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LOSSY TRANSMISSION LINES WITH JOSEPHSON JUNCTION – CONTINUOUS GENERALIZED SOLUTIONS

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ABSTRACT. The present paper is devoted to investigation of lossy transmission lines with Josephson junction. Such lines are described by a first order nonlinear hyperbolic system partial differential equations. We formulate a mixed problem for this system with boundary conditions generated by a circuit corresponding to Josephson junction. We present the mixed problem in a suitable operator form and obtain solution on a restricted domain. Then we continue this solution on the whole domain and call it a generalized one.

Key words: transmission lines, Josephson junction, nonlinear hyperbolic system, mixed problem.

AMS (MOS) Subject Classification. 35L50, 47H10, 58C30, 47N70

1. INTRODUCTION

A lot of papers have been devoted to the investigation of lossless transmission lines terminated by linear and nonlinear loads and their applications to RF-circuits (cf. for instance [1]–[14]). The problems for superconducting lossless transmission lines with Josephson junction have been investigated in [15]–[18]. In a recent paper [19], we have considered the mixed problem for such a system (a nonlinear hyperbolic system) reducing it to a fixed point problem of a suitable operator. Here we consider the same problem for a lossy transmission line with Josephson junction. From mathematical point of view, lossy transmission line system with Josephson junction is a nonlinear hyperbolic system plus a relation between Josephson flux and the voltage:

$$\begin{aligned} & C\frac{\partial u(x,t)}{\partial t}+\frac{\partial i(x,t)}{\partial x}+Gu(x,t)+j_0\sin\frac{2\pi\Phi(x,t)}{\Phi_0}=0,\\ & L\frac{\partial i(x,t)}{\partial t}+\frac{\partial u(x,t)}{\partial x}+Ri(x,t)=0, \qquad (1.1)\\ & \frac{\partial \Phi(x,t)}{\partial t}=u(x,t);(x,t)\in\Pi=\left\{(x,t)\in\Pi^2:(x,t)\in[0,\Lambda]\times[0,T]\right\}. \end{aligned}$$

Here $u(x,t)$, $i(x,t)$ and $\Phi(x,t)$ are unknown functions – voltage, current and Josephson flux; L, C, G and R are prescribed specific parameters of the line; $\Lambda > 0$

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is its length; $v = 1/\sqrt{LC}$; j_0 is maximal Josephson current per unit length and $K_J = 1/\Phi_0$ is Josephson constant; $\Phi_0 = \hbar/(2e) = 2.10^{15}\,W/m^2$ is flux induction quant. The lossy transmission line (cf. Fig. 1) is terminated by a circuit at the right-hand end corresponding to Josephson junction. First, we assume the resistive element R_1 (at right end) is a linear one in order to show of how to overcome the difficulty generated by sine nonlinearity. In contrast to lossless case, above system (1.1) is more complicated, but we are able to reduce it to lossless case and then solve it in an analogous way.

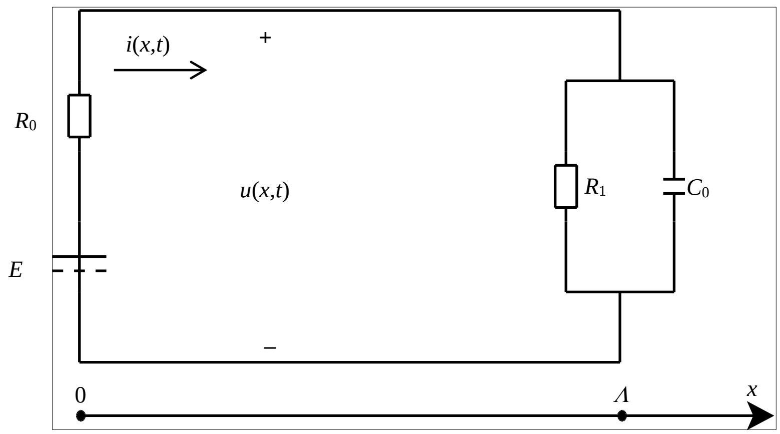

FIGURE 1. Lossy transmission lines with Josephson junction

For (1.1) one can formulate the following mixed (initial-boundary value) problem: to find the unknown functions $u(x,t)$ and $i(x,t)$ in Π satisfying initial conditions

$$u(x,0) = u_0(x),\ i(x,0) = i_0(x),\ x \in [0,\Lambda] \tag{1.2}$$

and boundary conditions

$$E(t) - u(0,t) - R_0 i(0,t) = 0,\ C_0\frac{du(\Lambda,t)}{dt} = i(\Lambda,t) - \frac{1}{R_1}u(\Lambda,t), t \in [0,T]. \tag{1.3}$$

Here, $i_0(x), u_0(x)$ are prescribed initial functions – the current and voltage at the initial instant, $E(t)$ is a prescribed source function, R_0 is its resistance, R_1 and C_0 are specific parameters (positive) of the elements of the circuit.

We demonstrate of how to overcome the difficulty caused by the sine-function. For our fixed point method, sine-function is not a "bad" nonlinearity. We use an operator presentation of the mixed problem for hyperbolic system in a diagonal form (cf. [13]). Choosing a suitable function space, and introducing suitable weighted metrics, we prove existence of generalized continuous solutions of (1.1)–(1.3) by a fixed point method. In order to have a strict contractive operator, we cut the domain Π. On the new obtained domain the operator in question is already contractive and this implies the existence of a unique solution. We obtain a sequence of subdomains whose union is Π. To every subdomain a unique solution corresponds. The obtained sequence is not necessary convergent. That is why, we choose a convergent subsequence by extending some results from [20]. We propose a constructive way of defining of such a sequence and the method can be applied to compact operators.

2. A DIAGONALIZATION OF THE OBTAINED FIRST ORDER PARTIAL DIFFERENTIAL EQUATION SYSTEM

Rewrite system (1.1) in the form

$$\frac{\partial u(x,t)}{\partial t} + \frac{1}{C}\frac{\partial i(x,t)}{\partial x} + \frac{G}{C}u(x,t) = -\frac{j_0}{C}\sin\left(2\pi K_J \int_0^t u(x,s)ds\right),$$

$$\frac{\partial i(x,t)}{\partial t} + \frac{1}{L}\frac{\partial u(x,t)}{\partial x} + \frac{R}{L}i(x,t) = 0,$$

and it takes the matrix form:

$$\begin{bmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial i}{\partial t} \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial i}{\partial x} \end{bmatrix} + \begin{bmatrix} \frac{G}{C} & 0 \\ 0 & \frac{R}{L} \end{bmatrix} \begin{bmatrix} u \\ i \end{bmatrix} = \begin{bmatrix} -\frac{j_0}{C}\sin\left(2\pi K_J \int_0^t u(x,s)ds\right) \\ 0 \end{bmatrix}. \tag{2.1}$$

Denoting by $A = \begin{bmatrix} 0 & \frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}$, $A_1 = \begin{bmatrix} \frac{G}{C} & 0 \\ 0 & \frac{R}{L} \end{bmatrix}$, $U = \begin{bmatrix} u \\ i \end{bmatrix}$, $\frac{\partial U}{\partial t} = \begin{bmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial i}{\partial t} \end{bmatrix}$, $\frac{\partial U}{\partial x} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial i}{\partial x} \end{bmatrix}$, $\Gamma = \begin{bmatrix} -\frac{j_0}{C}\sin\left(2\pi K_J \int_0^t u(x,s)ds\right) \\ 0 \end{bmatrix}$ we rewrite (2.1) in the form

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + A_1 U = \Gamma. \tag{2.2}$$

To transform matrix $A = \begin{bmatrix} 0 & 1/C \\ 1/L & 0 \end{bmatrix}$ in diagonal form we have to solve the characteristic equation: $\begin{vmatrix} -\lambda & 1/C \\ 1/L & -\lambda \end{vmatrix} = 0$ whose roots are $\lambda_1 = \frac{1}{\sqrt{LC}}$, $\lambda_2 = -\frac{1}{\sqrt{LC}}$. We find eigen-vectors as a solution of the systems:

$$\left| \begin{array}{l} -\frac{1}{\sqrt{LC}}\xi_1 + \frac{1}{L}\xi_2 = 0\,, \\ \frac{1}{C}\xi_1 - \frac{1}{\sqrt{LC}}\xi_2 = 0 \end{array} \right. \quad \text{and} \quad \left| \begin{array}{l} \frac{1}{\sqrt{LC}}\xi_1 + \frac{1}{L}\xi_2 = 0\,, \\ \frac{1}{C}\xi_1 + \frac{1}{\sqrt{LC}}\xi_2 = 0\,. \end{array} \right.$$

Eigen-vectors are $\left(\xi_1^{(1)}, \xi_2^{(1)}\right) = \left(\sqrt{C}, \sqrt{L}\right)$, $\left(\xi_1^{(2)}, \xi_2^{(2)}\right) = \left(-\sqrt{C}, \sqrt{L}\right)$.

Denote by H the matrix formed by the eigen-vectors $H = \begin{bmatrix} \sqrt{C} & \sqrt{L} \\ -\sqrt{C} & \sqrt{L} \end{bmatrix}$. Its inverse is $H^{-1} = \begin{bmatrix} \frac{1}{2\sqrt{C}} & -\frac{1}{2\sqrt{C}} \\ \frac{1}{2\sqrt{L}} & \frac{1}{2\sqrt{L}} \end{bmatrix}$. Then $A^{\text{can}} = HAH^{-1} = \begin{bmatrix} \frac{1}{\sqrt{LC}} & 0 \\ 0 & -\frac{1}{\sqrt{LC}} \end{bmatrix}$.

