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SOME RESULTS ON PERIODIC SOLUTIONS FOR EVEN ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper presents some new results for the existence of a unique 2π-periodic solution of even order differential equations. Here the assumption in [3, J. H. Chen and D. O'Regan, On periodic solutions for even order differential equations, *Nonlinear Anal.*, (2007), doi: 10.1016/j.na.2007.06.013] that maximal solution of an initial value problem exists is removed.

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1. INTRODUCTION

In this paper, we continue our study on the existence and uniqueness of periodic solutions for the following boundary value problem

$$\begin{cases} \left(g(t)u^{(k)}\right)^{(k)} + \sum_{j=1}^{k-1} \alpha_j u^{(2j)} + (-1)^{k+1} h(t,u) = e(t), \\ u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0, 1, \dots, 2k-1, \end{cases} \tag{1.1}$$

where $t \in [0, 2\pi]$, $u \in \Re^n$, $g(t) \in C^k(\Re)$, $e(t) \in C^1(\Re^n)$, $h(t,u) \in C^1(\Re \times \Re^n)$ are 2π-periodic in t and $\alpha_j, j = 1, \dots, k-1$ are constants.

Throughout this paper we use the following assumption:

(A1) The Jacobian matrix $h_u = (h_{iu_j})$ is a symmetric matrix, $g \in C^k(\Re)$ satisfies $0 < M_1 \le g(t) \le M_2$ on $\Re$ for some constants M_1 and M_2.

In [3], we related (1.1) to an initial value problem, and a new set of sufficient conditions for the existence of a unique 2π-periodic solution of (1.1) was given. We

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showed that the results of [2, 4, 6, 8, 9, 10] are consequences of Theorem 3.1 (i.e., see Theorem 1.1 in this paper) in [3].

Let $\delta : \Re_+ \to \Re_+ \backslash \{0\}$ be defined by

$$\delta(s) = \max_{\|u\| \le s, t \in [0,2\pi]} \left\{ \left(\min_{1 \le i \le n} \{ b_i(u) - \tau(N_i), \omega(N_i + 1) - b_i(u) \} \right)^{-1} \right\}, \tag{1.2}$$

where $b_i(u)$ are the eigenvalues of h_u, $i = 1, 2, \dots, n$, and

$$\tau(N_i) = M_2 N_i^{2k} + \sum_{j=1}^{k-1} (-1)^{j-k} \alpha_j N_i^{2j},$$

$$\omega(N_i + 1) = M_1 (N_i + 1)^{2k} + \sum_{j=1}^{k-1} (-1)^{j-k} \alpha_j (N_i + 1)^{2j}, \quad i = 1, \dots, n,$$

where N_i are nonnegative integers. Without loss of generality, it is always assumed that $\tau(N_i)$, $\omega(N_i)$ are positive nondecreasing sequences in i, respectively, and $\tau(N_i) < \tau(N_{i+1})$, $\omega(N_i + 1) < \omega(N_{i+1} + 1)$.

Theorem 1.1. [3] *Assume that assumption* **(A1)** *holds, and the eigenvalues of* h_u *satisfy*

$$0 < \tau(N_i) < \lambda_i(h_u) < \omega(N_i + 1), \quad i = 1, \dots, n.$$

Suppose also that, for arbitrary $\eta, c_0 \in \Re_+$, *the maximal solution* y *of the initial value problem*

$$\begin{cases} y'(r) = \eta \delta(y(r)), & r \in [0, 1], \\ y(0) = c_0, \end{cases} \tag{1.3}$$

is defined on $[0, 1]$. *Then there exists a unique* 2π*-periodic solution to system* (1.1).

The purpose of this paper is to continue the investigation which began in [3]. Some new sufficient conditions for the existence of a unique 2π-periodic solution of (1.1) are given. In particular we show that the assumption [3] that the existence of the maximal solution of the initial value problem (1.3) is unnecessary.

