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MULTISCALE MODELING OF THE VISCOELASTICITY OF A RUBBER ROD UNDER TENSILE DEFORMATION

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ABSTRACT. A molecular based model for the viscoelasticity of rubber is developed using a stick-slip continuous molecular model. A corresponding nonlinear continuum model is given for tensile deformation. A linearized version of the model is studied for qualitative properties of the model. In our model cross-linked(CC)-system of molecules restrict the motion of entrapped or physically constrained(PC)-molecules. The dynamics of the PC-molecules is modeled by reptation in which the CC-molecules act as constraint boxes and the PC-molecules have to reptate in between the CC-molecules. We assume that a CC-unit cell is placed at each point of the rubber continuum with an entrapped PC-cell inside it. The deformation of the CC-cell causes a deformation of the PC-system which relaxes after removal of the deformation. In the relaxation process the PC-molecules act as internal variables affecting the relaxation process of the CC-system. The Rouse model for relaxing polymers is incorporated into the stick-slip model presented by Johnson and Stacer [28] for describing the dynamics of the entrapped molecule for a short time right after instantaneous step-strain of the constraining CC-cell.

1. INTRODUCTION

In the paper by Banks, et. al. [1], [2], [3], [4], [14] model was developed based on molecular models of Johnson and Stacer [28] and Doi and Edwards [24], where strain energy density functions were used to characterize the stress distribution for tensile and shear deformations. In this paper we use a microscopic description of the stress tensor following Doi and Edwards [24] to characterize the stress distribution for a general deformation. In this approach we enforce reptation following the approach of Johnson and Stacer [28] adhering more to the architecture of the constrained polymer and relating its deformation more closely to the constraining CC-cell. In addition the physical parameter of both the polymers in the CC-cell and the PC-molecule can be more readily reflected in the model and the relaxation process of both the PC-molecules as well as the CC-molecules are better described.

The proposed model for a single polymer strand is represented by a series of beads (or nodes) separated by springs, governed by Hooke's Law. The Rouse Model of polymer elasticity was proposed to model the dynamics of polymers by the Brownian

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motion of these nodes. Such a model can be used to represent the dynamics of a system of chemically cross-linked polymers.

To develop the model we treat each constrained molecule as a chain of beads connected by springs representing intermolecular potential. Subsequent to an instantaneous step strain of the CC-cell the constrained molecule relaxes following the Rouse model for a short time. To enforce reptation we follow the idea of Johnson and Stacer [28]. That is, at each point of the rubber continuum we place a unit cell in the rubber continuum with an entrapped PC-molecule thereby relating the deformation of the entrapped molecule to that of the CC-cell.

As an application of the general stress formula that could be developed for rubber based on the microscopic approach we give a model of the viscoelasticity of rubber under tensile deformation. We also give qualitative properties of the model.

2. MODELING OF THE DYNAMICS OF THE PC-MOLECULAR CHAIN

We model a typical PC-molecule by a chain of N-beads connected by a spring. Let $R_n = (R_1, R_2, \ldots, R_N)$ be the position vectors of the beads in the chain. In the model we proceed to develop the dynamics of such a chain for a short period of time after instantaneous step deformation is given by the Rouse model where the motion of the beads be described by the Langevin equation [24]:

$$\frac{\partial}{\partial t} R_n(t) = \sum_m H_{nm} \cdot \left(-\frac{\partial U}{\partial R_m} + f_m(t) \right) + \frac{1}{2} k_B T \sum_m \frac{\partial}{\partial R_m} \cdot H_{nm}, \tag{1}$$

where $f_m(t)$ is a random force term, k_B is Boltzmann's constant, T is the temperature, and the mobility tensor and the interaction potential, are chosen to be

$$H_{nm} = \frac{\delta_{nm}}{\zeta} I,$$

$$U = \frac{k}{2} \sum_{n=2}^{N} \|(R_n - R_{n-1})\|^2,$$

respectively, with

$$k = \frac{3 k_B T}{b^2}, \tag{2}$$

where b is the effective segment bond length at equilibrium and ζ is the friction constant of the polymer sample.