Introduce new variables

$$Z=\left[\begin{array}{c}V(x,t)\\ I(x,t)\end{array}\right],\ H=\left[\begin{array}{cc}\sqrt{C} & \sqrt{L}\\ -\sqrt{C} & \sqrt{L}\end{array}\right],\ U=\left[\begin{array}{c}u(x,t)\\ i(x,t)\end{array}\right],\ Z=HU,\ (U=H^{-1}Z)$$

or $\left|\begin{array}{l}V(x,t)=\sqrt{C}\,u(x,t)+\sqrt{L}\,i(x,t)\\ I(x,t)=-\sqrt{C}\,u(x,t)+\sqrt{L}\,i(x,t)\end{array}\right.$ and $\left|\begin{array}{l}u(x,t)=\frac{V(x,t)}{2\sqrt{C}}-\frac{I(x,t)}{2\sqrt{C}}\\ i(x,t)=\frac{V(x,t)}{2\sqrt{L}}+\frac{I(x,t)}{2\sqrt{L}}\end{array}\right.\Rightarrow$

$$\Rightarrow\left|\begin{array}{l}u(0,t)=\frac{V(0,t)}{2\sqrt{C}}-\frac{I(0,t)}{2\sqrt{C}}\\ i(0,t)=\frac{V(0,t)}{2\sqrt{L}}+\frac{I(0,t)}{2\sqrt{L}}\end{array}\right.\text{ and }\left|\begin{array}{l}u(\Lambda,t)=\frac{V(\Lambda,t)}{2\sqrt{C}}-\frac{I(\Lambda,t)}{2\sqrt{C}}\\ i(\Lambda,t)=\frac{V(\Lambda,t)}{2\sqrt{L}}+\frac{I(\Lambda,t)}{2\sqrt{L}}.\end{array}\right. \tag{2.3}$$

Substituting $U=H^{-1}Z$ in (2.2) we obtain

$$\frac{\partial\left(H^{-1}Z\right)}{\partial t}+A\frac{\partial\left(H^{-1}Z\right)}{\partial x}+A_1\left(H^{-1}Z\right)=\Gamma.$$

Since H^{-1} is a constant matrix we obtain:

$$H^{-1}\frac{\partial Z}{\partial t}+\left(AH^{-1}\right)\frac{\partial Z}{\partial x}+\left(A_1H^{-1}\right)Z=\Gamma.$$

After multiplication from the left by H we obtain

$$\frac{\partial Z}{\partial t}+\left(HAH^{-1}\right)\frac{\partial Z}{\partial x}+\left(HA_1H^{-1}\right)Z=H\Gamma, \tag{2.4}$$

where $HA_1H^{-1}=\left[\begin{array}{cc}\sqrt{C} & \sqrt{L}\\ -\sqrt{C} & \sqrt{L}\end{array}\right]\left[\begin{array}{cc}\frac{G}{C} & 0\\ 0 & \frac{R}{L}\end{array}\right]\left[\begin{array}{cc}\frac{1}{2\sqrt{C}} & -\frac{1}{2\sqrt{C}}\\ \frac{1}{2\sqrt{L}} & \frac{1}{2\sqrt{L}}\end{array}\right]=$

$$=\left[\begin{array}{cc}\frac{1}{2}\left(\frac{G}{C}+\frac{R}{L}\right) & \frac{1}{2}\left(-\frac{G}{C}+\frac{R}{L}\right)\\ \frac{1}{2}\left(-\frac{G}{C}+\frac{R}{L}\right) & \frac{1}{2}\left(\frac{G}{C}+\frac{R}{L}\right)\end{array}\right]$$

and

$$H\Gamma=\left[\begin{array}{cc}\sqrt{C} & \sqrt{L}\\ -\sqrt{C} & \sqrt{L}\end{array}\right].\left[\begin{array}{c}-\frac{j_0}{C}\sin\left(2\pi K_J\int_0^t u(x,s)ds\right)\\ 0\end{array}\right]=$$

$$=\left[\begin{array}{c}-\dfrac{j_0}{\sqrt{C}}\sin\left(2\pi K_J\displaystyle\int_0^t\frac{V(x,s)-I(x,s)}{2\sqrt{C}}ds\right)\\ \frac{j_0}{\sqrt{C}}\sin\left(2\pi K_J\int_0^t\frac{V(x,s)-I(x,s)}{2\sqrt{C}}ds\right)\end{array}\right].$$

Then (2.4) can be rewritten as:

$$\left[\begin{array}{c}\frac{\partial V}{\partial t}\\ \\ \frac{\partial I}{\partial t}\end{array}\right]+\left[\begin{array}{cc}\frac{1}{\sqrt{LC}} & 0\\ 0 & -\frac{1}{\sqrt{LC}}\end{array}\right]\left[\begin{array}{c}\frac{\partial V}{\partial x}\\ \\ \frac{\partial I}{\partial x}\end{array}\right]+\left[\begin{array}{cc}\frac{1}{2}\left(\frac{R}{L}+\frac{G}{C}\right) & \frac{1}{2}\left(\frac{R}{L}-\frac{G}{C}\right)\\ \frac{1}{2}\left(\frac{R}{L}-\frac{G}{C}\right) & \frac{1}{2}\left(\frac{R}{L}+\frac{G}{C}\right)\end{array}\right]\left[\begin{array}{c}V\\ I\end{array}\right]=$$

$$=\left[\begin{array}{c}-\frac{j_0}{\sqrt{C}}\sin\left(\pi K_J\int_0^t\frac{V(x,s)-I(x,s)}{\sqrt{C}}ds\right)\\ \frac{j_0}{\sqrt{C}}\sin\left(\pi K_J\int_0^t\frac{V(x,s)-I(x,s)}{\sqrt{C}}ds\right)\end{array}\right],$$

or introducing the notations $\alpha=\frac{1}{2}\left(\frac{R}{L}+\frac{G}{C}\right)$; $h=\frac{1}{2}\left(\frac{R}{L}-\frac{G}{C}\right)$, we get (for $V=V(x,t),I=I(x,t)$):

$$\begin{array}{l}\frac{\partial V}{\partial t}+\frac{1}{\sqrt{LC}}\frac{\partial V}{\partial x}=-\alpha V-hI-\frac{j_0}{\sqrt{C}}\sin\left(\pi K_J\int_0^t\frac{V(x,s)-I(x,s)}{\sqrt{C}}ds\right),\\ \frac{\partial I}{\partial t}-\frac{1}{\sqrt{LC}}\frac{\partial I}{\partial x}=-hV-\alpha I+\frac{j_0}{\sqrt{C}}\sin\left(\pi K_J\int_0^t\frac{V(x,s)-I(x,s)}{\sqrt{C}}ds\right).\end{array} \tag{2.5}$$

We put $V(x,t) = e^{-\alpha t}W(x,t)$ and $I(x,t) = e^{-\alpha t}J(x,t)$ and then system (2.5) becomes (for $W = W(x,t)$, $J = J(x,t)$):

$$\begin{aligned} &\frac{\partial W}{\partial t} + \frac{1}{\sqrt{LC}}\frac{\partial W}{\partial x} = -hJ - \frac{j_0}{\sqrt{C}}e^{\alpha t}\sin\left(\frac{\pi}{\Phi_0}\int_0^t \frac{e^{-\alpha s}W(x,s) - e^{-\alpha s}J(x,s)}{\sqrt{C}}ds\right),\\ &\frac{\partial J}{\partial t} - \frac{1}{\sqrt{LC}}\frac{\partial J}{\partial x} = -hW + \frac{j_0}{\sqrt{C}}e^{\alpha t}\sin\left(\frac{\pi}{\Phi_0}\int_0^t \frac{e^{-\alpha s}W(x,s) - e^{-\alpha s}J(x,s)}{\sqrt{C}}ds\right). \end{aligned} \tag{2.6}$$

It is natural to look for bounded solution of the above system, that is,

$$|W(x,t)| \le W_0 e^{\mu t}, |J(x,t)| \le J_0 e^{\mu t}, t \in [0,T].$$

The new initial conditions we obtain from (1.2) – for $x \in [0,\Lambda]$:

$$\begin{aligned} &W(x,0) = \sqrt{C}u(x,0) + \sqrt{L}i(x,0) = \sqrt{C}u_0(x) + \sqrt{L}i_0(x) = W_0(x),\\ &J(x,0) = -\sqrt{C}u(x,0) + \sqrt{L}i(x,0) = -\sqrt{C}u_0(x) + \sqrt{L}i_0(x) = J_0(x). \end{aligned} \tag{2.7}$$