2. REFORMULATION

Consider the linear operator $L : \mathcal{D}(L) \to \mathcal{X}$ where

$$Lu = (-1)^k \left(g(t) u^{(k)} \right)^{(k)} + (-1)^k \sum_{j=1}^{k-1} \alpha_j u^{(2j)}$$

and a continuously Fréchet differentiable operator $N : \mathcal{D}(L) \to \mathcal{X}$ which is defined by

$$(N(u))(t) = -h(t, u(t)), \quad t \in [0, 2\pi]. \tag{2.1}$$

Then (1.1) is reformulated as

$$Lu + N(u) = (-1)^k e(t). \tag{2.2}$$

Let $\mathcal{X} = \mathcal{L}_n^2[0, 2\pi]$ be the set of all vector-valued functions $u(t) = (u_i(t))_{n\times 1}$ on $[0, 2\pi]$ such that $u_i \in \mathcal{L}^2[0, 2\pi]$ for $i = 1, \ldots, n$. Then $\mathcal{X}$ is a Hilbert space with the following inner product:

$$\langle u, v\rangle = \int_0^{2\pi} u^T(t)v(t)dt,$$

and we denote by $\|\cdot\|$ the norm induced by this inner product. Also, if

$$\mathcal{D}(L) = \Big\{u(t) = (u_1(t), \ldots, u_n(t))^T \mid u^{(i)}(0) = u^{(i)}(2\pi), i = 0, 1, \ldots, 2k-1\ ,$$
$$u_i^{(2k-1)}(t) \text{ absolutely continuous on } [0, 2\pi], \text{ and } u_i^{(2k)}(t) \in \mathcal{L}^2[0, 2\pi]\Big\}, \qquad (2.3)$$

then L is a closed self-adjoint operator on $\mathcal{D}(L)$. Therefore, $\mathcal{D}(L)$ is a Banach space with respect to the graph norm $|||\cdot||| : \mathcal{X} \to \Re$ defined by $|||u||| = \|u\| + \|Lu\|$ (see [3, 5, 8]).

In the sequel E and F will be a Banach space.

Lemma 2.1. [1, 7, p. 175] *$f : E \to F$ is a homeomorphism of E onto F if and only if f is a local homeomorphism and a closed map.*

3. EXISTENCE AND UNIQUENESS

As shown in Section 2, the boundary value problem (1.1) is equivalent to the operator equation

$$G(u) = Lu + N(u) = (-1)^k e(t), \qquad u \in \mathcal{D}(L).$$

Suppose that assumption **(A1)** holds. Let $Q(u(t)) = (h_{iu_j}(t, u))$. Then

$$(N'(u)v)(t) = -(h_{iu_j}(t, u))v(t) = -Q(u)v(t), \quad u, v \in \mathcal{D}(L), t \in [0, 2\pi],$$

and $G'_u = L + N'(u) = L - Q(u)$, where $Q(u)$ is a symmetric matrix.

Let $b_1(u), \ldots, b_n(u)$ be eigenvalues of $Q(u)$, and suppose there exist

$$\tau(N_i) = M_2 N_i^{2k} + \sum_{j=1}^{k-1} (-1)^{j-k}\alpha_j N_i^{2j}$$

and

$$\omega(N_i + 1) = M_1(N_i + 1)^{2k} + \sum_{j=1}^{k-1} (-1)^{j-k}\alpha_j (N_i + 1)^{2j}, \quad i = 1, \ldots, n,$$

such that

$$0 < \tau(N_i) < b_i(u) < \omega(N_i + 1), \quad i = 1, \ldots, n, \qquad (3.1)$$

where $u \in \mathcal{D}(L)$, and $N_i, i = 1, 2, \ldots, n$ are nonnegative integers.

Lemma 3.1. *Suppose condition* (3.1) *holds. Then N is Lipschitz continuous on $\mathcal{D}(L)$.*

Proof. If $u, v \in \mathcal{D}(L)$ then

$$h_u(t,u) - h_u(t,v) = Q(\xi)(u-v),$$

where $\xi(t) = v(t) + \theta(t)(u(t) - v(t)), \quad 0 < \theta(t) < 1.$

By (3.1), we have

$$< (u-v), Q(\xi)(u-v) > \leq b_n(\xi)\|u-v\|^2 \leq \omega(N_n+1)\|u-v\|^2.$$

Hence, if $u \neq v$, we have $\frac{\langle u-v, Q(\xi)(u-v)\rangle}{||u-v||^2} \leq \omega(N_n+1)$ so $||Q(\xi)|| \leq \omega(N_n+1)$, and as a result $\|h_u(t,u) - h_u(t,v)\| \leq \omega(N_n+1)\|u-v\|$. □

