$,

(A_2) $ug(u) > 0$, $vf(v) > 0$ for $u, v \neq 0$ and $g(u) \leq u$, $f(v) \leq v$ for $u, v \geq 0$; $\lim_{u\to 0} \frac{g(u)}{u} = 1 = \lim_{v\to 0} \frac{f(v)}{v}$.

With the system of equations (3.1), we associate the linear system of equations

$$\begin{aligned} &(x(t) - r_0 x(t-\tau))' + q_0 x(t-\sigma) = 0, \quad t \neq \tau_k,\ k \in \mathbb{N} \\ &\Delta(x(\tau_k) - r_0 x(\tau_k - \tau)) + p_0 x(\tau_k - \sigma) = 0, \quad k \in \mathbb{N}. \end{aligned} \tag{3.2}$$

In this section our aim is to establish conditions for the oscillation of solutions of the system (3.1) in terms of the oscillation of solutions of the limiting equations (3.2). We note that the associated characteristic equation for the system (3.2) is given by

$$-\lambda\left(1 - \frac{p_0}{q_0}\lambda\right)^L + r_0\lambda e^{\lambda\tau}\left(1 - \frac{p_0}{q_0}\lambda\right)^{L-s} + q_0 e^{\lambda\sigma} = 0. \tag{3.3}$$

By Theorem 2.1, (3.2) admits an oscillatory solution in the form (1.3) if and only if (3.3) has at least one real root λ with $\lambda > \frac{q_0}{p_0}$.

Theorem 3.1. *Assume that* (3.3) *has no real roots in* $\left[-\frac{q_0}{p_0}, \frac{q_0}{p_0}\right]$. *Furthermore, assume that* (A_1) *and* (A_2) *hold. Then the system* (3.1) *admits oscillatory solutions.*

Proof. Suppose that (3.1) doesn't admit oscillatory solution and let $y(t)$ be a nonoscillatory solution of (3.1). Then there exists $t_0 \geq \max\{\sigma, \tau\}$ such that $y(t) > 0$, $y(t-\tau) > 0$ and $y(t-\sigma) > 0$, for $t \geq t_0$. If we set

$$z(t) = y(t) - r(t)g(y(t-\tau)),$$

then the system (3.1) becomes

$$\begin{aligned} z'(t) &= -q(t)f(y(t-\sigma)) \leq 0, \quad t \neq \tau_k,\, k \in \mathbb{N}, \\ \Delta z(\tau_k) &= -p(\tau_k)f(y(\tau_k - \sigma)) \leq 0, \quad k \in \mathbb{N}, \end{aligned}$$

for $t \geq t_1 > t_0$. As a result, $z(t)$ is nonincreasing on $[t_2, \infty)$, $t_2 > t_1$. We claim that $z(t)$ is bounded, for $t \geq t_2$. If not, there exists $\{\eta_n\}$ such that $\lim_{n\to\infty} \eta_n = \infty$, $\lim_{n\to\infty} x(\eta_n) = \infty$ and $x(\eta_n) = \max_{t_2 \leq s \leq \eta_n} x(s)$. Consequently,

$$\begin{aligned} z(\eta_n) &= y(\eta_n) - r(\eta_n)g(y(\eta_n - \tau)) \\ &\geq y(\eta_n) - r(\eta_n)y(\eta_n - \tau) \\ &\geq (1 - r(\eta_n))y(\eta_n) \\ &\to +\infty \text{ as } n \to \infty \end{aligned}$$

implies that $z(t)$ is non-decreasing, a contradiction. So, our claim holds and $z(t)$ is bounded ultimately. We assert that $\lim_{t\to\infty} y(t) = 0$. Integrating (3.1) from t_2 to ∞, we obtain

$$\int_{t_2}^{\infty} q(s)f(y(s-\sigma))ds - \sum_{t_2 \leq \tau_k < \infty} \Delta z(\tau_k) = z(t_2) - \lim_{t\to\infty} z(t),$$

that is,

$$\int_{t_2}^{\infty} q(s)f(y(s-\sigma))ds + \sum_{t_2 \leq \tau_k < \infty} p(\tau_k)f(y(\tau_k - \sigma)) < \infty. \tag{3.4}$$

