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ON FUZZY FRACTIONAL ORDER DERIVATIVES AND DARBOUX PROBLEM FOR IMPLICIT DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, the Banach contraction principle and a fixed point theorem for absolute retract spaces are used to investigate the existence of fuzzy solutions for Caputo fractional implicit differential equations.

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1. INTRODUCTION

Fractional calculus is a generalization of differentiation and integration to an arbitrary order. The first work devoted exclusively to the subject of fractional calculus is the book by Oldham and Spanier [17]. A rigorous study of fractional calculus can be found in [20]. Fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, economics, and science. We can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [8, 10, 11, 12, 18, 21]. In recent years, there has been a significant development in fractional differential equations. One can see the monographs of Abbas *et al.* [1, 2], Kilbas *et al.* [15], Lakshmikantham *et al.* [16], and the references therein. Agarwal *et al.* [5] proposed the concept of solution for fractional differential equation with uncertainty. They considered the Riemman-Liouville's differentiability to solve fuzzy fractional differential equations; which is a combination of the Hukuhara difference and Riemman-Liouville's derivative; see also Arshad and Lupulescu [7] and the references therein.

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In this paper, we investigate the solution of Caputo's fuzzy fractional differential equations. This paper is organized as follow. In Section 2 we recall some basic knowledge of fuzzy calculus and fractional calculus. In Section 3 several basic concepts and properties of fuzzy fractional calculus are presented. We use the fixed point approach. For an application of the above cited approach to fuzzy differential equations see [6].

Our aim in this paper is to study the existence of solution for fuzzy partial hyperbolic differential equations involving Caputo derivatives. The Banach contraction principle and a fixed point theorem for absolute retract spaces are used to investigate the existence of fuzzy solutions for the fractional differential equation

$$({}^cD_0^q u)(x,y) = f(x,y,u(x,y),({}^cD_0^q u)(x,y)), \text{ if } (x,y) \in J := [0,a] \times [0,b], \tag{1.1}$$

$$\begin{cases} u(x,0) = \varphi(x), & x \in [0,a], \\ u(0,y) = \psi(y), & y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{1.2}$$

where $a, b > 0$, ${}^cD_0^q$ is the Caputo's fractional derivative of order $q = (q_1, q_2) \in (0,1] \times (0,1]$, $f : J \times E^n \times E^n \to E^n$ is a given continuous function, $\varphi : [0,a] \to E^n, \psi : [0,b] \to E^n$ are given absolutely continuous functions with $\psi(0) = \varphi(0)$.

We present two results for the problem (1.1)–(1.2); the first one is based on the Banach contraction principle and the second one on a fixed point theorem for absolute retract spaces.

To our knowledge, fuzzy solutions for the problem (1.1)–(1.2) have not been considered yet. So the present paper initiates the concept of fuzzy solutions for implicit fractional differential equations.

2. PRELIMINARIES

We introduce notations, definitions, and preliminary facts which are used throughout this paper.

Definition 2.1. Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function $A : X \to [0,1]$ and A(x), called the membership function of fuzzy set A, is interpreted as the degree of membership of element x in fuzzy set A for each $x \in$ X.

The value zero is used to represent complete non-membership, the value one is used to represent complete membership and values between them are used to represent intermediate degrees of membership. Let $P_k(\mathbb{R}^n)$ denote the collection of all nonempty compact convex subsets of $\mathbb{R}^n$ and define the addition and scalar multiplication in $P_k(\mathbb{R}^n)$ as usual. Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorff metric

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\}$$

where $d(b,A) = \inf\{d(b,a) : a \in A\}$. It is clear that $(P_k(\mathbb{R}^n), H_d)$ is a complete metric space [13].

Denote by

$$E^n = \{y : \mathbb{R}^n \to [0,1] \text{ such that they satisfy } (i) \text{ to } (iv) \text{ mentioned below}\},$$

(i) y is normal, that is there exists an $x_0 \in \mathbb{R}^n$ such that $y(x_0) = 1$;
(ii) y is fuzzy convex, that is for $x, z \in \mathbb{R}^n$ and $0 < \lambda \leq 1$,

$$y(\lambda x + (1-\lambda)z) \geq \min[y(x), y(z)];$$

(iii) y is upper semi-continuous;
(iv) $[y]^0 = \overline{\{x \in \mathbb{R}^n : y(x) > 0\}}$ is compact.