If we use the parameters defined above for the mobility tensor, H_{nm}, and for the interaction potential, U, then equation (1), for the cases when $n = 2, 3, \ldots, N-1$, can be written as

$$\zeta \frac{dR_n}{dt} = -k(2R_n - R_{n+1} - R_{n-1}) + f_n. \tag{3}$$

For the special cases of the extreme ends of the polymer, i.e., the cases when $n = 1$ and $n = N$, we see that (respectively)

$$\zeta \frac{dR_1}{dt} = -k(R_1 - R_2) + f_1, \tag{4}$$

$$\zeta \frac{dR_N}{dt} = -k(R_N - R_{N-1}) + f_N. \tag{5}$$

The term, f_n is a randomly distributed force, which takes into consideration the Brownian motion of the beads. Assume that the random force, f_n, is distributed according to a Gaussian distribution, which is determined by the following moments

$$\langle f_n(t) \rangle = 0,$$

$$\langle f_{n\alpha}(t) f_{m\beta}(t') \rangle = 2\zeta k_B T \delta_{nm} \delta_{\alpha\beta} \delta(t - t'). \tag{6}$$

If we regard n as a continuous variable, it is possible to rewrite Equation (3) using a continuous derivative as

$$\zeta \frac{\partial R_n}{\partial t} = k \frac{\partial^2 R_n}{\partial n^2} + f_n \tag{7}$$

$$\frac{\partial R_n}{\partial n}\Big|_{n=0} = \frac{\partial R_n}{\partial n}\Big|_{n=N} = 0, \tag{8}$$

under the assumption that $R_0 = R_1$ and $R_{N+1} = R_N$.

Define

$$b_n = R_{n+1} - R_n. \tag{9}$$

Then, from (4)–(8), we have

$$\frac{db_n}{dt} = -\frac{3kT}{\zeta b^2} \sum_{k=1}^{N-1} A_{nk} b_k + f_{n+1}(t) - f_n(t), \tag{10}$$

where

$$A_{nk} = 2\delta_{nk} - \delta_{n+1,k} - \delta_{n-1,k} \tag{11}$$

Suppose the rubber medium is subjected to a deformation where the configuration gradient is A. Then, we can write A in a unique way as a product of a stretch tensor E, and a rotation R as

$$A = ER \tag{12}$$

Let V_1, V_2, V_3, be the unit length eigenvectors corresponding to the eigenvalues λ_1, λ_2, λ_3 of E.

Suppose at time t_0 the portion of the PC-molecule between the n-th bead and $(n + 1)$-th bead is contained in a (CC)-cell of dimension L_{ni} in the V_i-direction. Corresponding to this CC-cell we write the vector

$$b_n^{CC}(t_0) = L_{n1}(t_0)V_1 + L_{n2}(t_0)V_2 + L_{n3}(t_0)V_3. \tag{13}$$

For the vector $b_n(t_0)$ we write

$$b_n(t_0) = l_{n1}(t_0)V_1 + l_{n2}(t_0)V_2 + l_{n3}(t_0)V_3. \tag{14}$$

Then,

$$E \cdot b_n^{CC}(t_0) = \lambda_1 L_{n1}(t_0)V_1 + \lambda_2 L_{n2}(t_0)V_2 + \lambda_3 L_{n3}(t_0)V_3 \tag{15}$$

$$E \cdot b_n(t_0) = \lambda_1 l_{n1}(t_0)V_1 + \lambda_2 l_{n2}(t_0)V_2 + \lambda_3 l_{n3}(t_0)V_3. \tag{16}$$

Note that

$$\lambda_i l_{ni}(t_0) - l_{ni}(t_0) = \frac{l_{ni}(t_0)}{L_{ni}(t_0)}(\lambda_i L_{ni}(t_0) - L_{ni}(t_0)). \tag{17}$$

Thus,

$$\Delta l_{ni}(t_0) = \frac{l_{ni}(t_0)}{L_{ni}(t_0)}(\Delta L_{ni}(t_0)). \tag{18}$$

Let

$$U_{mn}(t_0) = \sqrt{\frac{2}{N}} \sin \frac{mn\pi}{N}, \quad m, n = 1, 2, \ldots, N-1, \tag{19}$$

and

$$a_m = 4\sin^2(\frac{m\pi}{2N}), \quad m, n = 1, 2, \ldots, N-1. \tag{20}$$

Then U_{mn} is an orthogonal matrix.