The new boundary conditions we obtain from substituting (2.3) into (1.3):

$$E(t) - \frac{V(0,t)}{2\sqrt{C}} + \frac{I(0,t)}{2\sqrt{C}} - R_0\frac{V(0,t)}{2\sqrt{L}} - R_0\frac{I(0,t)}{2\sqrt{L}} = 0,$$

$$C_0\frac{d}{dt}\left(\frac{V(\Lambda,t)}{2\sqrt{C}} - \frac{I(\Lambda,t)}{2\sqrt{C}}\right) = \frac{V(\Lambda,t)}{2\sqrt{L}} + \frac{I(\Lambda,t)}{2\sqrt{L}} - \frac{1}{R_1}\frac{V(\Lambda,t)}{2\sqrt{C}} + \frac{1}{R_1}\frac{I(\Lambda,t)}{2\sqrt{C}}, t \in [0,T],$$

or in view of $Z_0 = \sqrt{L/C}$, we obtain

$$\begin{aligned} &V(0,t) = \frac{2\sqrt{C}Z_0}{Z_0+R_0}E(t) + \frac{Z_0-R_0}{Z_0+R_0}I(0,t),\\ &\frac{dI(\Lambda,t)}{dt} = \frac{dV(\Lambda,t)}{dt} - \frac{R_1-Z_0}{C_0Z_0R_1}V(\Lambda,t) - \frac{R_1+Z_0}{C_0Z_0R_1}I(\Lambda,t), t \in [0,T]. \end{aligned}$$

Substituting $V(x,t) = e^{-\alpha t}W(x,t)$, $I(x,t) = e^{-\alpha t}J(x,t)$ in the above equations we obtain

$$\begin{aligned} &W(0,t) = \frac{2\sqrt{C}Z_0}{Z_0+R_0}E(t)\,e^{\alpha t} + \frac{Z_0-R_0}{Z_0+R_0}J(0,t),\\ &\frac{dJ(\Lambda,t)}{dt} - \alpha J(\Lambda,t) = \frac{dW(\Lambda,t)}{dt} - \alpha W(\Lambda,t) - \frac{R_1-Z_0}{R_1Z_0C_0}W(\Lambda,t) - \frac{R_1+Z_0}{R_1Z_0C_0}J(\Lambda,t) \end{aligned}$$

or

$$\begin{aligned} &W(0,t) = \frac{2\sqrt{C}Z_0}{Z_0+R_0}E(t)e^{\alpha t} + \frac{Z_0-R_0}{Z_0+R_0}\,J(0,t),\\ &\frac{dJ(\Lambda,t)}{dt} = \frac{dW(\Lambda,t)}{dt} - \left(\frac{R_1-Z_0}{R_1Z_0C_0} + \alpha\right)W(\Lambda,t) - \left(\frac{R_1+Z_0}{R_1Z_0C_0} - \alpha\right)J(\Lambda,t). \end{aligned}$$

We present the last boundary conditions in an integral form

$$\begin{aligned} &W(0,t) = \frac{2\sqrt{C}\,Z_0}{Z_0+R_0}E(t)e^{\alpha t} + \frac{Z_0-R_0}{Z_0+R_0}\,J(0,t),\\ &J(\Lambda,t) = W(\Lambda,t) - \left(\frac{R_1-Z_0}{R_1Z_0C_0} + \alpha\right)\int_0^t W(\Lambda,s)ds - \left(\frac{R_1+Z_0}{R_1Z_0C_0} - \alpha\right)\int_0^t J(\Lambda,s)ds. \end{aligned}$$

3. AN OPERATOR FORMULATION OF THE MIXED PROBLEM

Let us formulate the mixed problem: to find a solution $(W(x,t), J(x,t))$ of the system

$$\begin{aligned} &\frac{\partial W}{\partial t} + \frac{1}{\sqrt{LC}}\frac{\partial W}{\partial x} = -hJ - \frac{j_0}{\sqrt{C}}e^{\alpha t}\sin\left(\frac{\pi}{\Phi_0}\int_0^t e^{-\alpha s}\frac{W(x,s)-J(x,s)}{\sqrt{C}}ds\right),\\ &\frac{\partial J}{\partial t} - \frac{1}{\sqrt{LC}}\frac{\partial J}{\partial x} = -hW + \frac{j_0}{\sqrt{C}}e^{\alpha t}\sin\left(\frac{\pi}{\Phi_0}\int_0^t e^{-\alpha s}\frac{W(x,s)-J(x,s)}{\sqrt{C}}ds\right) \end{aligned} \tag{3.1}$$

satisfying the initial conditions

$$W(x,0) = W_0(x), J(x,0) = J_0(x), x \in [0, \Lambda] \tag{3.2}$$

and boundary conditions

$$\begin{aligned} &W(0,t) = \gamma\sqrt{C}E(t)e^{\alpha t} + \beta J(0,t), \\ &J(\Lambda,t) = W(\Lambda,t) - (\gamma_1+\alpha)\int_0^t W(\Lambda,s)ds - (\gamma_2-\alpha)\int_0^t J(\Lambda,s)ds, \\ &\qquad\qquad t \in [0, T], \end{aligned} \tag{3.3}$$

where $\gamma = \frac{2Z_0}{Z_0+R_0}; \beta = \frac{Z_0-R_0}{Z_0+R_0}; \gamma_1 = \frac{R_1-Z_0}{R_1Z_0C_0}; \gamma_2 = \frac{R_1+Z_0}{R_1Z_0C_0}$.

Remark 3.1 Since we prove the existence of a continuous generalized solution, we assume that the Conformity Condition **(CC)** is satisfied:

$$W(0,0) = \gamma\sqrt{C}E(0) + \beta J(0,0), J(\Lambda,0) = W(\Lambda,0).$$

Indeed, the following conditions

$$J(0,0) = W(0,0) = 0, E(0) = 0, J_0(\Lambda) = W_0(\Lambda) \tag{CC}$$

implies **(CC)**.

Prior to formulating an operator corresponding to the mixed problem, we consider the Cauchy problem for the characteristics ($v = 1/\sqrt{LC}$) (cf. [19]):

$$\frac{d\xi}{d\tau} = \frac{1}{\sqrt{LC}} = v, \quad \xi(t) = x\ \forall (x,t) \in \Pi \Rightarrow \varphi_W(\tau;x,t) = v\tau + x - vt, \tag{3.4}$$

$$\frac{d\xi}{d\tau} = -\frac{1}{\sqrt{LC}} = -v, \xi(t) = x\,\forall (x,t) \in \Pi \Rightarrow \varphi_J(\tau;x,t) = -v\tau + x + vt. \tag{3.5}$$

The functions $\lambda_W(x,t) = v > 0$ and $\lambda_J(x,t) = -v < 0$ are continuous and imply a uniqueness to the left from t_0 of the solution $x = \varphi_W(t;x_0,t_0)$ for $dx/dt = v, x(t_0) = x_0$ and respectively $x = \varphi_J(t;x_0,t_0)$ for $dx/dt = -v, x(t_0) = x_0$.

Denote by $\chi_W(x,t)$ the smallest value of τ such that the solution $\varphi_W(\tau;x,t) = v\tau + x - vt$ of (3.4) still belongs to Π and respectively the solution $\varphi_J(\tau;x,t) = -v\tau + x + vt$ of (3.5) – by $\chi_J(x,t)$. If $\chi_W(x,t) > 0$, then $\varphi_W(\chi_W(x,t);x,t) = 0$ or $\varphi_W(\chi_W(x,t);x,t) = \Lambda$; and respectively if $\chi_J(x,t) > 0$, then $\varphi_J(\chi_J(x,t);x,t) = 0$ or $\varphi_J(\chi_J(x,t);x,t) = \Lambda$. In our case,

$$\chi_W(x,t) = \begin{cases} \frac{vt-x}{v}, \text{for } vt-x>0 \\ 0\,, \text{ for } vt-x \le 0 \end{cases}; \chi_J(x,t) = \begin{cases} \frac{vt+x-\Lambda}{v}, \text{for } vt+x-\Lambda>0 \\ 0\,, \text{ for } vt+x-\Lambda \le 0 \end{cases}.$$

Remark 3.2 We notice that $\chi_W(x,t)$and $\chi_J(x,t)$ are retarded functions in t:

$$\chi_W(x,t) \le t - \frac{x}{v} \le t, \chi_J(x,t) \le t + \frac{x-\Lambda}{v} \le t.$$

It is easy to see that $\varphi_W(\tau;x,t) = v\tau + x - vt \Rightarrow \varphi_W(0;x,t) = x - vt$ and

$$\varphi_J(\tau;x,t) = -v\tau + x + vt \Rightarrow \varphi_J(0;x,t) = x + vt.$$

Introduce the sets

$\Pi_{in,W} = \{(x,t) \in \Pi : \chi_W(x,t) = 0\} \equiv \{(x,t) \in \Pi : \ x - vt \geq 0\}$,

$\Pi_{in,J} = \{(x,t) \in \Pi : \chi_J(x,t) = 0\} \equiv \{(x,t) \in \Pi : \ x + vt - \Lambda \leq 0\}$,

$\Pi_{0W} = \left\{(x,t) \in \Pi : \chi_W(x,t) > 0, \varphi_W(\chi_W(x,t); x,t) = \frac{v(vt-x)}{v} + x - vt = 0\right\}$,

$\Pi_{\Lambda J} = \left\{(x,t) \in \Pi : \chi_J(x,t) > 0, \ \varphi_J(\chi_J(x,t); x,t) = -v\frac{vt+x-\Lambda}{v} + x + vt = \Lambda\right\}$.