For $0 < \alpha \leq 1$, we denote $[y]^\alpha = \{x \in \mathbb{R}^n : y(x) \geq \alpha\}$. Then from (i) to (iv), it follows that the $\alpha-$level sets $[y]^\alpha \in P_k(\mathbb{R}^n)$.

We define the supremum metric D on E^n by

$$D(u,\overline{u}) = \sup_{0<\gamma\leq 1} H_d([u]^\gamma, [\overline{u}]^\gamma)$$

for all $u, \overline{u} \in E^n$. (E^n, D) is a complete metric space [19].

The supremum metric H_1 on the space of continuous fuzzy-valued functions from J into E^n denoted by $C(J,E^n)$ is defined by

$$H_1(u,\overline{u}) = \sup_{(x,y)\in J} D(u(x,y), \overline{u}(x,y)).$$

$(C(J,E^n), H_1)$ is a complete metric space.

We define $\hat{0} \in E^n$ as $\hat{0}(x) = 1$ if $x = 0$ and $\hat{0}(x) = 0$ if $x \neq 0$.

Proposition 2.2. *If $x \in E^n$ then the following properties hold:*

(i) $[u]^\beta \subset [u]^\alpha$ *if* $0 \leq \beta \leq \alpha \leq 1$*;*
(ii) *if* $\{\alpha_n\} \subset [0,1]$ *is a nondecreasing sequence that converges to* $\hat{\alpha}$ *, then*

$$[u]^\alpha = \bigcap_{n\geq 1} [u]^{\alpha_n}.$$

Conversely, if $A_\alpha = \{[u_1^\alpha, u_2^\alpha]; \alpha \in [0,1]\}$ *is a family of closed real intervals satisfying (i) and (ii), then* $\{A_\alpha\}$ *defines a fuzzy number* $u \in E^n$ *such that* $[u]^\alpha = A_\alpha$*.*

Proposition 2.3. *If $\varphi : [0,a] \to E^n, \psi : [0,b] \to E^n$ are given absolutely continuous functions, then for all level $\alpha \in [0,1]$ we have*

$$[\varphi(x)]^\alpha = [\varphi_1^\alpha(x), \varphi_2^\alpha(x)], \quad 0 \leq \alpha \leq 1,$$

and

$$[\psi(y)]^\alpha = [\psi_1^\alpha(y), \psi_2^\alpha(y)], \quad 0 \leq \alpha \leq 1.$$

Definition 2.4. Let $x, y \in E^n$. If there exists $z \in E^n$ such that $x = y + z$, then z is called the H-difference of x and y and is denoted by $z = x \ominus y$.

We denote by $L^1(T, E^n)$ the space of all fuzzy function $f : I \to E^n$ that are Lebesgue integrable on the bounded interval I. Also, by $AC(J)$ we denote the space of absolutely continuous functions $f : I \to E^n$.

Definition 2.5. Let $f : [a,b] \to E^n$, $x_0 \in (a,b)$ and $\Phi(x) = \frac{1}{\Gamma(1-q)} \int_a^x \frac{f(t)}{(x-t)^q}\, dt$. We say that f is Riemann-Liouville H-differentiable of order $0 \leq q \leq 1$ at x_0, if there exists an element ${}^{RL}D^q_{a^+} f(x_0) \in C([0,a], E^n) \bigcap L^1([0,a], E^n)$, $0 \leq q \leq 1$, such that for all $h > 0$,

(1)

$${}^{RL}D^q_{a^+} f(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0 + h) \ominus \Phi(x_0)}{h} = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0 - h)}{h}$$

or

(2)

$${}^{RL}D^q_{a^+} f(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0 + h)}{-h} = \lim_{h \to 0^+} \frac{\Phi(x_0 - h) \ominus \Phi(x_0)}{-h}.$$

For sake of simplicity, we say that a fuzzy-valued function f is $[1,q]$-differentiable if it is differentiable as in case (1), and is $[2,q]$-differentiable if it is differentiable as in case (2).