Set

$$q_m = \sum_{n=1}^{N-1} U_{nm} b_n. \tag{21}$$

Then,

$$b_n = \sum_{n=1}^{N-1} U_{nk} q_k, \tag{22}$$

and

$$q_m'(t) = -\frac{3kT}{\zeta b^2} a_m q_m + h_m(t), \tag{23}$$

where

$$h_m(t) = \sum_{l=1}^{N-1} U_{lm}(f_{l+1}(t) - f_l(t)). \tag{24}$$

For ease of notation we set

$$C_m = -\frac{3kT}{\zeta b^2} a_m. \tag{25}$$

Immediately after the rubber medium is subjected to the above deformation at $t = t_0$ we have, for a short interval of time $t_0 < t < t_1$,

$$q_m(t) = U_{rp} b_r(t_0) + e^{-C_p(t-t_0)} U_{rp} \frac{l_{ri(t_0)}}{L_{ri}(t_0)} \Delta L_{ri}(t_0) V_i + \int_{t_0}^{t} e^{-C_p(t-s)} h_p(s) ds, \tag{26}$$

where we sum over repeated indices ($r = 1, 2, \ldots, N-1$; $i = 1, 2, 3$).

If the rubber medium is again subjected to instantaneous step deformation at time t_1, then for a short interval of time $t_1 < t < t_2$, we have

$$q_m(t) = U_{rp} b_r(t_0) + U_{rp} \frac{l_{ri(t_0)}}{L_{ri}(t_0)} \Delta L_{ri}(t_0) e^{-C_p(t-t_0)} V_i$$
$$U_{rp} \frac{l_{ri(t_1)}}{L_{ri}(t_1)} \Delta L_{ri}(t_1) e^{-C_p(t-t_1)} V_i + \int_{t_0}^{t} e^{-C_p(t-s)} h_p(s) ds, \qquad (27)$$

From (27) we infer, letting $\Delta_n = t_n - t_{n-1}$ tend to zero, that

$$q_m(t) = U_{rp} b_r(t_0) + \int_{t_0}^{t} U_{rp} \frac{l_{ri}(s)}{L_{ri}(s)} \frac{dL_{ri}(s)}{ds} e^{-C_p(t-s} V_i + \int_{t_0}^{t} e^{-C_p(t-s)} h_p(s) ds, \qquad (28)$$

From (28), using (22), we have

$$b_n(t) = b_n(t_0) + \int_{t_0}^{t} U_{np} U_{rp} \frac{l_{ri}(s)}{L_{ri}(s)} \frac{dL_{ri}(s)}{ds} e^{-C_p(t-s} V_i$$
$$+ \int_{t_0}^{t} e^{-C_p(t-s)} U_{np} U_{rp} (f_{r+1}(s) - f_r(s)) ds. \qquad (29)$$

3. STRAIN ENERGY DENSITY

Microscopic stress tensor can be written based directly on (29). In this paper we develop a macroscopic model for the stress in the rubber medium based on (29). Toward this goal we use stress energy density function making use of (29). The stress energy function W consists of a portion W_{cc} from the CC-system of molecules and a portion W_{PC} from the PC-system of molecules. Thus, from (13) and (14) above the strain energy density at the n-th bead has the form

$$W = \sum_{n=1}^{N-1} W_{CC}^{n}(L_{n1}, L_{n2}, L_{n3+}) + \sum_{n=1}^{N-1} W_{PC}^{n}(l_{n1}, l_{n2}, l_{n3+}) \qquad (30)$$

The Cauchy stress in the principal direction V_j is given by

$$\tau_j = \sum_{n=1}^{N-1} L_{nj} \frac{\partial W_{CC}^n}{\partial L_{nj}} + \sum_{n=1}^{N-1} \sum_{i=1}^{3} \frac{\partial W_{PC}^n}{\partial l_{ni}} \frac{\partial l_{ni}}{\partial L_{nj}} - P, \qquad (31)$$

where P is the hydrostatic pressure.