$\Pi_{0J} = \{(x,t) \in \Pi : \chi_J(x,t) > 0, \varphi_J(\chi_J(x,t); x,t) = 0\} = \emptyset$,

$\Pi_{\Lambda W} = \{(x,t) \in \Pi : \chi_W(x,t) > 0, \varphi_W(\chi_W(x,t); x,t) = \Lambda\} = \emptyset$,

Prior to presenting problem (3.1) in an operator form we introduce

$$\Phi_W(W,J)(x,t) = \begin{cases} W_0(x-vt))\,,\ (x,t) \in \Pi_{in,\,W}, \\ \Phi_{0W}\,(W,J)\,(\chi_W(x,t))\,,\ (x,t) \in \Pi_{0\,W}, \end{cases}$$

$$\Phi_J(W,J)(x,t) = \begin{cases} J_0(x+vt))\,,\ (x,t) \in \Pi_{in,\,J}, \\ \Phi_{\Lambda J}\,(W,J)\,(\chi_J(x,t))\,,\ (x,t) \in \Pi_{\Lambda\,J}, \end{cases}$$

or

$$\Phi_W(W,J)(x,t) = \begin{cases} W_0(x-vt),\ (x,t) \in \Pi_{in,\,W}, \\ \gamma\sqrt{C}E(\chi_W)e^{\alpha\,\chi_W} + \beta\,J(0,\chi_W)\,,\ (x,t) \in \Pi_{0\,W}, \end{cases}$$

$$\Phi_J(W,J)(x,t) = \begin{cases} J_0(x+vt))\,,\ (x,t) \in \Pi_{in,J}, \\ W(\Lambda,\chi_{_J}) - (\gamma_1+\alpha)\int_0^{\chi_J} W(\Lambda,s)ds - (\gamma_2-\alpha)\int_0^{\chi_J} J(\Lambda,s)ds, \\ \qquad (x,t) \in \Pi_{\Lambda J}. \end{cases}$$

We assign to the above mixed problem the following system of operator equations

$$W = B_W(W,J),\ J = B_J(W,J),$$

where

$$B_W(W,J)(x,t) := \Phi_W(W,J)(x,t) - h\int_{\chi_W}^t J(x,\tau)d\tau - \\ - \frac{j_0}{\sqrt{C}}\int_{\chi_W}^t e^{\alpha\tau}\sin\left(\frac{\pi}{\Phi_0}\int_0^\tau e^{-\alpha\,s}\frac{W(x,s)-J(x,s)}{\sqrt{C}}ds\right)d\tau,$$

$$B_J(W,J)(x,t) := \Phi_J(W,J)(x,t) - h\int_{\chi_J}^t W(x,\tau)d\tau \\ + \frac{j_0}{\sqrt{C}}\int_{\chi_J}^t e^{\alpha\tau}\sin\left(\frac{\pi}{\Phi_0}\int_0^t e^{-\alpha s}\frac{W(x,s)-J(x,s)}{\sqrt{C}}ds\right)d\tau.$$

We introduce the function sets

$$M_W = \{W \in C(\Pi) : |W(x,t)| \leq W_0 e^{\mu t}\}\,,\ M_J = \{J \in C(\Pi) : |J(x,t)| \leq J_0 e^{\mu t}\},$$

where $W_0,\ J_0,\ \mu$ are positive constants chosen below.

It is easy to verify that the set $M_W \times M_J$ turns out into a complete metric space with respect to the metric: $\rho((W,J),(\overline{W},\overline{J})) = \max\left\{\rho(W,\overline{W}),\rho(J,\overline{J})\right\}$, where

$$\rho(W,\overline{W}) = \sup\left\{e^{-\mu t}\left|W(x,t)-\overline{W}(x,t)\right| : (x,t) \in \Pi\right\},$$

$$\rho(J,\overline{J}) = \sup\left\{e^{-\mu t}\left|J(x,t)-\overline{J}(x,t)\right| : (x,t) \in \Pi\right\}.$$

4. EXISTENCE-UNIQUENESS OF CONTINUOUS GENERALIZED SOLUTION ON A SUBDOMAIN

Consider a mixed problem on the domain

$\Pi_\varepsilon = \left\{(x,t)\in\Pi^2 : (x,t)\in[0,\Lambda-\varepsilon]\times[0,T]\right\}, 0<\varepsilon<\Lambda.$

Theorem 4.1. Let the following conditions be fulfilled for sufficiently small initial data W_{00}, J_{00}:

4.1) $|E(t)|\,e^{\alpha t}\le E_0, t\in[0,\infty); |W_0(x)|\le W_{00}; |J_0(x)|\le J_{00},\ x\in[0,\Lambda];$

4.2) $\max\left\{W_{00};\ |\gamma|\sqrt{C}E_0+|\beta|\ J_0\right\}+\dfrac{1}{2}|h|\dfrac{\Lambda}{v}J_0+\dfrac{j_0}{\sqrt{C}}\dfrac{\Lambda}{v}\le W_0;$

4.3) $\max\left\{J_{00};\ W_0+\dfrac{|\gamma_1+\alpha|\,W_0+|\gamma_2-\alpha|\,J_0}{\mu}e^{\mu T-\frac{\mu\varepsilon}{v}}\right\}+\dfrac{1}{2}|h|\,W_0\dfrac{\Lambda}{v}+\dfrac{j_0}{\sqrt{C}}\dfrac{\Lambda}{v}\le J_0;$

$|\beta|+\dfrac{1}{2\mu}|h|+\dfrac{2j_0\pi}{\mu^2\Phi_0 C}<1; e^{-\frac{\mu\varepsilon}{v}}+\dfrac{|\gamma_1+\alpha|+|\gamma_2-\alpha|}{\mu}e^{-\frac{\mu\varepsilon}{v}}+\dfrac{1}{2\mu}|h|\,e^{-\frac{\mu\varepsilon}{v}}+\dfrac{2\pi j_0}{\mu^2\Phi_0 C}<1.$

Then there exists a unique $C(\Pi_\varepsilon)$-solution of (3.1).

Proof:

First step: We establish that the operator B maps the set $M_W\times M_J$ into itself.

We notice that $B_W(x,t)$ and $B_J(x,t)$ are continuous functions.

First we have to show $|B_W(W,J)(x,t)|\le W_0e^{\mu t}$, $|B_J(W,J)(x,t)|\le J_0e^{\mu t}$.

Indeed,

$$|B_W(W,J)(x,t)|\le|\Phi_W(x,t)|+|h|\int_{\chi_{W(x,t)}}^t|J(x,\tau)|\,d\tau+ + \tfrac{j_0\pi(W_0+J_0)}{\Phi_0 C}\int_{\chi_W(x,t)}^t e^{\alpha\tau}\int_0^\tau e^{-\alpha s}e^{\mu s}ds d\tau\le$$

$$\le e^{\mu t}\left\{\begin{array}{c}W_{00}\\ |\gamma|\sqrt{C}E_0+|\beta|\,J_0\end{array}\right\}+\tfrac{J_0}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\tfrac{e^{\mu t}}{\mu}+\tfrac{j_0\pi(W_0+J_0)}{\Phi_0 C}\int_{\chi_W(x,t)}^t e^{\alpha\tau}\tfrac{e^{(\mu-\alpha)\tau}-1}{\mu-\alpha}d\tau\le$$

$$\le e^{\mu t}\left\{\begin{array}{c}W_{00}\\ |\gamma|\sqrt{C}E_0+|\beta|\,J_0\end{array}\right\}+\tfrac{J_0}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\tfrac{e^{\mu t}}{\mu}+\tfrac{j_0\pi(W_0+J_0)}{\Phi_0 C(\mu-\alpha)}\tfrac{e^{\mu t}-e^{\mu\chi_W}}{\mu}\le$$

$$\le e^{\mu t}\left(\max\left\{W_{00};\ |\gamma|\sqrt{C}E_0+|\beta|\ J_0\right\}+\tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\tfrac{J_0}{\mu}+\tfrac{j_0\pi(W_0+J_0)}{\Phi_0 C\mu(\mu-\alpha)}\right)\le W_0e^{\mu t},$$

and analogously in view of $\chi_J(x,t)\le t+\frac{x-\Lambda}{v}\le t+\frac{\Lambda-\varepsilon-\Lambda}{v}=t-\frac{\varepsilon}{v}$,

$$|B_J(W,J)(x,t)|\le\left\{\begin{array}{c}J_0(x+vt)\\ |W(\Lambda,\chi_J)|+|\gamma_1+\alpha|\int_0^{\chi_J}|W(\Lambda,s)|\,ds+|\gamma_2-\alpha|\int_0^{\chi_J}|J(\Lambda,s)|\,ds\end{array}\right\}+$$
$$+\tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\int_{\chi_J}^t|W(x,\tau)|\,d\tau+$$
$$+\tfrac{j_0}{\sqrt{C}}\int_{\chi_J}^t e^{\alpha\tau}\sin\left(\tfrac{\pi}{\Phi_0\sqrt{C}}\int_0^\tau e^{-\alpha s}\left(|W(x,s)|+|J(x,s)|\right)ds\right)d\tau\le$$

