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EXISTENCE AND UNIQUENESS THEOREM FOR FUZZY INTEGRAL EQUATION OF FRACTIONAL ORDER

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ABSTRACT. We present an existence and uniqueness theorem for integral equations of fractional order involving fuzzy set valued mappings of a real variable whose values are normal, convex, upper semicontinuous and compactly supported fuzzy sets in $\mathbb{R}^n$. The method of successive approximation is the main tool in our analysis.

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1. INTRODUCTION

Dubois and Prade [8, 9] introduced the concept of integration of fuzzy functions. Alternative approaches were later suggested by Goetschel and Voxman [11], Kaleva [13], Nanda [17] and others. While Goetschel and Voxman preferred a Riemann integral type approach, Kaleva chose to define the integral of fuzzy function, using the Lebesgue-type concept of integration. For more information about integration of fuzzy functions and fuzzy integral equations, for instance, see ([1]–[5], [8]–[11], [13]–[17], [19], [21], [23]) and references therein. On the other hand, the first serious attempt to give a logical definition of a fractional derivative is due to Liouville, see [12] and references therein. Now, the fractional calculus topic is enjoying growing interest among scientists and engineers, see [6, 7, 12, 18, 20, 22].

By means of the fuzzy integral due to Kaleva [13], we investigate the fractional fuzzy integral equation, for the fuzzy set-valued mappings of a real variable whose values are normal, convex, upper semi-continuous and compactly supported fuzzy sets in $\mathbb{R}^n$. This equation takes the form

$$y(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s, y(s))}{(t-s)^{1-\alpha}}\, ds, \quad t \in [0, T], \tag{1.1}$$

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where $f : [0,T] \to E^n$ and $g : [0,T] \times E^n \to E^n$, the definition of E^n is given in Section 2.

Definition 1.1. Let $f \in \mathrm{L}^1(a,b)$, $0 \leq a < b < \infty$, and let $\alpha > 0$ be a real number. The fractional integral of order α of Riemann-Liouville type is defined by (see [20, 22])

$$\mathrm{I}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds.$$

Rewrite Eq.(1.1) in the form

$$y(t) = f(t) + \mathrm{I}^\alpha g(t, y(t)), \quad t \geq 0, \tag{1.2}$$

where I^α is the standard Riemann-Liouville fractional integral operator.

In this paper, we prove an existence and uniqueness theorem of a solution to the fuzzy integral equation (1.2). The method of successive approximation is the main tool in our analysis.

2. AUXILIARY FACTS AND RESULTS

This section is devoted to collect some definitions and results which will be needed further on.

Definition 2.1. Let X be a nonempty set. A *fuzzy set* A in X is characterized by its membership function $\mathrm{A} : \mathrm{X} \to [0,1]$ and $\mathrm{A}(x)$, called the membership function of fuzzy set A, is interpreted as the degree of membership of element x in fuzzy set A for each $x \in \mathrm{X}$.

The value zero is used to represent complete non-membership, the value one is used to represent complete membership and values between them are used to represent intermediate degrees of membership.

Example 2.2. The membership function of fuzzy set of real numbers, close to zero, can be defined as follows

$$\mathrm{A}(x) = \frac{1}{1+x^3}.$$

Using this function, we can determine the membership grade of each real number in this fuzzy set, which signifies the degree to which that number is close to zero. For instance, the number 3 is assigned a grade of 0.035, the number 1 a grade of 0.5 and the number 0 a grade of 1.

Example 2.3. Let the membership function of fuzzy set of real numbers, close to one defined as follows

$$\mathrm{B}(x) = \exp(-\gamma(x-1)^2),$$

where γ is a positive real number.

Let $P_k(\mathbb{R}^n)$ denote the collection of all nonempty compact convex subsets of $\mathbb{R}^n$ and define the addition and scalar multiplication in $P_k(\mathbb{R}^n)$ as usual. Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorff metric

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\}$$

where $d(b,A) = \inf\{d(b,a) : a \in A\}$. It is clear that $(P_k(\mathbb{R}^n), d)$ is a complete metric space [14].

A fuzzy set $u \in E^n$ is a function $u : \mathbb{R}^n \to [0,1]$ for which

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex, i.e., for $x,\ y \in \mathbb{R}^n$ and $\beta \in [0,1]$,

$$u(\beta x + (1-\beta)y) \geq \min(u(x), u(y))$$

(iii) u is upper semi-continuous, and

(iv) the closure of $\{x \in \mathbb{R}^n\ :\ u(x) > 0\}$, denoted by $[u]^0$, is compact.