Using (17)

$$\tau_j = \sum_{n=1}^{N-1} [L_{nj} \frac{\partial W_{CC}^n}{\partial L_{nj}} + l_{nj} \frac{\partial W_{PC}^n}{\partial l_{nj}}] - P \qquad (32)$$

4. DYNAMIC MODEL

In this section we develop a dynamic model for a rubber rod under tensile deformation. Consider a deformation of the rubber rod of the form

$$(x, y, z) \longrightarrow \left(x + u(x, t), y - \frac{1}{2}u_x(x, t)y, z - \frac{1}{2}u_x(x, t)z\right), \quad 0 < y^2 \ll 1, 0 < z^2 \ll 1.$$

Assuming $u_{xx} = o(u_x)$, we have

$$L_{n1} \approx 1 + u_x, \quad L_{n2} = L_{n3} \approx 1 - \frac{1}{2}u_x, \quad \text{and} \quad V_i = e_i, \quad i = 1, 2, 3$$

where e_i is a unit vector in the i-th direction.

Next, in (31), set $\tau_2 = \tau_3 = 0$, and eliminating P, we get from (31)

$$\tau_1 = \sum_{n=1}^{N-1} \left[L_{n1}\frac{\partial W_{CC}^n}{\partial L_{n1}} - L_{n2}\frac{\partial W_{CC}^n}{\partial L_{n2}} + l_{n1}\frac{\partial W_{PC}^n}{\partial l_{n1}} - l_{n2}\frac{\partial W_{PC}^n}{\partial l_{n2}}\right] \tag{33}$$

From the deformation considered we enforce incompressibility by requiring

$$L_{n2} = L_{n3} \approx \frac{1}{\sqrt{L_{n1}}}, \qquad l_{n2} = l_{n3} \approx \frac{1}{\sqrt{l_{n1}}}$$

In addition, since we have tensile deformation we have $l_{ni} = b_{ni}$.

$$\begin{aligned} b_{n1} &= b_{n1}(t_0) + \int_{t_0}^{t} U_{np}U_{rp}\frac{b_{r1}(s)}{L_{r1}(s)}\frac{dL_{r1}(s)}{ds}e^{-C_p(t-s)}ds \\ &= \int_{t_0}^{t} e^{-C_p(t-s)}U_{np}U_{rp}(f_{r+1}(s) - f_{r+1}(s))ds \quad n = 1, \ldots, N-1. \end{aligned} \tag{34}$$

Next, write $1 + \partial_x u(x, t)$ for $L_{ri}(t)$, $r = 1, 2, \ldots, N-1$, and $1 + \partial_x u_n^{PC}(x, t)$ for $b_{ni}(t)$. Now, setting $t_0 = 0$, we use (34) to write

$$\begin{aligned} 1 + \partial_x u_n^{PC}(x, t) = 1 &+ \partial_x u_n^{PC}(x, 0) \\ &+ \int_0^t U_{nq}U_{rq}\frac{1 + \partial_x u_r^{PC}(x, s)}{1 + \partial_x u_x(x, s)}\partial_{sx}^2 u(x, s)e^{-C_q(t-s)}ds \\ &+ \int_0^t U_{nq}U_{rq}e^{-C_q(t-s)}(f_{r+1}(s) - f_{r+1}(s))ds \end{aligned} \tag{35}$$