$$\le\left\{\begin{array}{c}J_{00}\\ W_0e^{\mu\chi_J}+|\gamma_1+\alpha|\,W_0\frac{e^{\mu\chi_J}-1}{\mu}+|\gamma_2-\alpha|\,\frac{e^{\mu\chi_J}-1}{\mu}\end{array}\right\}+$$
$$+\tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|W_0\tfrac{e^{\mu t}-e^{\mu\chi_J}}{\mu}+\tfrac{j_0\pi(W_0+J_0)}{\Phi_0 C}\int_{\chi_J}^t e^{\alpha\tau}\int_0^\tau e^{-\alpha s}e^{\mu s}ds d\tau\le$$

$$\le\left\{\begin{array}{c}J_{00}\\ W_0e^{\mu\left(t-\frac{\varepsilon}{v}\right)}+\frac{|\gamma_1+\alpha|W_0+|\gamma_2-\alpha|J_0}{\mu}e^{\mu\left(t-\frac{\varepsilon}{v}\right)}\end{array}\right\}+$$
$$+\tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|W_0\tfrac{e^{\mu t}}{\mu}+\tfrac{j_0\pi(W_0+J_0)}{\Phi_0 C}\int_{\chi_J}^t e^{\alpha\tau}\tfrac{e^{(\mu-\alpha)\tau}-1}{\mu-\alpha}d\tau\le$$

$$\le e^{\mu t}\left(\max\left\{J_{00};W_0e^{-\frac{\mu\varepsilon}{v}}+\tfrac{|\gamma_1+\alpha|W_0+|\gamma_2-\alpha|J_0}{\mu}e^{-\frac{\mu\varepsilon}{v}}\right\}+\right.$$

$$+\tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\tfrac{W_0}{\mu}+\tfrac{j_0\pi(W_0+J_0)}{\Phi_0 C\mu(\mu-\alpha)}\Big)\le J_0\,e^{\mu t}.$$

The operator B is a strict contraction since

$$\begin{aligned}
&\left|B_W(W,J)(x,t)-B_W(\overline{W},\overline{J})(x,t)\right|\le\\
&\le\left|\Phi_W(W,J)(x,t)-\Phi_W(\overline{W},\overline{J})(x,t)\right|+\tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\int_{\chi_W(x,t)}^{t}\left|J(x,\tau)-\overline{J}(x,\tau)\right|e^{-\mu\tau}e^{\mu\tau}d\tau+\\
&+\tfrac{j_0\pi}{\Phi_0 C}\int_{\chi_W(x,t)}^{t}e^{\alpha\tau}\left(\int_0^{\tau}e^{-\alpha s}\left|W(x,s)-\overline{W}(x,s)\right|ds+\int_0^{\tau}e^{-\alpha s}\left|J(x,s)-\overline{J}(x,s)\right|ds\right)d\tau\le\\
&\le|\beta|\left|J(0,\chi_W(x,t))-\overline{J}(0,\chi_W(x,t))\right|e^{-\mu\chi_W(x,t)}e^{\mu\chi_W(x,t)}+\\
&+\tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\rho(J,\overline{J})\int_{\chi_W(x,t)}^{t}e^{\mu\tau}d\tau+\\
&+\tfrac{j_0\pi}{\Phi_0 C}\left(\int_{\chi_W(x,t)}^{t}\left(\int_0^{\tau}\left|W(x,s)-\overline{W}(x,s)\right|e^{-\mu s}e^{\mu s}e^{-\alpha s}ds+\right.\right.\\
&\qquad\qquad\left.\left.+\int_0^{\tau}\left|J(x,s)-\overline{J}(x,s)\right|e^{-\mu s}e^{\mu s}e^{-\alpha s}ds\right)d\tau\right)\le\\
&\le|\beta|\,\rho(J,\overline{J})e^{\mu\chi_W(x,t)}+\tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\rho(J,\overline{J})\tfrac{e^{\mu t}-e^{\mu\chi_W}}{\mu}+\\
&+\tfrac{j_0\pi\left(\rho(W,\overline{W})+\rho(J,\overline{J})\right)}{\Phi_0 C}\int_{\chi_W(x,t)}^{t}e^{\alpha\tau}\int_0^{\tau}e^{\mu s}e^{-\alpha s}ds\,d\tau\le\\
&\le\left(|\beta|\,e^{\mu t}+\tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\tfrac{e^{\mu t}-e^{\mu\chi_W}}{\mu}+\right.\\
&\qquad\qquad\left.+\tfrac{2j_0\pi}{\mu^2\Phi_0 C}\int_{\chi_W(x,t)}^{t}e^{\alpha\tau}\tfrac{e^{(\mu-\alpha)\tau}-1}{\mu-\alpha}d\tau\right)\rho((W,J),(\overline{W},\overline{J}))\le\\
&\le e^{\mu t}\left(|\beta|+\tfrac{1}{2\mu}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|+\tfrac{2j_0\pi}{\mu(\mu-\alpha)\Phi_0 C}\right)\rho\left((W,J),(\overline{W},\overline{J})\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\rho(B_W(W,J),B_W(\overline{W},\overline{J}))\le\\
&\qquad\le\left(|\beta|+\tfrac{1}{2\mu}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|+\tfrac{2j_0\pi}{\mu(\mu-\alpha)\Phi_0 C}\right)\rho((W,J),(\overline{W},\overline{J}))\equiv K_V\rho((W,J),(\overline{W},\overline{J})).
\end{aligned}$$

For the second component we obtain for $x\in[0;\Lambda-\varepsilon]$:

$$\begin{aligned}
&\left|B_J(W,J)(x,t)-B_J(\overline{W},\overline{J})(x,t)\right|\le\left|W(\Lambda,\chi_J)-\overline{W}(\Lambda,\chi_J)\right|+\\
&+|\gamma_1+\alpha|\int_0^{\chi_J}\left|W(\Lambda,s)-\overline{W}(\Lambda,s)\right|ds+|\gamma_2-\alpha|\int_0^{\chi_J}\left|J(\Lambda,s)-\overline{J}(\Lambda,s)\right|ds+\\
&+\tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\int_{\chi_J}^{t}\left|W(x,\tau)-\overline{W}(x,\tau)\right|d\tau+\\
&+\tfrac{\pi j_0}{\Phi_0 C}\int_{\chi_J}^{t}e^{\alpha\tau}\left|\int_0^{\tau}e^{-\alpha s}\left(W(x,s)-\overline{W}(x,s)+J(x,s)-\overline{J}(x,s)\right)ds\right|d\tau\le\\
&\le\left|W(\Lambda,\chi_J)-\overline{W}(\Lambda,\chi_J)\right|e^{-\mu\chi_J}e^{\mu\chi_J}+|\gamma_1+\alpha|\int_0^{\chi_J}\left|W(\Lambda,s)-\overline{W}(\Lambda,s)\right|e^{\mu s}e^{-\mu s}ds+\\
&+|\gamma_2-\alpha|\int_0^{\chi_J}\left|J(\Lambda,s)-\overline{J}(\Lambda,s)\right|e^{-\mu s}e^{\mu s}e^{-\alpha s}ds+\\
&+\tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\int_{\chi_J}^{t}\left|W(x,s)-\overline{W}(x,s)\right|e^{-\mu s}e^{\mu s}e^{-\alpha s}ds+\\
&+\tfrac{\pi j_0}{\Phi_0 C}\int_{\chi_J}^{t}e^{\alpha\tau}\left(\int_0^{\tau}\left(\left|W(x,s)-\overline{W}(x,s)\right|e^{-\mu s}e^{\mu s}e^{-\alpha s}+\right.\right.\\
&\qquad\qquad\left.\left.+\left|J(x,s)-\overline{J}(x,s)\right|e^{-\mu s}e^{\mu s}e^{-\alpha s}\right)ds\right)d\tau\le\\
&\le\rho(W,\overline{W})e^{\mu\chi_J}+|\gamma_1+\alpha|\,\rho(W,\overline{W})\int_0^{\chi_J}e^{\mu s}ds+|\gamma_2-\alpha|\,\rho(J,\overline{J})\int_0^{\chi_J}e^{\mu s}ds+\\
&+\tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\rho(W,\overline{W})\int_{\chi_J}^{t}e^{\mu s}ds+\tfrac{\pi j_0}{\Phi_0 C}\left(\rho(W,\overline{W})+\rho(J,\overline{J})\right)\int_{\chi_J}^{t}\tfrac{e^{\mu\tau}}{\mu-\alpha}d\tau\le\\
&\le\rho(W,\overline{W})e^{\mu\chi_J}+|\gamma_1+\alpha|\,\rho(W,\overline{W})\tfrac{e^{\mu\chi_J}-1}{\mu}+|\gamma_2-\alpha|\,\rho(J,\overline{J})\tfrac{e^{\mu\chi_J}-1}{\mu}+\\
&+\tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\rho(W,\overline{W})\tfrac{e^{\mu\chi_J}-1}{\mu}+\tfrac{\pi j_0}{\Phi_0 C(\mu-\alpha)}\left(\rho(W,\overline{W})+\rho(J,\overline{J})\right)\tfrac{e^{\mu t}-e^{\mu\chi_J}}{\mu}\le
\end{aligned}$$

$$\leq \rho((W,J),(\overline{W},\overline{J}))\left[e^{\mu\left(t+\frac{x-\Lambda}{v}\right)}+\frac{|\gamma_1+\alpha|}{\mu}e^{\mu\left(t+\frac{x-\Lambda}{v}\right)}+\frac{|\gamma_2-\alpha|}{\mu}e^{\mu\left(t+\frac{x-\Lambda}{v}\right)}+\right.$$
$$\left.+\frac{1}{2\mu}\left|\frac{R}{L}-\frac{G}{C}\right|e^{\mu\left(t+\frac{x-\Lambda}{v}\right)}+e^{\mu t}\frac{2\pi j_0}{\mu(\mu-\alpha)\Phi_0 C}\right]\leq$$
$$\leq e^{\mu t}\rho((W,J),(\overline{W},\overline{J}))\left(e^{-\frac{\mu\varepsilon}{v}}+\frac{|\gamma_1+\alpha|+|\gamma_2-\alpha|}{\mu}e^{-\frac{\mu\varepsilon}{v}}+\right.$$
$$\left.+\frac{1}{2\mu}\left|\frac{R}{L}-\frac{G}{C}\right|e^{-\frac{\mu\varepsilon}{v}}+\frac{2\pi j_0}{\mu(\mu-\alpha)\Phi_0 C}\right).$$