Note that $g \in C^k(\Re)$ satisfies $0 < M_1 \leq g(t) \leq M_2$ on $\Re$ for some constants M_1 and M_2 (see assumption **(A1)**). Then, the eigenvalues of the operator L satisfy $\lambda_i(L) \in [\mu_i, \nu_i]$, where $\mu_i \in [\tau_1(N_i), \tau(N_i)]$ and $\nu_i \in [\omega(N_i+1), \omega_1(N_i+1)]$; here

$$\tau_1(N_i) = M_1 N_i^{2k} + \sum_{j=1}^{k-1}(-1)^{j-k}\alpha_j N_i^{2j}$$

and

$$\omega_1(N_i+1) = M_2(N_i+1)^{2k} + \sum_{j=1}^{k-1}(-1)^{j-k}\alpha_j(N_i+1)^{2j}, \quad i = 1, \ldots, n,$$

where $N_i, i = 1, 2, \ldots, n$ are nonnegative integers. Thus, for each fixed point $t \in [0, 2\pi]$, zero is not an eigenvalue of the following eigenvalue problem

$$Lu - Q(u_0)u = \gamma u, \tag{3.2}$$

where $u_0 \in \mathcal{D}(L)$ is fixed. Hence, $L - Q(u_0)$ is invertible at u_0 for each fixed point $t \in [0, 2\pi]$. If the eigenvalues of $Q(u_0)$ are ordered according to $b_1(u_0) \leq b_2(u_0) \leq \cdots \leq b_n(u_0)$, then by the spectral theorem [5, 8, 9],

$$\begin{aligned} \|(L - Q(u_0))^{-1}\| &= \{\text{distance of 0 from the spectrum of } L - Q(u_0)\}^{-1} \\ &\leq \left(\min_{1\leq i\leq n}\{b_i(u_0) - \tau(N_i), \omega(N_i+1) - b_i(u_0)\}\right)^{-1}. \end{aligned} \tag{3.3}$$

That is, for each $u \in \mathcal{D}(L)$, $G'(u)$ is invertible and

$$\|G'(u)^{-1}\| \leq \left(\min_{1\leq i\leq n}\{b_i(u) - \tau(N_i), \omega(N_i+1) - b_i(u)\}\right)^{-1}. \tag{3.4}$$

Theorem 3.2. *Assume that assumption* **(A1)** *holds, and that the eigenvalues of* $Q(u)$ *satisfy* (3.1) *for all* $u \in \mathcal{D}(L)$. *Then there exists a unique function* $u \in \mathcal{D}(L)$ *satisfying the operator equation* $Lu + N(u) = (-1)^k e(t)$ *for arbitrary* $e(t) \in \mathcal{X}$, *i.e., there exists a unique* 2π*-periodic solution to system* (1.1).

Proof. Since zero is not an eigenvalue of $G'_u = L + N'(u)$ for all $u \in \mathcal{D}(L)$, it follows that G'_u is invertible at each u. Hence $L + N(u)$ is a local homeomorphism.

Next, let $u_k \in \mathcal{D}(L), k = 1, 2, \dots$ be such that

$$u_k \to u,\ \ k \to \infty \tag{3.5}$$

and

$$G(u_k) \to y,\ \ k \to \infty \tag{3.6}$$

where $u, y \in \mathcal{X}$. From Lemma 3.1 and (3.5), we know that $h_u(t, u_k) \to h_u(t, u)$ as $k \to \infty$. Hence, $L(u_k) \to y + h_u(t, u)$ as $k \to \infty$. Since L is closed on $\mathcal{D}(L)$, it follows that $u \in \mathcal{D}(L)$. That is, $G(u) = y$, i.e., G is closed on $\mathcal{D}(L)$. Therefore, by Lemma 2.1, G is a homeomorphism of $\mathcal{D}(L)$ onto $\mathcal{X}$. Thus, for each $e \in \mathcal{X}$, there exists a unique 2π-periodic solution u to system (1.1). The proof is complete. □

Remark 3.3. Theorem 3.2 shows that the assumption of the existence of the maximal solution of the initial value problem in Theorem 1.1 (i.e., the main result of [3]) is unnecessary.

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