Let $x : I \to E^n$ be a fuzzy function; we have

$$[x(t)]^\alpha = [x_1^\alpha(t), x_2^\alpha(t)],\ t \in I,\ \alpha \in [0,1].$$

Let $x \in C([0,a]; E^n) \bigcap L^1([0,a]; E^n)$, and define the fuzzy fractional primitive of order q for x by

$$I^q_{a+} x(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} x(s) ds,\ t \in [0,a],$$

and let

$$[I^q x(t)]^\alpha = \frac{1}{\Gamma(q)} \left[\int_0^t (t-s)^{q-1} x_1^\alpha(s) ds, \int_0^t (t-s)^{q-1} x_2^\alpha(s) ds \right], \quad t \in [0,a].$$

Also, the following properties are obvious:

(i) $I^q(cx)(t) = cI^q x(t)$ for each $c \in E^n$

(ii) $I^q(x+y)(t) = I^q x(t) + I^q y(t)$.

Theorem 2.6 ([4]). *Let $f(t) \in C([0,a], E^n) \bigcap L^1([0,a], E^n)$, $t \in [0,a]$, $0 < q \leq 1$, and $0 \leq \alpha \leq 1$. The Caputo H-differentiable function of fuzzy-valued is defined as following:*

i) *The case ${}^c[(i)-q]$:*

$$ {}^cD^q_{a+} f^\alpha(t) = \left[{}^cD^q_{a+} f^\alpha_1(t), {}^c D^q_{a+} f^\alpha_2(t)\right]. $$

ii) *The case ${}^c[(ii)-q]$*

$$ ({}^cD^q_{a+} f)(t;\alpha) = \left[{}^cD^q_{a+} f^\alpha_2(t), {}^c D^q_{a+} f^\alpha_1(t)\right], $$

where

$$ \begin{aligned} {}^cD^q_{a+} f^\alpha_1(t) &= {}^{RL}D^q_{a+}\left[f^\alpha_1(t) - \sum_{k=0}^{n-1} \frac{x^k}{k!} f_1^{\alpha(k)}(a)\right](x) \\ &= \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-q-1} \left[f^\alpha_1(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f_1^{\alpha(k)}(a)\right] dt, \end{aligned} $$

and

$$ \begin{aligned} {}^cD^q_{a+} f^\alpha_2(t) &= {}^{RL}D^q_{a+}\left[f^\alpha_2(t) - \sum_{k=0}^{n-1} \frac{x^k}{k!} f_2^{\alpha(k)}(a)\right](x) \\ &= \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-q-1} \left[f^\alpha_2(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f_2^{\alpha(k)}(a)\right] dt. \end{aligned} $$

Theorem 2.7 ([14]). *Let $x : [0,a] \to E^n$ be a Caputo fractional differentiable function and let*

$$ [u(t)]^\alpha = [u^\alpha_1(t), u^\alpha_2(t)]. $$

Then the boundary function $u^\alpha_1(t)$, $u^\alpha_2(t)$ are Caputo differentiable and:

1) *Case ${}^c[(i)-q]$, we have*

$$ [{}^cD^q u(t)]^\alpha = [{}^cD^q u^\alpha_1(t), {}^c D^q u^\alpha_2(t)]. $$

2) *Case ${}^c[(ii)-q]$, we have*

$$ [{}^cD^q u(t)]^\alpha = [{}^cD^q u^\alpha_2(t), {}^c D^q u^\alpha_1(t)]. $$

Definition 2.8 ([22]). Let $q = (q_1, q_2) \in (0,\infty) \times (0,\infty)$, $\theta = (0,0)$ and $u \in L^1(J)$. The left-sided mixed Riemann-Liouville integral of order q of u is defined by

$$ (I^q_\theta u)(x,y) = \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} u(s,t)\,dt\,ds. $$

In particular,

$$ (I^q_\theta u)(x,y) = u(x,y), \quad (I^\sigma_\theta u)(x,y) = \int_0^x \int_0^y u(s,t)\,dt\,ds $$

for almost all $(x,y) \in J$, where $\sigma = (1,1)$.

For instance, $I_\theta^q u$ exists for all $q_1, q_2 \in (0, \infty)$, if $u \in L^1(J)$. Note also that if $u \in C(J)$, then $(I_\theta^q u) \in C(J)$, and moreover

$$(I_\theta^q u)(x, 0) = (I_\theta^q u)(0, y) = 0, \quad x \in [0, a] \; y \in [0, b].$$

Definition 2.9. A space Z is called an absolute retract (written $Z \in AR$) if Z is metrizable and for any metrizable space W and any embedding $h : Z \to W$ the set $h(Z)$ is a retract of W.