It follows that

$$\rho(B_J(W,J),B_J(\overline{W},\overline{J}))\leq\left(e^{-\frac{\mu\varepsilon}{v}}+\frac{|\gamma_1+\alpha|+|\gamma_2-\alpha|}{\mu}e^{-\frac{\mu\varepsilon}{v}}+\right.$$
$$\left.+\frac{1}{2\mu}|h|\,e^{-\frac{\mu\varepsilon}{v}}+\frac{2\pi j_0}{\mu(\mu-\alpha)\Phi_0 C}\right)\rho((W,J),(\overline{W},\overline{J}))\equiv K_J\rho((W,J),(\overline{W},\overline{J})),$$

and finally

$$\rho\left((B_W(W,J),B_J(W,J)),\left(B_W(\overline{W},\overline{J}),B_J(\overline{W},\overline{J})\right)\right)\leq\max\{K_W;K_J\}\rho((W,J),(\overline{W},\overline{J})).$$

Therefore, the unique fixed point of B (cf. [22]) is a unique generalized continuous solution belonging to $\Pi_\varepsilon=\{(x,t)\in[0,\Lambda-\varepsilon]\times[0,T]\}$.

Theorem 4.1 is thus proved.

5. EXISTENCE OF SOLUTION OF THE MIXED PROBLEM

Let us introduce the sets

$$M_W^l=\{W\in M_W:|W(x,t-W(\bar{x},\bar{t})|\leq l_W(|x-\bar{x}|+|t-\bar{t}|)\,;x,\bar{x}\in[0,\Lambda],t,\bar{t}\in[0,T]\},$$
$$M_J^l=\{J\in M_J:|J(x,t)-J(\bar{x},\bar{t})|\leq l_J(|x-\bar{x}|+|t-\bar{t}|)\,;x,\bar{x}\in[0,\Lambda],t,\bar{t}\in[0,T]\}\,.$$

For every $\varepsilon=1/n$ in accordance of Theorem 4.1 we obtain a unique solution (W_n,J_n) on $M_{W,n}^l\times M_{J,n}^l$. The set $M_{W,n}^l\left(resp.M_{J,n}^l\right)$ consists of all restrictions of functions from $M_W^l\left(resp.M_J^l\right)$on $[0;\Lambda-(1/n)\,]\times[0;T]$ for every sufficiently large $n\in N$. We extend functions $(W_n,J_n)\in M_{W,n}^l\times M_{J,n}^l$ on the whole domain $\Pi=[0,\Lambda]\times[0,T]$ such that the extensions belong to $M_W^l\times M_J^l$. For instance,

$$\tilde{W}_n(x,t)=\begin{cases} W_n(x,t),\ (x,t)\in[0;\Lambda-(1/n)]\times[0;T]\,,\\ W_n(\Lambda-(1/n)\,,t),\ (x,t)\in[\Lambda-(1/n)\ ;\Lambda]\times[0;T]\,,\end{cases}$$

$$\tilde{J}_n(x,t)=\begin{cases} J_n(x,t),\ (x,t)\in[0;\Lambda-(1/n)]\times[0;T]\,,\\ J_n(\Lambda-(1/n)\,,t),\ (x,t)\in[\Lambda-(1/n)\ ;\Lambda]\times[0;T]\,.\end{cases}$$

Now we are able to state the problem for existence of $\lim\limits_{n\to\infty}\left(\tilde{W}_n,\tilde{J}_n\right)$ in the topology of $M_W^l\times M_J^l$. But we are not sure that the last sequence is convergent.

In what follows we give a constructive way to form a convergent subsequence of $\left(\tilde{W}_n,\tilde{J}_n\right)$.

Let us consider the conditions:

E1) $3\left(|\beta|\,l_J+\frac{J_0}{2}\left|\frac{R}{L}-\frac{G}{C}\right|+\frac{j_0}{\sqrt{C}}\right)\leq l_W$;

E2) $l_W+|\gamma_1+\alpha|\,W_0\,+|\gamma_2-\alpha|\ J_0+\frac{2j_0}{\sqrt{C}}\leq l_J$.

They imply that $\{B_W(W,J)(x,t), B_J(W,J)(x,t)\}$ forms an equicontinuous family of functions.

Indeed,

$$|W_n(x,t)-W_n(\bar{x},\bar{t})| = |B_W(W_n,J_n)(x,t)-B_W(W_n,J_n)(\bar{x},\bar{t})| \leq$$
$$\leq |\Phi_W(W_n,J_n)(x,t)-\Phi_W(W_n,J_n)(\bar{x},\bar{t})|+$$
$$+\tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\left|\int_{\bar{t}}^{t} J(x,\tau)d\tau\right| + \tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\left|\int_{\chi_W(\bar{x},\bar{t})}^{\chi_W(x,t)} J(x,\tau)d\tau\right|+$$
$$+\tfrac{j_0}{\sqrt{C}}\left|\int_{\bar{t}}^{t}\sin\left(\tfrac{\pi}{\Phi_0\sqrt{C}}\int_0^\tau (W(x,s)-J(x,s))\,ds\right)d\tau\right|+$$
$$+\tfrac{j_0}{\sqrt{C}}\left|\int_{\chi_W(\bar{x},\bar{t})}^{\chi_W(x,t)}\sin\left(\tfrac{\pi}{\Phi_0\sqrt{C}}\int_0^\tau (W(x,s)-J(x,s))\,ds\right)d\tau\right| \leq$$
$$\leq |\beta|\,|J(0,\chi_W(x,t))-J(0,\chi_W(\bar{x},\bar{t}))| + \tfrac{J_0}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|(|t-\bar{t}|+|\chi_W(x,t)-\chi_W(\bar{x},\bar{t})|)+$$
$$+\tfrac{j_0}{\sqrt{C}}(|\,t-\bar{t}|+|\chi_W(x,t)-\chi_W(\bar{x},\bar{t})|) \leq$$
$$\leq |\beta|\,l_J|\chi_W(x,t)-\chi_W(\bar{x},\bar{t})| + \left(\tfrac{J_0}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|+\tfrac{j_0}{\sqrt{C}}\right)(|t-\bar{t}|+|\chi_W(x,t)-\chi_W(\bar{x},\bar{t})|) \leq$$
$$\leq |\beta|\,l_J\left(|t-\bar{t}|+\tfrac{1}{v}|x-\bar{x}|\right) + \left(\tfrac{J_0}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|+\tfrac{j_0}{\sqrt{C}}\right)\left(2\,|t-\bar{t}|+\tfrac{1}{v}|x-\bar{x}|\right) \leq$$
$$\leq 3\left(|\beta|\,l_J+\tfrac{J_0}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|+\tfrac{j_0}{\sqrt{C}}\right)(|t-\bar{t}|+|x-\bar{x}|) \leq l_W\,(|t-\bar{t}|+|x-\bar{x}|)$$