For $0 < \gamma \leq 1$, the $\alpha-$level set $[u]^\gamma$ is define by $[u]^\gamma = \{x \in \mathbb{R}^n\ :\ u(x) \geq \gamma\}$. Then from $(i) - (iv)$, it follows that $[u]^\gamma \in P_k(\mathbb{R}^n)$ for all $0 \leq \gamma \leq 1$.

By Zadeh's extension principle, we can define addition and scalar multiplication in E^n as follows:

$$\begin{aligned} [u+v]^\gamma &= [u]^\gamma + [v]^\gamma, \\ [\lambda\, u]^\gamma &= \lambda\, [u]^\gamma, \end{aligned}$$

where $u,\ v \in E^n$, $\lambda \in \mathbb{R}$ and $0 \leq \gamma \leq 1$. Define $\hat{0}\ :\ \mathbb{R}^n \to [0,1]$ by

$$\hat{0}(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

We call $\hat{0}$ the null element of E^n.

Let $D : E^n \times E^n \longrightarrow [0,\infty)$ be define by

$$D(u,v) = \sup_{0\leq\gamma\leq 1} d\left([u]^\gamma,\ [v]^\gamma\right)$$

where d is the Hausdorff metric defined in $P_k(\mathbb{R}^n)$. Then (E^n, D) is a complete metric space [21]. Also, we know that [13]

(1) $D(u+w, v+w) = D(u,v)$ for $u,\ v,\ w \in E^n$

(2) $D(\lambda u, \lambda v) = |\lambda|\ D(u,v)$ for all $u,\ v \in E^n$ and $\lambda \in \mathbb{R}$

Now, we recall some definitions and theorems concerning integrability properties for the set-valued mapping of a real variable whose values are in (E^n, D) [13, 21].

Definition 2.4. A mapping $F : J \to E^n$ is strongly measurable if for $\gamma \in [0,1]$ the set-valued mapping $F_\gamma : J \to P_k(\mathbb{R}^n)$ defined by $F_\gamma(t) = [f(t)]^\gamma$ is Lebesgue measurable, when $P_k(\mathbb{R}^n)$ is endowed with the topology generated by the Hausdorff metric d.

Definition 2.5. A mapping $F : J \to E^n$ is called strongly bounded if there exists an integrable function h such that $\|y\| \leq h(t)$ for all $y \in F_0(t)$.

Definition 2.6. Let $F :\ J \to E^n$. The integral of F over J, defined by $\int\limits_J F(t)\, dt$, is defined levelwise by

$$\begin{aligned} \left(\int_J F(t)\, dt\right)^\gamma &= \int_J F_\gamma(t)\, dt \\ &= \{f(t)\, dt \mid f : J \to \mathbb{R}^n \text{ is a measurable selection for } F_\gamma\} \end{aligned}$$

A strongly measurable and integrably bounded mapping $F : J \to E^n$ ia said to be integrable over J if $\int\limits_J F(t)\, dt \in E^n$.

Theorem 2.7. If $F :\ J \to E^n$ is strongly measurable and integrably bounded, then F is integrable.

Theorem 2.8. If $F :\ J \to E^n$ is continuous then it is integrable.

Theorem 2.9. If $F :\ J \to E^n$ is integrable and $b \in J$. Then

$$\int_{t_0}^{t_0+a} F(t)\, dt = \int_{t_0}^{b} F(t)\, dt + \int_{b}^{t_0+a} F(t)\, dt$$

Theorem 2.10. If $F,\ G :\ J \to E^n$ be integrable and $\lambda \in \mathbb{R}$. Then

(1) $\int\limits_J (F(t) + G(t))\, dt = \int\limits_J F(t)\, dt + \int\limits_J G(t)\, dt$,
(2) $\int_J \lambda\, F(t)\, dt = \lambda \int\limits_J F(t)\, dt$,
(3) $D(F, G)$ is integrable,
(4) $D\left(\int\limits_J F(t)\, dt, \int\limits_J G(t)\, dt\right) \leq \int\limits_J D(F(t), G(t))\, dt$.

3. MAIN THEOREM

In this section, we will study Eq.(1.2) assuming that the following assumptions are satisfied, Let L and T be positive numbers:

(a_1) $f : [0,T] \to E^n$ is continuous and bounded.
(a_2) $g : [0,T] \times E^n \to E^n$ is continuous and satisfies the Lipschitz condition, i.e.,

$$D\left(g(t, y_2(t)), g(t, y_1(t))\right) \leq L\ D\left(y_2(t), y_1(t)\right),\ t \in [0,T],$$

where $y_i : [0,T] \to E^n$, $i = 1,\ 2$.
(a_3) $g(t, \hat{0})$ is bounded on $[0,T]$.