Setting $\partial_x u_n^{PC}(x, 0) = 0$,

$$\begin{aligned} \partial_x u_n^{PC}(x, t) \approx &\int_0^t U_{nq}U_{rq}(1 + \partial_x u_r^{PC} - \partial_x u)\partial_{sx}^2 u(x, s)e^{-C_q(t-s)}ds \\ &+ \int_0^t U_{nq}U_{rq}e^{-C_q(t-s)}(f_{r+1}(s) - f_{r+1}(s))ds \end{aligned} \tag{36}$$

$$\partial_x u_n^{PC}(x,t) - \partial_x u(x,t) \approx -\partial_x u(x,0)e^{-C_q t} - \int_0^t U_{nq}U_{rq}\partial_x u(x,s)e^{-C_q(t-s)} + \int_0^t U_{nq}U_{rq}(\partial_x u_r^{PC} - \partial_x u)\partial_{sx}^2 u(x,s)e^{-C_q(t-s)}ds + \int_0^t U_{nq}U_{rq}e^{-C_q(t-s)}(f_{r+1}(s) - f_{r+1}(s))ds \tag{37}$$

Given $u(x,t)$ we can solve the linear integral equation (37) for $\partial_x u_n^{PC}(x,t) - \partial_x u(x,t)$, and hence, for $\partial_x u_n^{PC}(x,t)$, $n = 1, 2 \ldots, N-1$.

An approximation for $\partial_x u_n^{PC}(x,t)$ can be obtained from the linear system of integral equations

$$\partial_x u_n^{PC}(x,t) = \partial_x u(x,t) - \int_0^t C_q U_{nq} \sum_{r=1}^{N-1} U_{rq}\partial_x u(x,s)e^{-C_q(t-s)} + \int_0^t U_{nq}U_{rq}(\partial_x u_r^{PC} - \partial_x u)\partial_{sx}^2 u(x,s)e^{-C_q(t-s)}ds - \partial_x u(x,0)e^{-C_q t} + \int_0^t U_{nq}U_{rq}e^{-C_q(t-s)}(f_{r+1}(s) - f_{r+1}(s))ds, \quad n = 1, \ldots, N-1 \tag{38}$$

The stress σ_{n1} at bead n is given by $\frac{\tau_1(L_{n1}, l_{n1})}{l_{n1}}$. Adding contributions to the stress from all the beads we have

$$\sum_n \sigma_{n1} = \sum_n \frac{\tau_1(1+\partial_x u, 1+\partial_x u_n^{PC})}{1+\partial_x u_n^{PC})} \approx A\partial_x u + \sum_n B_n \partial_x u_n^{PC}, \quad A \geq 0, \quad B_n \geq 0. \tag{39}$$

Then, from momentum balance we write

$$\rho\frac{\partial^2 u}{\partial t^2} - A\partial_x^2 u - \sum_n \partial_x^2 u_n^{PC} = q \tag{40}$$

Using (38) and (40) we have

$$\rho\frac{\partial^2 u}{\partial t^2} - A\partial_x^2 u - B\partial_x^2 u + \int_0^t C_q \sum_n B_n U_{nq} \sum_r U_{rq}\partial_x^2 u(x,s)e^{-C_q(t-s)} = q \tag{41}$$

Next we proceed to get some understanding of the qualitative behaviour of (41).

Consider the Sturm-Liouville problem

$$-y'' - \lambda y = 0$$
$$y(a) - h_0 y'(a) = 0, \;\; h_0 \geq 0$$
$$y(b) + h_1 y'(b) = 0, \;\; h_1 \geq 0$$

We know that there is a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \nearrow \infty$ and corresponding orthonormal eigenfunctions $\psi_1, \psi_2, \ldots, \psi_n, \ldots$.