In view of the conditions on q and f, (3.4) implies that $\liminf_{t\to\infty} y(t) = 0$. By Lemma 1.5.2 [6], it follows that $\lim_{t\to\infty} z(t) = 0$. Let $\varepsilon \in (0, 1 - r_0)$ be given. Then for sufficiently large t, it follows that $r(t) \leq r_0 + \varepsilon < 1$. Now,

$$\begin{aligned} 0 &= \lim_{t\to\infty} z(t) = \limsup_{t\to\infty} z(t) \\ &\geq \limsup_{t\to\infty} (y(t) - (r_0 + \varepsilon)\, y(t-\tau)) \\ &\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} (-(r_0 + \varepsilon)\, y(t-\tau)) \\ &= (1 - r_0 - \varepsilon) \limsup_{t\to\infty} y(t) \end{aligned}$$

implies that $\limsup_{t\to\infty} y(t) = 0$ and hence $\lim_{t\to\infty} y(t) = 0$. As $\{y(\tau_k - 0)\}_1^\infty$ and $\{y(\tau_k + 0)\}_1^\infty$ are sequence of real values, and continuity of y implies that

$$\lim_{k\to\infty} y(\tau_k - 0) = 0, \quad \lim_{k\to\infty} y(\tau_k + 0) = 0$$

due to $\liminf_{t\to\infty} y(t) = 0$ and $\limsup_{t\to\infty} y(t) = 0$ respectively. Hence, for the system (3.1), $\lim_{t\to\infty} y(t) = 0 = \lim_{k\to\infty} y(\tau_k)$. Let's set

$$Q(t) = q(t)\frac{f(y(t-\sigma))}{y(t-\sigma)}, \quad P(t) = p(t)\frac{f(y(t-\sigma))}{y(t-\sigma)}, \quad R(t) = r(t)\frac{g(y(t-\tau))}{y(t-\tau)}$$

for sufficiently large t. Then it is easy to see that $\lim_{t\to\infty} Q(t) = q_0$, $\lim_{t\to\infty} R(t) = r_0$ and $\liminf_{t\to\infty} P(t) = \liminf_{t\to\infty} p(t) \lim_{t\to\infty} \frac{f(y(t-\sigma))}{y(t-\sigma)} = p_0$. Ultimately, the system (3.1) becomes

$$\begin{aligned}
&(y(t) - R(t)y(t-\tau))' + Q(t)y(t-\sigma) = 0, \quad t \neq \tau_k,\ k \in \mathbb{N} \\
&\Delta(y(\tau_k) - R(\tau_k)y(\tau_k - \tau)) + P(\tau_k)y(\tau_k - \sigma) = 0, \quad k \in \mathbb{N},
\end{aligned} \tag{3.5}$$

for large t. Let $0 < \varepsilon_1 < q_0$ be given such that $Q(t) \geq q_0 - \varepsilon_1$, for any large t. Then integrating (3.5) from t to T $(T > t)$, we get

$$z(t) = z(T) + \int_t^T Q(s)y(s-\sigma)ds - \sum_{t\leq\tau_k<T} \Delta z(\tau_k),$$

that is,

$$\begin{aligned}
y(t) &= R(t)y(t-\tau) + \int_t^\infty Q(s)y(s-\sigma)ds + \sum_{t\leq\tau_k<\infty} P(\tau_k)y(\tau_k - \sigma) \\
&\geq (r_0 - \varepsilon)y(t-\tau) + (q_0 - \varepsilon_1)\int_t^\infty y(s-\sigma)ds + p_0 \sum_{t\leq\tau_k<\infty} y(\tau_k - \sigma),
\end{aligned}$$

for any large t. Let $Y = BC([t^*, \infty), \mathbb{R})$ be the space of all real valued bounded continuous functions defined on $\mathbb{R}$ such that Y is a Banach space with respect to the sup norm defined by

$$\|x\| = \sup_{t\geq t^*} |x(t)|.$$

Let

$$S = \{x \in Y : 0 \leq x(t) \leq 1,\ t \geq t^*\}.$$

Clearly, S is a closed and convex subspace of Y. For $\rho = \max\{\sigma, \tau\}$ and $y \in S$, we define

$$Tx(t) = \begin{cases} Tx(t^* + \rho), & t \in [t^*, t^* + \rho] \\ \frac{\alpha}{y(t)}[(r_0 - \varepsilon)x(t-\tau)y(t-\tau) + (q_0 - \varepsilon_1)\int_t^\infty x(s-\sigma)y(s-\sigma)ds & \\ \quad +p_0 \sum_{t\leq\tau_k<\infty} x(\tau_k - \sigma)y(\tau_k - \sigma)], & t \geq t^* + \rho, \end{cases}$$

where $\alpha < 1$. Clearly, $Tx(t) \geq 0$ for $t \geq t^*$ and

$$\begin{aligned} Tx(t) &\leq \frac{\alpha}{y(t)} \left[(r_0 - \varepsilon) y(t-\tau) + (q_0 - \varepsilon_1) \int_t^\infty y(s-\sigma) ds + p_0 \sum_{t \leq \tau_k < \infty} y(\tau_k - \sigma) \right] \\ &\leq \alpha < 1 \end{aligned}$$