Theorem 2.10 ([9]). *Let $X \in AR$ and $F : X \to X$ a continuous and completely continuous map. Then F has a fixed point.*

3. THE MAIN RESULTS

Let us start by defining what we mean by a solution of the problem (1.1)–(1.2).

Definition 3.1. A function $u \in C(J)$ such that $u(x, y)$, $D_{(0,x)}^{q_1} u(x, y)$, $D_{(0,y)}^{q_2} u(x, y)$, and $D_{(0,0)}^{q} u(x, y)$ are continuous for $(x, y) \in J$ and $I_{(0,0)}^{1-q} u(x, y) \in AC(J)$ is said to be a solution of (1.1)–(1.2) if u satisfies equation (1.1) and conditions (1.2) on J.

For the existence of solutions for the problem (1.1)–(1.2) we need the following lemma.

Lemma 3.2 ([3]). *Let $f(x, y, u, z) : J \times E^n \times E^n \to E^n$ be continuous. Then problem* (1.1)–(1.2) *is equivalent to the problem*

$$g(x, y) = f(x, y, \mu(x, y) + I_0^q g(x, y), g(x, y)),$$

and if $g \in C(J)$ is a solution of this equation, then $u(x, y) = \mu(x, y) + I_0^q g(x, y)$, where

$$\mu(x, y) = \varphi(x) + \psi(y) - \varphi(0).$$

Next, we present conditions for the existence and uniqueness of a solution of problem (1.1)–(1.2).

Theorem 3.3. *Assume that the following hypotheses hold:*

(H_1) *$f : J \times E^n \times E^n \to E^n$ is a continuous function;*
(H_2) *there exist constants $k > 0$, and $0 < l < 1$ such that*

$D(f(x, y, u, z); f(x, y, v, w)) \leq kD(u; v) + lD(z; w)$, *for any $u,\ v \in E^n$ and $(x, y) \in J$.*

If

$$\frac{k a^{q_1} b^{q_2}}{(1 - l)\Gamma(q_1 + 1)\Gamma(q_2 + 1)} < 1, \tag{3.1}$$

then there exists a unique solution to IVP (1.1)–(1.2) *on J.*

Proof. Transform the problem (1.1)–(1.2) into a fixed point problem. Consider the operator $N : C(J) \to C(J)$ defined by,

$$N(u)(x,y) = \mu(x,y) + I^q_{(0,0)} g(x,y),$$

where $g \in C(J)$ is such that

$$g(x,y) = f(x,y,u(x,y),g(x,y)).$$

By Lemma 3.2, the problem of finding the solutions of the IVP (1.1)–(1.2) is reduced to finding the solutions of the operator equation $N(u) = u$. To apply Theorem 2.7, we proceed as follows.

Case ${}^c[i-q]$ **H-differentiability**: The problem (1.1)–(1.2) is equivalent to the fractional differential system

$$\begin{cases} ({}^cD^q_0 u^\alpha_1)(x,y) = f_\alpha(x,y,u^\alpha_1(x,y),u^\alpha_2(x,y)), & \text{if } (x,y) \in J, \\ u^\alpha_1(x,0) = \varphi^\alpha_1(x),\ u^\alpha_1(0,y) = \psi^\alpha_1(y), & \text{if } x \in [0,a],\ y \in [0,b], \end{cases} \tag{3.2}$$

and

$$\begin{cases} ({}^cD^q_0 u^\alpha_2)(x,y) = g_\alpha(x,y,u^\alpha_1(x,y),u^\alpha_2(x,y)), & \text{if } (x,y) \in J, \\ u^\alpha_2(x,0) = \varphi^\alpha_2(x),\ u^\alpha_2(0,y) = \psi^\alpha_2(y), & \text{if } x \in [0,a],\ y \in [0,b]. \end{cases} \tag{3.3}$$