and

$$|J_n(x,t)-J_n(\bar{x},\bar{t})| = |B_J(W_n,J_n)(x,t)-B_J(W_n,J_n)(\bar{x},\bar{t})| \leq$$
$$\leq |\Phi_J(W_n,J_n)(x,t)-\Phi_J(W_n,J_n)(\bar{x},\bar{t})|+$$
$$+\tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\left|\int_{\bar{t}}^{t} W(x,\tau)d\tau\right| + \tfrac{1}{2}\left|\tfrac{R}{L}-\tfrac{G}{C}\right|\left|\int_{\chi_W(\bar{x},\bar{t})}^{\chi_W(x,t)} W(x,\tau)d\tau\right|+$$
$$+\tfrac{j_0}{\sqrt{C}}\left|\int_{\chi_J(x,t)}^{t}\sin\left(\tfrac{\pi}{\Phi_0\sqrt{C}}\int_0^\tau (W_n(x,s)-J_n(x,s))\,ds\right)d\tau-\right.$$
$$\left.-\int_{\chi_J(\bar{x},\bar{t})}^{\bar{t}}\sin\left(\tfrac{\pi}{\Phi_0\sqrt{C}}\int_0^\tau (W_n(x,s)-J_n(x,s))\,ds\right)d\tau\right| \leq$$
$$\leq |W_n(\Lambda-(1/n)\,,\chi_J(x,t))-W_n(\Lambda-(1/n)\,,\chi_J(\bar{x},\bar{t}))|+$$
$$+|\gamma_1+\alpha|\left|\int_{\chi_J(\bar{x},\bar{t})}^{\chi_J(x,t)} W_n(\Lambda-(1/n)\,,s)ds\right| + |\gamma_2-\alpha|\left|\int_{\chi_J(\bar{x},\bar{t})}^{\chi_J(x,t)} J_n(\Lambda-(1/n)\,,s)ds\right|+$$
$$+\tfrac{j_0}{\sqrt{C}}\left|\int_{\bar{t}}^{t}\sin\left(\tfrac{\pi}{\Phi_0\sqrt{C}} int_0^\tau (W_n(x,s)-J_n(x,s))\,ds\right)d\tau\right|+$$
$$+\tfrac{j_0}{\sqrt{C}}\left|\int_{\chi_J(\bar{x},\bar{t})}^{\chi_J(x,t)}\sin\left(\tfrac{\pi}{\Phi_0\sqrt{C}}\int_0^\tau (W_n(x,s)-J_n(x,s))\,ds\right)d\tau\right| \leq$$
$$\leq l_W\,|\chi_J(x,t)-\chi_J(\bar{x},\bar{t})| + |\gamma_1+\alpha|\,W_0\,|\chi_J(x,t)-\chi_J(\bar{x},\bar{t})|+$$
$$+|\gamma_2-\alpha|\,J_0\,|\chi_J(x,t)-\chi_J(\bar{x},\bar{t})| + \tfrac{j_0}{\sqrt{C}}|t-\bar{t}| + \tfrac{j_0}{\sqrt{C}}|\chi_J(x,t)-\chi_J(\bar{x},\bar{t})| \leq$$
$$\leq \left(l_W+|\gamma_1+\alpha|\,W_0+|\gamma_2-\alpha|\,J_0+\tfrac{j_0}{\sqrt{C}}\right)|\chi_J(x,t)-\chi_J(\bar{x},\bar{t})| + \tfrac{j_0}{\sqrt{C}}|t-\bar{t}| \leq$$
$$\leq \left(l_W+|\gamma_1+\alpha|\,W_0+|\gamma_2-\alpha|\,J_0+\tfrac{j_0}{\sqrt{C}}\right)\left(|t-\bar{t}|+\tfrac{1}{v}|x-\bar{x}|\right) + \tfrac{j_0}{\sqrt{C}}|t-\bar{t}| \leq$$
$$\leq \left(l_W+|\gamma_1+\alpha|\,W_0+|\gamma_2-\alpha|\,J_0+\tfrac{2j_0}{\sqrt{C}}\right)(|t-\bar{t}|+|x-\bar{x}|) \leq l_J\,(|t-\bar{t}|+|x-\bar{x}|)\,.$$

We extend every function W_n and J_n on Π in such a way that the obtained extensions $\tilde{W}_n(x,t), \tilde{J}_n(x,t)$ form a family of equicontinuous functions. This can be done in the following way:

$$\tilde{W}_n(x,t) = \begin{cases} W_n(x,t), & (x,t)\in[0;\Lambda-(1/n)]\times[0;T], \\ W_n(\Lambda-(1/n)\,,t), & (x,t)\in[\Lambda-(1/n)\,;\Lambda]\times[0;T], \end{cases}$$

$$\tilde{J}_n(x,t) = \begin{cases} J_n(x,t), & (x,t) \in [0;\Lambda - (1/n)] \times [0;T], \\ J_n(\Lambda - (1/n)\,,t), & (x,t) \in [\Lambda - (1/n)\,;\Lambda] \times [0;T]. \end{cases}$$

6. CHOICE OF SUBSEQUENCE

The Arzela-Ascoli theorem does not give a constructive way for defining a convergent subsequence. That is why we form a convergent subsequence generalizing some results from [20] to the case of sequences of functions. First, we briefly recall some basic results from Chapter IV of [20].

Let $\{y_k\}_{k=1}^{\infty}$ be an infinite sequence and $\{a_{n\,k}\}_{n,k=1}^{\infty}$ be an infinite matrix. Let us form the sequence $\tilde{y}_n = \sum_{k=1}^{\infty} a_{n\,k}\, y_k$. If $\lim \tilde{y}_n$ exists then it is called a *generalized limit.*

The following theorem is valid: *the necessary and sufficient conditions that* $\tilde{y}_n \to y$ *whenever* $y_n \to y$ *are that:*

(a) $\sum_{k=1}^{\infty} |a_{n\,k}| \le M$ *for every* $n > n_0$*; (b)* $\lim\limits_{n\to\infty} a_{n\,k} = 0$ *for every fixed* k*; (c)* $\sum_{k=1}^{\infty} a_{n\,k} \equiv A_n \underset{n\to\infty}{\to} 1.$

Infinite matrices satisfying *(a), (b), (c)* are called *T-matrices.* A transformation of a sequence by a T-matrix is called *regular* one. Let $\{y_k\}_{k=1}^{\infty}$ be a bounded divergent sequence and let $U = \lim\sup\, y_n$, $L = \lim\inf\, y_n$.

Theorem 6.1 [20] Every number y between L and U is a generalized limit of this sequence for some positive T-matrix.

The purpose of this section is to extend Theorem 6.1 for bounded sequence of equicontinuous functions.

Let $\{I_k(x,t)\}_{k=1}^{\infty}$ be a family such that $I_k(x,t) \in M_J^l(\Pi)$. In what follows we consider the convergence with respect to the norm $\|I\| = \max\{|I(x,t)| : (x,t) \in \Pi\}$. Obviously$\|I\|_\mu = \max\{e^{-\mu t}\,|I(x,t)| : (x,t) \in \Pi\} \le \|I\|$. In fact we need only pointwise convergence.

Define functions $I_U(x,t) = \lim\sup I_k(x,t)$, $I_L(x,t) = \lim\inf I_k(x,t)$, $(x,t) \in \Pi$.

If $I_U(x,t) = I_L(x,t)$, then we put $I(x,t) = \lim\limits_{k\to\infty} I_k(x,t) = I_U(x,t) = I_L(x,t)$.

Theorem 6.2. For every function satisfying the inequalities

$$I_L(x,t) < I(x,t) < I_U(x,t) \tag{6.1}$$

there is a positive T-matrix such that $I(x,t) = \lim\limits_{n\to\infty} \sum_{k=1}^{\infty} a_{n\,k}(x,t)\, I_k(x,t)$.

Proof: Let us choose two subsequences of $\{I_k(x,t)\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} a_k(x,t) = I_U(x,t),\ \lim_{k\to\infty} b_k(x,t) = I_L(x,t)$$

and for instance

$\|I_U - a_1\| \le \frac{1}{2}, \|I_L - b_1\| \le \frac{1}{2},$

$\|I_U - a_2\| \le \frac{1}{2^2}, \|I_L - b_2\| \le \frac{1}{2^2}, \ldots, \|I_U - a_k\| \le \frac{1}{2^k}, \|I_L - b_k\| \le \frac{1}{2^k}, ...$

It follows that $\sum_{k=1}^{\infty} \|I_U - a_k\| < \infty$, $\sum_{k=1}^{\infty} \|I_L - b_k\| < \infty$.

Let $\sum_{k=1}^{\infty} \|I_U - a_k\| = a, \sum_{k=1}^{\infty} \|I_L - b_k\| = b$ and let us put $a_k(x,t) = I_U(x,t) + \xi_k(x,t), b_k(x,t) = I_L(x,t) + \eta_k(x,t)$.

From (6.1) we obtain $I_U(x,t) - I(x,t) = p(x,t) > 0, I(x,t) - I_L(x,t) = q(x,t) > 0$. Obviously, $p(x,t) + q(x,t) > 0$ on Π which implies $p(x,t) + q(x,t) \ge a_{pq} > 0$. Then

$$\frac{\sum_{k=1}^n q(x,t)a_k(x,t) + \sum_{k=1}^n p(x,t)b_k(x,t)}{n(p(x,t)+q(x,t))} =$$
$$= \frac{\sum_{k=1}^n q(x,t)(I_U(x,t)+\xi_k(x,t)) + \sum_{k=1}^n p(x,t)(I_L(x,t)+\eta_k(x,t))}{n(p(x,t)+q(x,t))} =$$
$$= \frac{\sum_{k=1}^n qI_U + \sum_{k=1}^n q\xi_k + \sum_{k=1}^n pI_L + \sum_{k=1}^n p\eta_k}{n(p+q)} =$$
$$= \frac{\sum_{k=1}^n (I-I_L)I_U + \sum_{k=1}^n (I_U-I)I_L}{n(p+q)} + \frac{\sum_{k=1}^n q\xi_k + \sum_{k=1}^n p\eta_k}{n(p+q)} =$$
$$= \frac{n(I-I_L)I_U + n(I_U-I)I_L}{n(p+q)} + \frac{\sum_{k=1}^n q\xi_k + \sum_{k=1}^n p\eta_k}{n(p+q)} =$$
$$= \frac{I(I_U-I_L)}{I_U-I+I-I_L} + \frac{\sum_{k=1}^n q\xi_k + \sum_{k=1}^n p\eta_k}{n(p+q)} = I(x,t) + \frac{\sum_{k=1}^n q\xi_k + \sum_{k=1}^n p\eta_k}{n(p+q)}.$$

But

$$\left| \frac{\sum_{k=1}^n q(x,t)a_k(x,t) + \sum_{k=1}^n p(x,t)b_k(x,t)}{n(p(x,t)+q(x,t))} - I(x,t) \right| \le$$
$$\le \frac{\left|\sum_{k=1}^n q\xi_k + \sum_{k=1}^n p\eta_k\right|}{n(p(x,t)+q(x,t))} \le \frac{2I_0a + 2I_0b}{na_{pq}}.$$

Therefore, $\left\| \frac{\sum_{k=1}^n q(x,t)a_k(x,t) + \sum_{k=1}^n p(x,t)b_k(x,t)}{n(p(x,t)+q(x,t))} - I(x,t) \right\| \underset{n\to\infty}{\longrightarrow} 0$.