Now, we are in a position to state and prove our main result in paper

Theorem 3.1. Let the assumptions $(a_1)-(a_3)$ be satisfied. If

$$T < \left(\frac{\Gamma(\alpha+1)}{L}\right)^{\frac{1}{\alpha}},$$

then Eq.(1.2) has a unique solution y on $[0,T]$ and the successive iterations

$$\begin{aligned} y_0(t) &= f(t) \\ y_{n+1}(t) &= f(t) + \mathrm{I}^\alpha g(t, y_n(t)), \quad n = 0,\ 1,\ 2,\ ... \end{aligned} \tag{3.1}$$

are uniformly convergent to y on $[0,T]$.

Proof: First we prove that y_n are bounded on $[0,T]$. We have $y_0 = f(t)$ is bounded, thanks (a_1). Assume that y_{n-1} is bounded. From (3.1) we have

$$\begin{aligned} D(y_n(t),\hat{0}) &= D\left(f(t) + \mathrm{I}^\alpha g(t, y_{n-1}(t)),\ \hat{0}\right) \\ &\le D\left(f(t),\ \hat{0}\right) + D\left(\mathrm{I}^\alpha g(t, y_{n-1}(t)),\ \hat{0}\right) \\ &\le D\left(f(t),\ \hat{0}\right) + \frac{1}{\Gamma(\alpha)}\int_0^t D\left(\frac{g(s, y_{n-1}(s))}{(t-s)^{1-\alpha}}\ ,\ \hat{0}\right) ds \\ &\le D\left(f(t),\ \hat{0}\right) + \frac{1}{\Gamma(\alpha)} \sup_{0\le t\le T} D(g(t, y_{n-1}(t)),\ \hat{0}) \int_0^t \frac{ds}{(t-s)^{1-\alpha}}. \end{aligned}$$

But

$$\begin{aligned} D(g(t, y_{n-1}(t)),\ \hat{0}) &\le D(g(t, y_{n-1}(t)),\ g(t,\ \hat{0})) + D(g(t,\ \hat{0}),\ \hat{0}) \\ &\le L\ D(y_{n-1}(t),\ \hat{0}) + D(g(t,\ \hat{0}),\ \hat{0}). \end{aligned}$$

So

$$\begin{aligned} D(y_n(t),\hat{0}) &\le D\left(f(t),\ \hat{0}\right) + \frac{T^\alpha}{\Gamma(\alpha+1)} \sup_{0\le t\le T}[L\ D(y_{n-1}(t),\ \hat{0}) + D(g(t,\ \hat{0}),\ \hat{0})] \\ &\le D\left(f(t),\ \hat{0}\right) + \sup_{0\le t\le T} D(y_{n-1}(t),\ \hat{0}) + \frac{T^\alpha}{\Gamma(\alpha+1)} \sup_{0\le t\le T} D(g(t,\ \hat{0}),\ \hat{0}). \end{aligned}$$

This proves that y_n is bounded. Therefore, $\{y_n\}$ is a sequence of bounded functions on $[0,T]$.

Second we prove that y_n are continuous on $[0,T]$. For $0 \le t \le \tau \le T$, we have