We define the operator N by

$$N(v^\alpha_1)(x,y) = \mu^\alpha_1(x,y) + I^q_{(0,0)} g(x,y)$$

with

$$\mu^\alpha_1(x,y) = \varphi^\alpha_1(x) + \psi^\alpha_1(y) - \varphi_1(0).$$

Let $v, w \in C(J)$. Then, for $(x,y) \in J$, we have

$$\begin{aligned} D(N(v^\alpha_1)(x,y); N(w^\alpha_1)(x,y)) \ \leq\ & \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} \\ & \times D(g(s,t),h(s,t))dt\,ds, \end{aligned} \tag{3.4}$$

where $g, h \in C(J)$ such that

$$g(x,y) = f(x,y,v(x,y),g(x,y))$$

and

$$h(x,y) = f(x,y,w(x,y),h(x,y)).$$

By (H_2), we get

$$D(g(x,y),h(x,y)) \leq kD(v(x,y);w(x,y)) + lD(g(x,y),h(x,y)).$$

Then,

$$D(g(x,y),h(x,y)) \leq \frac{k}{1-l} D(v(x,y);w(x,y)).$$

Thus, (3.4) implies that

$D(N(v^\alpha_1)(x,y); N(w^\alpha_1)(x,y)) \leq$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y D(g(x,y),h(x,y))(x-s)^{q_1-1}(y-t)^{q_2-1}dt\,ds\\
&\leq \frac{k}{(1-l)\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1}D(v_1^\alpha(s,t);w_1^\alpha(s,t))dt\,ds\\
&\leq \frac{k}{(1-l)\Gamma(q_1)\Gamma(q_2)}H_1(v_1^\alpha;w_1^\alpha)\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1}dt\,ds\\
&\leq \frac{ka^{q_1}b^{q_2}}{(1-l)\Gamma(q_1+1)\Gamma(q_2+1)}H_1(v_1^\alpha;w_1^\alpha).
\end{aligned}$$

Consequently,

$$H_1(N(v_1^\alpha);N(w_1^\alpha)) \leq \frac{ka^{q_1}b^{q_2}}{(1-l)\Gamma(q_1+1)\Gamma(q_2+1)}H_1(v_1^\alpha;w_1^\alpha).$$

By (3.1), N is a contraction, and hence N has a unique fixed point by Banach's contraction principle.

We now transform the problem (3.3) into fixed a point problem; consider the operator N such that

$$N(v_2^\alpha)(x,y) = \mu_2^\alpha(x,y) + I^q_{(0,0)}g(x,y)$$

with

$$\mu_2^\alpha(x,y) = \varphi_2^\alpha(x) + \psi_2^\alpha(y) - \varphi_2(0).$$

The fixed point of the operator N is a solution of the problem (3.3). So in this case there is a unique solution to the problem (1.1)–(1.2).

Case $^c[ii-q]$ **H-differentiability**: The problem (1.1)–(1.2) is equivalent to the fractional differential system

$$\begin{cases} ({}^cD_0^q u_1^\alpha)(x,y) = g_\alpha(x,y,u_1^\alpha(x,y),u_2^\alpha(x,y)), & \text{if } (x,y)\in J,\\ u_1^\alpha(x,0)=\varphi_1^\alpha(x),\ u_1^\alpha(0,y)=\psi_1^\alpha(y), & \text{if } x\in[0,a],\ y\in[0,b], \end{cases} \tag{3.5}$$

and

$$\begin{cases} ({}^cD_0^q u_2^\alpha)(x,y) = f_\alpha(x,y,u_1^\alpha(x,y),u_2^\alpha(x,y)), & \text{if } (x,y)\in J,\\ u_2^\alpha(x,0)=\varphi_2^\alpha(x),\ u_2^\alpha(0,y)=\psi_2^\alpha(y), & \text{if } x\in[0,a],\ y\in[0,b]. \end{cases} \tag{3.6}$$

To transform the problem (3.5) into fixed point problem, consider the operator

$$N(u_1^\alpha)(x,y) = \mu_1^\alpha(x,y) \ominus I^q_{(0,0)}g_\alpha(s,t)$$

with

$$\mu_1^\alpha(x,y) = \varphi_1^\alpha(x) + \psi_1^\alpha(y).$$

By the same technique, we can prove that there exists at least one solution for the problem (1.1)–(1.2). This completes the proof of the theorem. □

Our second result in this section is based on a fixed point theorem for absolute retract spaces.