implies that $Tx \in S$ and $T : S \longrightarrow S$. For $x_1, x_2 \in S$,

$$\begin{aligned} |Tx_1(t) - Tx_2(t)| \leq & \frac{\alpha}{y(t)} [(r_0 - \varepsilon) y(t-\tau) |x_1(t-\tau) - x_2(t-\tau)| \\ & + (q_0 - \varepsilon_1) \int_t^\infty y(s-\sigma) |x_1(s-\sigma) - x_2(s-\sigma)| ds \\ & + p_0 \sum_{t \leq \tau_k < \infty} y(\tau_k - \sigma) |x_1(\tau_k - \sigma) - x_2(\tau_k - \sigma)|] \\ \leq & \frac{\alpha \|x_1 - x_2\|}{y(t)} \left[(r_0 - \varepsilon) y(t-\tau) + (q_0 - \varepsilon_1) \int_t^\infty y(s-\sigma) ds \right. \\ & \left. + p_0 \sum_{t \leq \tau_k < \infty} y(\tau_k - \sigma) \right] \\ \leq & \alpha \|x_1 - x_2\| \end{aligned}$$

implies that T is contraction. By the Banach's fixed point theorem, T has a unique fixed point in $[0, 1]$. Hence,

$$x(t) = \begin{cases} Tx(t^* + \rho), & t \in [t^*, t^* + \rho] \\ \frac{\alpha}{y(t)} [(r_0 - \varepsilon) x(t-\tau) y(t-\tau) + (q_0 - \varepsilon_1) \int_t^\infty x(s-\sigma) y(s-\sigma) ds & \\ \quad + p_0 \sum_{t \leq \tau_k < \infty} x(\tau_k - \sigma) y(\tau_k - \sigma)], & t \geq t^* + \rho. \end{cases}$$

Setting $w(t) = x(t)y(t)$, for $t \geq t^* + \rho$ we obtain

$$\begin{aligned} &(w(t) - \alpha(r_0 - \varepsilon) w(t-\tau))' + \alpha(q_0 - \varepsilon_1) w(t-\sigma) = 0, \quad t \neq \tau_k,\ k \in \mathbb{N} \\ &\Delta(w(\tau_k) - \alpha(r_0 - \varepsilon) w(\tau_k - \tau)) + p_0\, \alpha\, w(\tau_k - \sigma)) = 0, \quad k \in \mathbb{N}, \end{aligned} \tag{3.6}$$

that is, $w(t)$ is a positive solution of (3.6) whose characteristic equation is given by

$$-\lambda \left[1 - \frac{p_0 \alpha}{\alpha(q_0 - \varepsilon_1)} \lambda \right]^L + \alpha(r_0 - \varepsilon) \lambda e^{\lambda \tau} \left[1 - \frac{p_0 \alpha}{\alpha(q_0 - \varepsilon_1)} \lambda \right]^{L-s} + \alpha(q_0 - \varepsilon_1) e^{\lambda \sigma} = 0.$$

From Theorem 2.2, it follows that $w(t)$ is the positive solution of (3.6) if and only if

$$\lambda < \frac{\alpha(q_0 - \varepsilon_1)}{p_0 \alpha} = \frac{q_0 - \varepsilon_1}{p_0} < \frac{q_0}{p_0}$$

which then implies that (3.3) has a real root in $\left[-\frac{q_0}{p_0}, \frac{q_0}{p_0} \right]$ due to Theorem 2.12, a contradiction. This completes the proof of the theorem. $\square$

Example 3.2. Consider the system of equations

$$(y(t) - r(t)g(y(t-2)))' + q(t)f(y(t-1)) = 0, \quad t \neq \tau_k,\ t > 2,\ k \in \mathbb{N}$$
$$\Delta(y(\tau_k) - r(\tau_k)g(y(\tau_k - 2))) + p(\tau_k)f(y(\tau_k - 1)) = 0, \quad k \in \mathbb{N}, \tag{3.7}$$

where $r(t) = 0.172123227 + e^{-t}$, $q(t) = 0.1 + e^{-t}$, $p(t) = 0.2(2 + sint)$, $\tau_k = k, k \in \mathbb{N}$, and $g(u) = (2 - e^{-|u|})u = f(u)$. The limiting equation for (3.7) is given by

$$(x(t) - r_0 x(t-2))' + q_0 x(t-1) = 0, \quad t \neq \tau_k,\ k \in \mathbb{N}$$
$$\Delta(x(\tau_k) - r_0 x(\tau_k - 2)) + p_0 x(\tau_k - 1) = 0, \quad k \in \mathbb{N}, \tag{3.8}$$

where $r_0 = 0.172123227$, $q_0 = 0.1$ and $p_0 = 0.2$. If we choose $L = 5$ and $s = 6$, then from the characteristic equation of (3.8), it follows that $A = -1$, $\lambda = 1$ and

$$x(t) = e^{-t}(-1)^{i(2,t)}$$

is an oscillatory solution of (3.7). Hence, by Theorem 3.1, (3.7) admits an oscillatory solution.

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