$$\begin{aligned} D(y_n(t), y_n(\tau)) &\le D(f(t), f(\tau)) + \frac{1}{\Gamma(\alpha)} D\left(\int_0^t \frac{g(s, y_{n-1}(s))}{(t-s)^{1-\alpha}}\ ds, \int_0^\tau \frac{g(s, y_{n-1}(s))}{(\tau-s)^{1-\alpha}}\ ds\right) \\ &\le D\left(f(t), f(\tau)\right) + \frac{1}{\Gamma(\alpha)} D\left(\int_0^t \frac{g(s, y_{n-1}(s))}{(t-s)^{1-\alpha}}\ ds, \int_0^t \frac{g(s, y_{n-1}(s))}{(\tau-s)^{1-\alpha}}\ ds\right) \\ &\quad + \frac{1}{\Gamma(\alpha)} D\left(\int_t^\tau \frac{g(s, y_{n-1}(s))}{(\tau-s)^{1-\alpha}}\ ds,\ \hat{0}\right) \\ &\le D\left(f(t), f(\tau)\right) + \frac{1}{\Gamma(\alpha)}\int_0^t D\left(\frac{g(s, y_{n-1}(s))}{(t-s)^{1-\alpha}}, \frac{g(s, y_{n-1}(s))}{(\tau-s)^{1-\alpha}}\right) ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_t^\tau D\left(\frac{g(s,y_{n-1}(s))}{(\tau-s)^{1-\alpha}},\ \hat{0}\right)\,ds\\
\leq\ & D\left(f(t),f(\tau)\right)+\frac{1}{\Gamma(\alpha)}\sup_{0\leq t\leq T} D(g(t,y_{n-1}(t)),\ \hat{0})\\
&\int_0^t |(t-s)^{\alpha-1}-(\tau-s)^{\alpha-1}|\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\sup_{0\leq t\leq T} D(g(t,y_{n-1}(t)),\ \hat{0})\int_t^\tau \frac{ds}{(\tau-s)^{1-\alpha}}\,ds\\
\leq\ & D\left(f(t),f(\tau)\right)+\frac{1}{\Gamma(\alpha+1)}[|t-\tau|^\alpha-|t^\alpha-\tau^\alpha|]\\
&\sup_{0\leq t\leq T} D(g(t,y_{n-1}(t)),\ \hat{0})\\
&+\frac{1}{\Gamma(\alpha+1)}\ |t-\tau|^\alpha\ \sup_{0\leq t\leq T} D(g(t,y_{n-1}(t)),\ \hat{0})\\
\leq\ & D\left(f(t),f(\tau)\right)+\frac{1}{\Gamma(\alpha+1)}[2\ |t-\tau|^\alpha-|t^\alpha-\tau^\alpha|]\\
&\sup_{0\leq t\leq T} D(g(t,y_{n-1}(t)),\ \hat{0})\\
\leq\ & D\left(f(t),f(\tau)\right)+\frac{1}{\Gamma(\alpha+1)}[2\ |t-\tau|^\alpha-|t^\alpha-\tau^\alpha|]\\
&\sup_{0\leq t\leq T}[L\ D(g(y_{n-1}(t)),\ \hat{0})+D(g(t,\hat{0}),\ \hat{0})].
\end{aligned}
$$

The last inequality, by symmetry, is valid for all $t,\ \tau\in[0,T]$ regardless whether or not $t\leq\tau$. Thus, $D(y_n(t),y_n(\tau))\to 0$ as $t\to\tau$. Therefore, the sequence $\{y_n\}$ is continuous on $[0,T]$.

For $n\geq 1$, we have

$$
\begin{aligned}
D(y_{n+1}(t),y_n(t)) &= \frac{1}{\Gamma(\alpha)}\ D\left(\int_0^t \frac{g(s,y_n(s))}{(t-s)^{1-\alpha}}\ ds,\int_0^t \frac{g(s,y_{n-1}(s))}{(t-s)^{1-\alpha}}\ ds\right)\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t D\left(\frac{g(s,y_n(s))}{(t-s)^{1-\alpha}},\int_0^t \frac{g(s,y_{n-1}(s))}{(t-s)^{1-\alpha}}\right)\,ds\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t D\left(g(s,y_n(s)),g(s,y_{n-1}(s))\right)\frac{ds}{(t-s)^{1-\alpha}}\\
&\leq \frac{1}{\Gamma(\alpha)}\sup_{0\leq t\leq T} D(g(t,y_n(t)),g(t,y_{n-1}(t)))\int_0^t \frac{ds}{(t-s)^{1-\alpha}}\\
&\leq \frac{L\ T^\alpha}{\Gamma(\alpha+1)}\sup_{0\leq t\leq T} D(y_n(t),y_{n-1}(t))\\
&\leq \left(\frac{L\ T^\alpha}{\Gamma(\alpha+1)}\right)^2\sup_{0\leq t\leq T} D(y_{n-1}(t),y_{n-2}(t))\\
&\ \ \vdots\\
&\leq \left(\frac{L\ T^\alpha}{\Gamma(\alpha+1)}\right)^n\sup_{0\leq t\leq T} D(y_1(t),y_0(t)).
\end{aligned}
\tag{3.2}
$$

But

$$\begin{aligned} D(y_1(t), y_0(t)) &= \frac{1}{\Gamma(\alpha)} D\left(\int_0^t \frac{g(s,f(s))}{(t-s)^{1-\alpha}}\, ds, \hat{0}\right) \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t D\left(\frac{g(s,f(s))}{(t-s)^{1-\alpha}}, \hat{0}\right) ds \\ &\leq \frac{1}{\Gamma(\alpha)} \sup_{0\leq t\leq T} D(g(t,f(t)), \hat{0}) \int_0^t \frac{ds}{(t-s)^{1-\alpha}}. \end{aligned}$$