Theorem 3.4. *Assume that* (H_1) *and the following conditions hold:*

(H_3) *There exist* $A, B, C \in C(J, \mathbb{R}_+)$ *with* $\|C\|_\infty < 1$, *such that*

$$D(f(x,y,u(x,y),v(x,y));\hat{0}) \le A(x,y) + B(x,y)D(u(x,y);\hat{0}) + C(x,y)D(v(x,y);\hat{0});$$

(H_4) *For all* $(x,y) \in J$, *the set*

$$\left\{\mu(x,y) + I^q_{(0,0)} g(x,y),\ u \in \Theta\right\}$$

is a totally bounded subset of E^n, *where*

$$\Theta = \left\{u \in C(J,E^n):\ D(u(x,y),\hat{0}) \le M,\ (x,y) \in J\right\},$$

with M *a suitable positive constant,*

$$g(x,y) = f(x,y,u(x,y),g(x,y)),$$

and

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

If

$$\frac{\|B\|_\infty a^{q_1} b^{q_2}}{(1-\|C\|_\infty))\Gamma(q_1)\Gamma(q_2)} < 1,$$

then the IVP (1.1)–(1.2) *has at least one solution on* J.

Proof. It is clear that the solutions of the problem (1.1)–(1.2) are fixed points of the operator

$$N : C(J,E^n) \to C(J,E^n)$$

defined by:

$$N(u)(x,y) = \mu(x,y) + I^q_{(0,0)} g(x,y)$$

We make

$$\Theta = \{u \in C(J,E^n) \text{ and } D(u(x,y),\hat{0}) \le M,\ (x,y) \in J\}.$$

We see that Θ is a convex subset of the Banach space $C(J,E^n)$, so in particular Θ is an absolute retract.

Case $^c[i-q]$ **H-differentiability**: The problem (1.1)–(1.2) is equivalent to the fractional differential system

$$\begin{cases} ({}^cD^q_0 u^\alpha_1)(x,y) = g_\alpha(x,y,u^\alpha_1(x,y),u^\alpha_2(x,y)), & \text{if } (x,y) \in J,\\ u^\alpha_1(x,0) = \varphi^\alpha_1(x),\ u^\alpha_1(0,y) = \psi^\alpha_1(y), & \text{if } x \in [0,a],\ y \in [0,b], \end{cases} \tag{3.7}$$

and

$$\begin{cases} ({}^cD^q_0 u^\alpha_2)(x,y) = f_\alpha(x,y,u^\alpha_1(x,y),u^\alpha_2(x,y)), & \text{if } (x,y) \in J,\\ u^\alpha_2(x,0) = \varphi^\alpha_2(x),\ u^\alpha_2(0,y) = \psi^\alpha_2(y), & \text{if } x \in [0,a],\ y \in [0,b]. \end{cases} \tag{3.8}$$

We want to transform the problem (3.7) into a fixed point problem. Consider the operator $N : C(J,E^n) \to C(J,E^n)$ defined by

$$N(u^\alpha_1)(x,y) = \mu^\alpha_1(x,y) + I^q_{(0,0)} f_\alpha(x,y)$$

with

$$\mu_1^\alpha(x,y)=\varphi_1^\alpha(x)+\psi_1^\alpha(y)-\varphi_1(0).$$

Clearly the fixed points of the operator $N:C(J,E^n)\to C(J,E^n)$ are solutions of the problem (1.2). We shall show that $N:C(J,E^n)\to C(J,E^n)$ is continuous and completely continuous. The proof will be given in several steps.

Step 1. $N(\Theta)\subset\Theta$.

Let $u\in\Theta$ and $(x,y)\in J$. By (H_3), we have

$$\begin{aligned} D(g(x,y),\hat{0}) &\leq A(x,y)+B(x,y)D(u;\hat{0})+C(x,y)D(g(x,y);\hat{0})\\ &\leq \|A\|_\infty+\|B\|_\infty D(u(x,y);\hat{0})+\|C\|_\infty D(g(x,y);\hat{0}). \end{aligned}$$