Consider a solution of (41) in the form

$$u(t,x)=\sum_{n=1}^{\infty}u_n(t)\psi_n(x). \tag{42}$$

Writing

$$q(t,x)=\sum_{n=1}^{\infty}q_n(t)\psi_n(x)$$

we have

$$u_m''(t)+\frac{\lambda_m(A_m+B_m)}{\rho}u_m-\frac{\lambda_m C_q U_{nq}B_n}{\rho}\int_0^t u_m(s)e^{-C_q(t-s)}ds=\frac{q_m}{\rho} \tag{43}$$

Let $u_m^1=u_m,\quad u_m^2=u_m'$. Then,

$$\frac{d}{dt}\begin{pmatrix}u_m^1\\ \\ u_m^2\end{pmatrix}=\begin{pmatrix}u_m^2\\ \\ -\frac{\lambda_m(A+B)}{\rho}u_m^1+\frac{\lambda_m C_q U_{nq}B_n}{\rho}\int_0^t u_m(s)e^{-C_q(t-s)}ds+\frac{q_m}{\rho}\end{pmatrix} \tag{44}$$

Taking Laplace transform in (44) we have

$$\begin{aligned}\zeta\hat{u}_m^1-u_m^1(0)&=\hat{u}_m^2\\ \zeta\hat{u}_m^2-u_m^2(0)&=-\frac{\lambda_m(A+B)}{\rho}u_m^1+\frac{\lambda_m C_q U_{nq}B_n}{\rho}\frac{1}{\zeta+C_q}+\frac{q_m}{\rho}\end{aligned}$$

Let

$$\gamma_1=\sum\nolimits_n U_{n1}B_n$$

We need to investigate the roots of the equation

$$\zeta^3+C_1\zeta^2+\frac{1}{\rho}\lambda_m(A+B)\zeta+\frac{1}{\rho}\lambda_m[(A+B)C_1-C_1\gamma_1]=0 \tag{45}$$

By Routh-Hurwitz Theorem the roots of (45) all have negative real parts provided $AC_1+C_1(B-\gamma_1)>0$. Further, the roots will be to the left half a vertical line given by $x=-M, M>0$. Let

$$\begin{aligned}\gamma_q&=\sum\nolimits_n U_{nq}B_n\\ D_q&=\sum_{q=2}^{\infty}C_q\gamma_q e^{C_q(t-s)}\\ \Gamma(\rho,C_1,\lambda_m,\gamma_1,\zeta)&=\rho\zeta^2(\zeta+C_1)+\lambda_m(A+B)(\zeta+C_1)-\lambda_m C_1\gamma_1\\ H_1(\rho,C_1,\lambda_m,\gamma_1,\zeta)&=\frac{\Gamma}{\zeta(\zeta+C_1)}\\ H_2(\rho,C_1,\lambda_m,\gamma_1,\zeta)&=(u_m'(0)\zeta+u_m(0))H_1\\ \Theta(t)&=\sum_{q=2}^{\infty}D_q e^{-C_q t}\end{aligned}$$

Then,

$$H_1\hat{u}_m - (u'_m(0)\zeta + u_m(0)) = \frac{\lambda_m}{\rho}u'_m\hat{\Theta} + \frac{1}{\rho}\hat{q}_m$$

Taking inverse Laplace transform H_1 and H_2 are given by

$$H_1 = \sum_{i=1}^{3} \kappa_{mi} e^{\zeta_{mi}} t$$

$$H_2 = \sum_{i=1}^{3} \eta_{mi} e^{\zeta_{mi}} t$$

Then,

$$\begin{aligned} u_m(t) &= \int_0^t \left[\int_0^s \left(\sum_{i=1}^{3} \kappa_{mi} e^{\zeta_{mi}} (s-r) \right) u_m(r)dr \right] \Theta(t-s)ds \\ &\quad + \sum_{i=1}^{3} \eta_{mi} e^{\zeta_{mi}} t + \frac{1}{\rho} \sum_{i=1}^{3} \kappa_{mi} \int_0^t e^{\zeta_{mi}} (t-s) q_m(s)ds \\ &= \int_0^t u_m(s) \sum_{i=1}^{3} \sum_{q=2}^{\infty} \frac{\kappa_i D_q}{\lambda_i + C_q} (e^{\zeta_{mi}} (t-s) - e^{-C_q(t-s)})ds \\ &\quad + \sum_{i=1}^{3} \eta_{mi} e^{\zeta_{mi}} t + \frac{1}{\rho} \sum_{i=1}^{3} \kappa_{mi} \int_0^t e^{\zeta_{mi}} (t-s) q_m(s)ds \end{aligned} \tag{46}$$