Consequently $\lim\limits_{n\to\infty} \frac{\sum_{k=1}^n q(x,t)a_k(x,t) + \sum_{k=1}^n p(x,t)b_k(x,t)}{n(p+q)} = I(x,t)$.

It remains to construct a T-matrix. We define the n-th row of $\{a_{n\,k}(x,t)\}_{n,k=1}^{\infty}$ by taking $a_{n\,k} = 0$ if k is the subscript of a function $I_k(x,t)$ not occurring in $a_1(x,t)$, $a_2(x,t), ..., a_n(x,t)$ or in $b_1(x,t), b_2(x,t), ..., b_n(x,t)$.

If $I_k(x,t)$ is one of $a_1(x,t), a_2(x,t), ..., a_n(x,t)$ we take $a_{nk}(x,t) = \frac{q(x,t)}{n(p(x,t)+q(x,t))}$ and if $I_k(x,t)$ is one of $b_1(x,t), b_2(x,t), ..., b_n(x,t)$ we take $a_{nk}(x,t) = \frac{p(x,t)}{n(p(x,t)+q(x,t))}$.

Then $\sum_{k=1}^{\infty} a_{n\,k}(x,t) = 1$ for every n, and $\lim\limits_{n\to\infty} a_{n\,k}(x,t) = 0$ for every fixed k uniformly in (x,t).

Consequently $\{a_{n\,k}(x,t)\}_{n,k=1}^{\infty}$ is a positive T-matrix.

Theorem 6.1 is thus proved.

7. CONCLUSION

Here we collect all inequalities from the proof of Theorem 4.1. For sufficiently small W_{00},J_{00} and sufficiently large n we have:

$$|\gamma| \sqrt{C} E_0 + |\beta| \ J_0 + \frac{1}{2} \left| \frac{R}{L} - \frac{G}{C} \right| \frac{J_0}{\mu} + \frac{j_0 \pi \left(W_0 + J_0\right)}{\Phi_0 C \mu \left(\mu - \alpha\right)} \le W_0;$$

$$W_0 e^{-\frac{\mu}{nv}} + \frac{|\gamma_1 + \alpha| \, W_0 + |\gamma_2 - \alpha| \, J_0}{\mu} e^{-\frac{\mu}{nv}} + \frac{1}{2} \left| \frac{R}{L} - \frac{G}{C} \right| \frac{W_0}{\mu} + \frac{j_0 \pi \left(W_0 + J_0\right)}{\Phi_0 C \mu \left(\mu - \alpha\right)} \le J_0 \, ;$$

$$K_W = |\beta| + \frac{1}{2\mu}\left|\frac{R}{L} - \frac{G}{C}\right| + \frac{2j_0\pi}{\mu(\mu-\alpha)\Phi_0 C} < 1;$$

$$K_J = e^{-\frac{\mu}{nv}} + \frac{|\gamma_1+\alpha| + |\gamma_2-\alpha|}{\mu}e^{-\frac{\mu}{nv}} + \frac{1}{2\mu}\left|\frac{R}{L} - \frac{G}{C}\right|e^{-\frac{\mu}{nv}} + \frac{2\pi j_0}{\mu(\mu-\alpha)\Phi_0 C} < 1;$$

$$3\left(|\beta|\, l_J + \frac{J_0}{2}\left|\frac{R}{L} - \frac{G}{C}\right| + \frac{j_0}{\sqrt{C}}\right) \le l_W; l_W + |\gamma_1+\alpha|\, W_0 + |\gamma_2-\alpha|\; J_0 + \frac{2j_0}{\sqrt{C}} \le l_J.$$

Let us consider a Josephson transmission line (cf. [15]–[18]) with

$L = 2,5.10^{-9}\, H/m, C = 1,3.10^{-6} F/m,\, G = 480\, mho/m$, length $\Lambda = 3.10^{-4} m$, $\sqrt{C} = 1,14.10^{-3}$.

Then $v = 1/\sqrt{LC} = 1/\sqrt{2,5.10^{-9}.1,3.10^{-6}} = 1,75.10^7$ and

$T = \Lambda/v = 3.10^{-4}/\left(1,75.10^7\right) \approx 1,7.10^{-11}$ sec,

$Z_0 = \sqrt{L/C} = \sqrt{\left(2,5.10^{-9}\right)/\left(1,3.10^{-6}\right)} \approx 0,044\Omega$,

$\Phi_0 = 2.10^{-15} W/m^2; j_0 = 1,9A/m$.

Assume that a Heaviside condition is satisfied: $\frac{R}{L} - \frac{G}{C} = 0 \Rightarrow R = L\frac{G}{C} \approx 0,95$.

Then $\alpha = \frac{1}{2}\left(\frac{R}{L} + \frac{G}{C}\right) = \frac{R}{L} = \frac{0,95}{2,5.10^{-9}} \approx 3,8.10^8$.

Let us take $R_1 = 0,6\,\Omega$, $C_0 = 10^{-10} F$and $R_0 = R_1 = Z_0 = 0,044\Omega$. Then

$$\gamma = \frac{2Z_0}{Z_0+R_0} = 1\,; \beta = \frac{Z_0-R_0}{Z_0+R_0} = 0\,; \gamma_1 = \frac{R_1-Z_0}{R_1Z_0C_0} = 0\,; \gamma_2 = \frac{R_1+Z_0}{R_1Z_0C_0} = 45,5.10^{10};$$

$|\gamma_1+\alpha| = 3,8.10^8$; $|\gamma_2-\alpha| = 45,5.10^{10} - 0,38.10^9 \approx 3,84.10^{11}$.

Choose $\mu = 10^{12}, j_0\pi = 5,97$. If we choose the accuracy $\varepsilon = 10^{-5}$, then $e^{-\left(10^{12}.10^{-5}\right)/1,75.10^7} \approx e^{-0,571} \approx 0,564$ and the above inequalities for sufficiently small initial date become:

$1,14.10^{-3}E_0 + \frac{5,97}{2.10^{-15}.1,3.10^{-6}}\frac{W_0+J_0}{10^{12}(10^{12}-3,8.10^8)} \le W_0;$

$0,564\left(W_0 + \frac{3,8.10^8 W_0 + 3,84.10^{11}J_0}{10^{12}}\right) + \frac{5,97}{2.10^{-15}.1,3.10^{-6}}\frac{W_0+J_0}{10^{12}(10^{12}-3,8.10^8)} \le J_0;$

$K_W = \frac{5,97}{2.10^{-15}.1,3.10^{-6}}\frac{2}{10^{12}(10^{12}-3,8.10^8)} < 1;$

$K_J = 0,564 + \frac{3,8.10^8+3,84.10^{11}}{10^{12}}0,564 + \frac{2}{10^{12}(10^{12}-3,8.10^8)}\frac{5,97}{2.10^{-15}.1,3.10^{-6}} < 1;$

$\frac{5,7}{1,14.10^{-3}} \le l_W; l_W + 3,8.10^8 W_0 + 3,84.10^{11}J_0 + \frac{3,8}{1,14.10^{-3}} \le l_J$

or for $E_0 = W_0 = J_0 \approx 10^{-8}$ it follows

$\frac{1,14}{10^3} + 2\frac{2,29}{10^3} \le 1;\ \left(0,564 + \frac{2,13}{10^4} + \frac{2,3}{10^3}\right) + \left(\frac{2,17}{10} + \frac{2,3}{10^3}\right) \le 1;$

$K_W = \frac{4,6}{10^3} < 1; K_J = 0,786 < 1; 5.10^3 \le l_W;$

$l_W + 3,8.10^8.10^{-8} + 3,84.10^{11}.10^{-8} + 3,3.10^3 \le l_J.$

Finally we note that

$|W(x,t)| \le W_0 e^{\mu T} = W_0 e^{10^{12}.1,7.10^{-11}} = 10^{-8}e^{17} = 10^{-8}.2,4.10^7 \approx 0,24$ and

$|J(x,t)| \le J_0 e^{\mu T} = 10^{-8}e^{17} \approx 0,24.$

It should be noted that the actual physical quantities must be calculated by the formulas

$$\left| \begin{array}{l} u(x,t) = e^{-\alpha t} W(x,t) / \left(2\sqrt{L} \right) + e^{-\alpha t} J(x,t) / \left(2\sqrt{L} \right), \\ i(x,t) = e^{-\alpha t} W(x,t) / \left(2\sqrt{C} \right) - e^{-\alpha t} J(x,t) / \left(2\sqrt{C} \right). \end{array} \right.$$

The above example shows that we obtain a solution on the whole rectangle $[0,\Lambda]\times[0,T]$, that is, not for sufficiently small T.

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