Then

$$D(g(x,y),\hat{0})\leq \frac{\|A\|_\infty+\|B\|_\infty M}{1-\|C\|_\infty}.$$

Thus,

$$\begin{aligned} D(Nu_1^\alpha((x,y)),\hat{0}) &\leq D(\mu_1^\alpha(x,y),\hat{0})\\ &\quad + D\left(\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1}f_\alpha(s,t)dt\,ds,\hat{0}\right)\\ &\leq D(\mu_1^\alpha(x,y),\hat{0})\\ &\quad +\frac{1}{\Gamma(q_1)\Gamma(q_2)}\left(\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1}D(f_\alpha(s,t),\hat{0})dt\,ds\right)\\ &\leq D(\mu_1^\alpha(x,y),\hat{0})+\frac{\|A\|_\infty+M\|B\|_\infty}{(1-\|C\|_\infty)\Gamma(q_1)\Gamma(q_2)}a^{q_1}b^{q_2}\leq M. \end{aligned}$$

Step 2: *N is continuous.*

Let the sequence $[U_n(x;y)]^\alpha=[U_{n,1}^\alpha(x;y),U_{n,2}^\alpha(x;y)]$ in Θ be such that $U_{n,1}^\alpha\to U_1^\alpha$ is an element of Θ in $C(J,E^n)$ and $U_{n,2}^\alpha\to U_2^\alpha$ is an element of Θ in $C(J,E^n)$. Let $\{u_n\}$ be a sequence such that $u_n\to u$ in $C(J,E^n)$. Let $\eta>0$ be such that $H_1(u_n;\widehat{0})\leq\eta$. Then,

$$\begin{aligned} &D(N(u_{n,1}^\alpha)(x,y);N(u_1^\alpha)(x,y))\leq\\ &\leq\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y(x-s)^{q_1-1}(y-t)^{q_2-1}D(f(s,t,u_{n,1}^\alpha(s,t));f(s,t,u_1^\alpha(s,t)))dt\,ds\\ &\leq\frac{\|A\|_\infty+M\|B\|_\infty}{(1-\|C\|_\infty)q_1q_2\Gamma(q_1)\Gamma(q_2)}a^{q_1}b^{q_2}H_1(f(.,.,u_{n,1}^\alpha(.,.));f(.,.,u_1^\alpha(.,.))). \end{aligned}$$

Since f is a continuous function, we have

$$\begin{aligned} H_1(N(u_{n,1}^\alpha);N(u_1^\alpha)) &\leq\frac{\|A\|_\infty+M\|B\|_\infty}{\Gamma(q_1+1)\Gamma(q_2+1)}a^{q_1}b^{q_2}H_1(f(.,.,u_{n,1}^\alpha)(.,.);f(.,.,u_1^\alpha(.,.))\\ &\to 0 \text{ as } n\to\infty. \end{aligned}$$

Thus, N is a continuous.

Step 3: *The operator N is equicontinuous.*

Let $(x_1,y_1),(x_2,y_2)\in(0,a]\times(0,b]$, $x_1<x_2$, $y_1<y_2$, $u\in\Theta$. Then,

$$\begin{aligned}
&D(N(u_1^\alpha)(x_2,y_2);N(u_1^\alpha)(x_1,y_1))\le D(\mu(x_1,y_1);\mu(x_2,y_2))\\
&\quad+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^{x_1}\int_0^{y_1}[(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}-(x_1-s)^{q_1-1}(y_1-t)^{q_2-1}]\\
&\quad\times D(f(s,t);\widehat{0})dt\,ds+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}\\
&\quad D\left(f(s,t);\widehat{0}\right)dt\,ds\\
&\quad+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}D(f(s,t);\widehat{0})dt\,ds\\
&\quad+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}D(f(s,t);\widehat{0})dt\,ds\\
&\quad\le D(\mu(x_1,y_1);\mu(x_2,y_2))+\frac{\|A\|_\infty+\|B\|_\infty M}{(1-\|C\|_\infty)\Gamma(q_1)\Gamma(q_2)}\\
&\quad\times\int_0^{x_1}\int_0^{y_1}[(x_1-s)^{q_1-1}(y_1-t)^{q_2-1}-(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}]dt\,ds\\
&\quad+\frac{\|A\|_\infty+\|B\|_\infty M}{(1-\|C\|_\infty)\Gamma(q_1)\Gamma(q_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}dt\,ds\\
&\quad+\frac{\|A\|_\infty+\|B\|_\infty M}{(1-\|C\|_\infty)\Gamma(q_1)\Gamma(q_2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}dt\,ds\\
&\quad+\frac{\|A\|_\infty+\|B\|_\infty M}{(1-\|C\|_\infty)\Gamma(q_1)\Gamma(q_2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}dt\,ds\\
&\quad\le D(\mu(x_1,y_1);\mu(x_2,y_2))\\
&\quad+\frac{\|A\|_\infty+\|B\|_\infty M}{(1-\|C\|_\infty)\Gamma(q_1)\Gamma(q_2)}[2y_2^{q_2}(x_2-x_1)^{q_1}+2x_2^{q_1}(y_2-y_1)^{q_2}\\
&\quad+x_1^{q_1}y_1^{q_2}-x_2^{q_1}y_2^{q_2}-2(x_2-x_1)^{q_1}(y_2-y_1)^{q_2}].
\end{aligned}$$