Now consider the equation

$$w_m(t) = \int_0^t w_m(s) \sum_{i=1}^{3} \sum_{q=2}^{\infty} \frac{\kappa_i D_q}{\lambda_i + C_q} (e^{\zeta_{mi}} (t-s) - e^{-C_q(t-s)})ds + \sum_{i=1}^{3} \eta_{mi} e^{\zeta_{mi}} t \tag{47}$$

The integral equation (47) can be solved uniquely and

$$|w_m(t)| \leq L_m e^{-K_m t}$$

We also have

$$\begin{aligned} u_m(t) - w_m(t) &= \int_0^t (u_m(s) - w_m(s)) \sum_{i=1}^{3} \sum_{q=2}^{\infty} \frac{\kappa_i D_q}{\lambda_i + C_q} (e^{\zeta_{mi}} (t-s) - e^{-C_q(t-s)})ds \\ &\quad + \frac{1}{\rho} \sum_{i=1}^{3} \kappa_{mi} \int_0^t e^{\zeta_{mi}} (t-s) q_m(s)ds \end{aligned} \tag{48}$$

If q_m is white noise the second integral in (48) is a Wiener integral. Then,

$$E[u_m(t) - w_m(t)] = 0$$

Thus,

$$E[u_m(t)] = w_m(t)$$

Let $R(t,s)$ be the resolvent kernel for the integral equation for (47). Then,

$$\begin{aligned} u_m(t) - w_m(t) &= \frac{1}{\rho}\sum_{i=1}^{3}\kappa_{mi}\int_0^t e^{\zeta_{mi}}(t-s)q_m(s)ds \\ &= \int_0^t R(t,s)\frac{1}{\rho}\sum_{i=1}^{3}\kappa_{mi}\int_0^s e^{\zeta_{mi}}(s-r)q_m(r)dr \\ &= \int_0^t q_m(r)\int_r^t R(t,s)\frac{1}{\rho}\sum_{i=1}^{3}\kappa_{mi}e^{\zeta_{mi}}(s-r)ds\,dr \end{aligned} \tag{49}$$

The inner integral in (49) is bounded by a constant of the form

$$d_m\frac{1}{\rho}|\kappa_{mi}|(e^{Re(\zeta_{mi})t} + e^{Re(\zeta_{mi})r})$$

Now we see that

$$E(|u_m(t) - w_m(t)|^2) \leq \Delta_m t$$

$$E(|u_m(t)|^2) \leq |w_m(t)|^2 + \Delta_m t$$

5. NONLINEAR PROBLEM

To obtain higher order terms set

$$\varepsilon_n(x,t) = \partial_x u_n^{PC}(x,t)$$

and from (37) and (39) we have

$$\rho\frac{\partial^2 u}{\partial t^2} - \partial_x\sum_n \sigma_{n1}(\partial_x u, \varepsilon_1, \ldots, \varepsilon_n) = q,$$

$$\begin{aligned} \dot{\varepsilon}_n - (\partial^2_{tx}u)\varepsilon_n = \partial^2_{tx}u &- C_q U_{nq}\delta_q\partial_x u(x,t) \\ &+ \int_0^t C_q^2 U_{nq}\delta_q\partial_x u(x,s)ds \\ &+ \int_0^t C_q U_{nq}U_{rq}(\varepsilon_r(s) - \partial_x u(x,s))\partial^2_{sx}u(x,s))e^{-C_q(t-s)}ds \\ &+ f_{n+1}(s) - f_n(s) - \int_0^t e^{-C_q(t-s)}C_q U_{nq}U_{rq}(f_{r+1}(s) - f_r(s))ds \end{aligned}$$

where $\delta_q = \sum_r U_{rq}$ and

$$\sigma_{n1}(\partial_x u, \varepsilon_1, \ldots, \varepsilon_n) = \sum_n \frac{\tau_1(1+\partial_x u, 1+\varepsilon_n)}{(1+\varepsilon_n)}.$$

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