Thus

$$\sup_{0\leq t\leq T} D(y_1(t), y_0(t)) \leq \frac{T^\alpha}{\Gamma(\alpha+1)}[LM+N] := R,$$

where

$$M = \sup_{0\leq t\leq T} D(f(t), \hat{0}) \quad \text{and} \quad N = \sup_{0\leq t\leq T} D(g(t,\hat{0}), \hat{0}).$$

Therefore (3.2) takes the form

$$D(y_{n+1}(t), y_n(t)) \leq R\, \left(\frac{L\, T^\alpha}{\Gamma(\alpha+1)}\right)^n. \tag{3.3}$$

Next, we show that for each $t \in [0,T]$ the sequence $\{y_n(t)\}$ is a Cauchy sequence in E^n. Let $p,\ q$ be such that $q > p$ and $t \in [0,T]$. Then, by using (3.3), we have

$$\begin{aligned} D(y_q(t), y_p(t)) &\leq D(y_q(t), y_{q-1}(t)) + D(y_{q-1}(t), y_{q-2}(t)) + \ldots + D(y_{p+1}(t), y_p(t)) \\ &\leq R\, \left(\frac{L\, T^\alpha}{\Gamma(\alpha+1)}\right)^{q-1} + R\, \left(\frac{L\, T^\alpha}{\Gamma(\alpha+1)}\right)^{q-2} + \ldots + R\, \left(\frac{L\, T^\alpha}{\Gamma(\alpha+1)}\right)^{p} \\ &= R\, \left(\frac{L\, T^\alpha}{\Gamma(\alpha+1)}\right)^{q-1} \left[1 + \frac{\Gamma(\alpha+1)}{L\, T^\alpha} + \left(\frac{\Gamma(\alpha+1)}{L\, T^\alpha}\right)^2 + \ldots \right. \\ &\qquad \left. + \left(\frac{\Gamma(\alpha+1)}{L\, T^\alpha}\right)^{q-p-1}\right] \\ &= R\, \left(\frac{L\, T^\alpha}{\Gamma(\alpha+1)}\right)^{q-1} \left[\frac{1 - \left(\frac{\Gamma(\alpha+1)}{L\, T^\alpha}\right)^{q-p}}{1 - \frac{\Gamma(\alpha+1)}{L\, T^\alpha}}\right]. \end{aligned}$$

The right hand side of the last inequality tends to zero as $p,\ q \to \infty$. This implies that $\{y_n(t)\}$ is a Cauchy sequence. Consequently, the sequence $\{y_n(t)\}$ is convergent, thanks to the completeness of the metric space (E^n, D). If we denote $y(t) = \lim_{n\to\infty} y_n(t)$, then $y(t)$ satisfies (1.2). It is continuous and bounded on $[0,T]$.

To prove the uniqueness, let $x(t)$ be a continuous solution of (1.2) on $[0,T]$. Then

$$x(t) = f(t) + \mathrm{I}^\alpha g(t, x(t)), \quad t \geq 0.$$

Now, for $n \geq 1$, we have

$$\begin{aligned} D(x(t), y_n(t)) &= D\left(\mathrm{I}^\alpha g(t,x(t)), \mathrm{I}^\alpha g(t, y_n(t))\right) \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t D\left(\frac{g(s,x(s))}{(t-s)^{1-\alpha}}, \int_0^t \frac{g(s,y_n(s))}{(t-s)^{1-\alpha}}\right) ds \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t D\left(g(s,x(s)),g(s,y_n(s))\right)\frac{ds}{(t-s)^{1-\alpha}}\\
&\leq \frac{1}{\Gamma(\alpha)}\sup_{0\leq t\leq T} D(g(t,x(t)),g(t,y_n(t)))\int_0^t \frac{ds}{(t-s)^{1-\alpha}}\\
&\leq \frac{L\,T^{\alpha}}{\Gamma(\alpha+1)}\sup_{0\leq t\leq T} D(x(t),y_n(t))\\
&\quad \vdots\\
&\leq \left(\frac{L\,T^{\alpha}}{\Gamma(\alpha+1)}\right)^n \sup_{0\leq t\leq T} D(x(t),y_0(t)).
\end{aligned}
$$

Since $\frac{L\,T^{\alpha}}{\Gamma(\alpha+1)} < 1$

$$\lim_{n\to\infty} y_n(t) = x(t) = y(t), \quad t\in[0,T].$$

This completes the proof.

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