As $x_1\to x_2$, $y_1\to y_2$ the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that N is continuous and completely continuous. As above, can show that the operator N is a contraction, and so it has a unique fixed point. □

Case $^c[ii-q]$ **H-differentiability**: The problem (1.1)–(1.2) is equivalent to the fractional differential system

$$\begin{cases}({}^cD_0^q u_1^\alpha)(x,y)=g_\alpha(x,y,u_1(x,y)^\alpha,u_2^\alpha(x,y)), & \text{if } (x,y)\in J,\\ u_1^\alpha(x,0)=\varphi_1^\alpha(x),\ u_1^\alpha(0,y)=\psi_1^\alpha(y), & \text{if } x\in[0,a],\ y\in[0,b],\end{cases}\tag{3.9}$$

and

$$\begin{cases}({}^cD_0^q u_2^\alpha)(x,y)=f_\alpha(x,y,u_{1(x,y)}^\alpha,u_2^\alpha(x,y)), & \text{if } (x,y)\in J,\\ u_2^\alpha(x,0)=\varphi_2^\alpha(x),\ u_2^\alpha(0,y)=\psi_2(y), & \text{if } x\in[0,a],\ y\in[0,b].\end{cases}\tag{3.10}$$

To transform the problem (3.9) into fixed point problem, consider the operator

$$\begin{aligned} &N(u_1^\alpha)(x,y)\\ &= \mu_1^\alpha(x,y) \ominus \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} g_\alpha(s,t,u_1^\alpha(s,t),u_2^\alpha(s,t))dt\,ds. \end{aligned}$$

By the same technique we can prove that there exists at least one solution to the problem (3.9) and finally the IVP(1.1)–(1.2) has at least one solution.

4. AN EXAMPLE

As an application of our results we consider the following fuzzy partial hyperbolic functional differential equations of the form

$$({}^cD_0^q u)(x,y) = \frac{1}{(5e^{x+y+2})(1+D(u(x,y),\hat{0})+D({}^cD_0^q u(x,y),\hat{0}))},\ (x,y)\in[0,1]\times[0,1], \tag{4.1}$$

$$u(x,0)=x,\ x\in[0,1],\ u(0,y)=y^2,\ y\in[0,1]. \tag{4.2}$$

For $(x,y)\in[0,1]\times[0,1]$, set

$$f(x,y,u(x,y),{}^cD_0^q u(x,y)) = \frac{1}{(5e^{x+y+2})(1+D(u(x,y),\hat{0})+D({}^cD_0^q u(x,y),\hat{0}))}.$$

For each $u,\overline{u},v,\overline{v}\in E^1$ and $(x,y)\in[0,1]\times[0,1]$, we have

$$\begin{aligned} &D(f(x,y,u(x,y),v(x,y)); f(x,y,\overline{u}(x,y),\overline{v}(x,y)))\\ &\qquad\le \frac{1}{5e^2}(D(u(x,y);\overline{u}(x,y)) + D(v(x,y);\overline{v}(x,y))). \end{aligned}$$

Hence, condition (H_2) is satisfied with $k=l=\frac{1}{5e^2}$. We shall show that condition (3.1) holds with $a=b=1$. Indeed

$$\frac{ka^{q_1}b^{q_2}}{(1-l)\Gamma(q_1+1)\Gamma(q_2+1)} = \frac{1}{(5e^2-1)\Gamma(q_1+1)\Gamma(q_2+1)} < 1,$$

which is satisfied for each $(q_1,q_2)\in(0,1]\times(0,1]$. Consequently Theorem 3.3 implies that problem (4.1)–(4.2) has a unique solution defined on $[0,1]\times[0,1]$.

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