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POSITIVE SOLUTIONS FOR SYSTEMS OF SECOND ORDER FOUR-POINT NONLINEAR BOUNDARY VALUE PROBLEMS

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ABSTRACT. Intervals of the parameters λ and μ are determined for which there exist positive solutions of the system of four-point nonlinear boundary value problems, $u''(t)+\lambda a(t)f(v)=0$, $v''(t)+\mu b(t)g(u)=0$, for $0<t<1$, and satisfying, $u(0)=\alpha u(\xi), u(1)=\beta u(\eta), v(0)=\alpha v(\xi), v(1)=\beta v(\eta)$. A Guo-Krasnosel'skii fixed point theorem is applied.

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1. INTRODUCTION

We are concerned with determining values of λ and μ (eigenvalues) for which there exist positive solutions for the system of four-point boundary value problems,

$$\begin{aligned} u''(t)+\lambda a(t)f(v(t))=0, &\quad 0<t<1,\\ v''(t)+\mu b(t)g(u(t))=0, &\quad 0<t<1, \end{aligned} \tag{1.1}$$

$$\begin{aligned} u(0)=\alpha u(\xi), &\quad u(1)=\beta u(\eta),\\ v(0)=\alpha v(\xi), &\quad v(1)=\beta v(\eta), \end{aligned} \tag{1.2}$$

where $0<\xi<\eta<1, 0\le \alpha,\beta<1$, and

(A) $f,g\in C([0,\infty),[0,\infty))$;
(B) $a,b\in C([0,1],[0,\infty))$, and each does not vanish identically on any subinterval;
(C) All of

$$f_0:=\lim_{x\to 0^+}\frac{f(x)}{x},\qquad g_0:=\lim_{x\to 0^+}\frac{g(x)}{x},$$
$$f_\infty:=\lim_{x\to\infty}\frac{f(x)}{x}\quad\text{and}\quad g_\infty:=\lim_{x\to\infty}\frac{g(x)}{x},$$

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exist as positive real numbers.

There continues to be high research activity in the study of positive solutions for a variety of boundary value problems. Questions are of both theoretical and applied nature as found in [1, 5, 6, 7, 10, 13, 19, 14, 15]. A good deal of this interest has been directed toward scalar problems, but more recent concentration has been on positive solutions for systems of boundary value problems [11, 12, 16, 18, 20]. The existence of positive solutions for three-point boundary value problems also has been studied extensively in recent years. For some appropriate references we suggest [16] and [17].

Recently in [2], the existence of positive solutions was studied for the scalar second order four-point boundary value problem,

$$x''(t) + \lambda h(t) f(t, x(t)) = 0, \quad 0 < t < T \tag{1.3}$$

$$x(0) = \alpha x(\xi),\ x(1) = \beta x(\eta) \tag{1.4}$$

Moreover, Benchohra *et al.* [4] and Henderson and Ntouyas [8] studied the existence of positive solutions of systems of nonlinear eigenvalue problems. Also, Henderson and Ntouyas [9] dealt with the existence of positive solutions of systems of nonlinear eigenvalue problems for three-point boundary conditions. In this paper, we employ the methods used in some of the previous papers to extend those results to eigenvalue problems for systems of four-point boundary value problems (1.1), (1.2).

Again, a main tool in this paper involves application of the Guo-Krasnosel'skii fixed point theorem for operators leaving a Banach space cone invariant [7]. A Green's function plays a fundamental role in defining an appropriate operator on a suitable cone.

2. SOME PRELIMINARIES

In this section, we state some preliminary lemmas and the well-known Guo-Krasnosel'skii fixed point theorem.

Lemma 2.1. [3] *Let*

$$\delta := \alpha\xi(1-\beta) + (1-\alpha)(1-\beta\eta) \neq 0.$$

The Green's function for the boundary value problem

$$-u''(t) = 0, \quad 0 < t < 1 \tag{2.1}$$

$$u(0) = \alpha u(\xi), \quad u(1) = \beta u(\eta), \tag{2.2}$$

is given by

$$G(t,s)=\begin{cases} s\in[0,\xi]: \begin{cases} \frac{s}{\delta}[(1-\beta\eta)+(\beta-1)t], & s\le t;\\ \frac{t}{\delta}[(1-\beta\eta)+(\beta-1)s+\frac{(\delta-1+\beta\eta)(s-t)}{\delta}, & t\le s;\end{cases}\\ s\in[\xi,\eta]: \begin{cases} \frac{1}{\delta}[(1-\beta\eta)+(\beta-1)t](\alpha\xi-\alpha s+s), & s\le t;\\ \frac{1}{\delta}[(1-\beta\eta)+(\beta-1)s](\alpha\xi-\alpha t+t), & t\le s;\end{cases}\\ s\in[\eta,1]: \begin{cases} \frac{1-s}{\delta}(t-\alpha t+\alpha\xi)+(s-t), & s\le t;\\ \frac{1-s}{\delta}(\alpha\xi-\alpha t+t), & t\le s.\end{cases}\end{cases} \tag{2.3}$$

Lemma 2.2. [3] *Let $0\le\alpha<1/(1-\xi)$, $0\le\beta<1/\eta$. Then the Green's function $G(t,s)$ satisfies*

$$G(t,s)>0, \quad for \quad 0<s,t<1, \tag{2.4}$$

$$\min_{t\in[\xi,\eta]} G(t,s)\ge\gamma\max_{0\le t\le 1} G(t,s) \quad for \quad \xi\le t\le\eta,\ 0<s<1, \tag{2.5}$$

where γ is defined by

$$\gamma=\begin{cases} \min\left\{\frac{1-\eta}{1-\beta\eta},\frac{\alpha\xi+(1-\alpha)\eta}{\alpha\xi},\frac{1-\beta\eta}{\beta(1-\eta)},\frac{\xi}{1-\alpha+\alpha\xi}\right\}, & \alpha\beta\neq 0;\\ \min\left\{\frac{1-\eta}{1-\beta\eta},\frac{1-\beta\eta}{\beta(1-\eta)},\xi\right\}, & \alpha=0,\beta\neq 0;\\ \min\left\{1-\eta,\frac{\alpha\xi+(1-\alpha)\eta}{\alpha\xi},\frac{\xi}{1-\alpha+\alpha\xi}\right\}, & \alpha\neq 0,\beta=0;\\ \min\{1-\eta,\xi\}, & \alpha=\beta=0.\end{cases} \tag{2.6}$$

We note that a pair $(u(t),v(t))$ is a solution of eigenvalue problem (1.1), (1.2) if, and only if,

$$u(t)=\lambda\int_0^1 G(t,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds, \quad 0\le t\le 1,$$

where

$$v(t)=\mu\int_0^1 G(t,s)b(s)g(u(s))ds, \quad 0\le t\le 1.$$

Values of λ for which there are positive solutions (positive with respect to a cone) of (1.1), (1.2) will be determined via applications of the following fixed point theorem.

Theorem 2.3. *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P}\subset\mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let*

$$T:\mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)\to\mathcal{P}$$

be a completely continuous operator such that, either

(i) $||Tu||\le||u||, u\in\mathcal{P}\cap\partial\Omega_1,$ *and* $||Tu||\ge||u||, u\in\mathcal{P}\cap\partial\Omega_2,$ *or*
(ii) $||Tu||\ge||u||, u\in\mathcal{P}\cap\partial\Omega_1,$ *and* $||Tu||\le||u||, u\in\mathcal{P}\cap\partial\Omega_2.$

Then T has a fixed point in $\mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)$.

3. POSITIVE SOLUTIONS IN A CONE

In this section, we apply Theorem 2.3 to obtain solutions in a cone (that is, positive solutions) of (1.1), (1.2). For our construction, let $\mathcal{B} = C[0,1]$ with supremum norm, $\|\cdot\|$, and define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \left\{ x \in \mathcal{B} \mid x(t) \geq 0 \text{ on } [0,1], \text{ and } \min_{t\in[\xi,\eta,]} x(t) \geq \gamma\|x\| \right\}.$$

For our first result, define positive numbers L_1 and L_2 by

$$L_1 := \max\left\{ \left[\gamma\int_\xi^\eta \min_{0\leq s\leq 1} G(s,r)a(r)f_\infty dr\right]^{-1}, \left[\gamma\int_\xi^\eta \min_{0\leq s\leq 1} G(s,r)b(r)g_\infty dr\right]^{-1} \right\},$$

and

$$L_2 := \min\left\{ \left[\int_0^1 \max_{0\leq t\leq 1} G(t,r)a(r)f_0 dr\right]^{-1}, \left[\int_0^1 \max_{0\leq t\leq 1} G(t,r)b(r)g_0 dr\right]^{-1} \right\}.$$

Theorem 3.1. *Assume conditions* (A), (B) *and* (C) *are satisfied. Then, for each* λ, μ *satisfying*

$$L_1 < \lambda, \mu < L_2, \tag{3.1}$$

there exists a pair (u, v) *satisfying* (1.1), (1.2) *such that* $u(t) > 0$ *and* $v(t) > 0$ *on* $(0, 1)$.

Proof. Let λ, μ as in (3.1) and let $\epsilon > 0$ be chosen such that

$$\max\left\{ \left[\gamma\int_\xi^\eta \min_{0\leq s\leq 1} G(s,r)a(r)(f_\infty - \epsilon)dr\right]^{-1}, \left[\gamma\int_\xi^\eta \min_{0\leq s\leq 1} G(s,r)b(r)(g_\infty - \epsilon)dr\right]^{-1} \right\} \leq \lambda, \mu$$

and

$$\lambda, \mu \leq \min\left\{ \left[\int_0^1 \max_{0\leq t\leq 1} G(t,r)a(r)(f_0 + \epsilon)dr\right]^{-1}, \left[\int_0^1 \max_{0\leq t\leq 1} G(t,r)b(r)(g_0 + \epsilon)dr\right]^{-1} \right\}.$$

Define an integral operator $T : \mathcal{P} \to \mathcal{B}$ by

$$Tu(t) := \lambda\int_0^1 G(t,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds, \quad u \in \mathcal{P}. \tag{3.2}$$

We seek suitable fixed points of T in the cone $\mathcal{P}$.

By Lemma 2.2, $T\mathcal{P} \subset \mathcal{P}$. In addition, standard arguments show that T is completely continuous.

Now, from the definitions of f_0 and g_0, there exists an $H_1 > 0$ such that

$$f(x) \leq (f_0 + \epsilon)x \text{ and } g(x) \leq (g_0 + \epsilon)x, \quad 0 < x \leq H_1.$$

Let $u \in \mathcal{P}$ with $\|u\| = H_1$. We first have

$$\begin{aligned}
\mu \max_{0\le s\le 1} \int_0^1 G(s,r)b(r)g(u(r))dr &\le \mu \int_0^1 \max_{0\le s\le 1} G(s,r)b(r)g(u(r))dr \\
&\le \mu \int_0^1 \max_{0\le s\le 1} G(s,r)b(r)(g_0+\epsilon)u(r)dr \\
&\le \mu \int_0^1 \max_{0\le s\le 1} G(s,r)b(r)dr(g_0+\epsilon)\|u\| \\
&\le \|u\| \\
&= H_1.
\end{aligned}$$

As a consequence, we next have

$$\begin{aligned}
Tu(t) &= \lambda \int_0^1 G(t,s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right)ds \\
&\le \lambda \int_0^1 \max_{0\le t\le 1} G(t,s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right)ds \\
&\le \lambda \int_0^1 \max_{0\le t\le 1} G(t,s)a(s)(f_0+\epsilon)\mu \int_0^1 G(s,r)b(r)g(u(r))drds \\
&\le \lambda \int_0^1 \max_{0\le t\le 1} G(t,s)a(s)(f_0+\epsilon)H_1 ds \\
&\le H_1 \\
&= \|u\|.
\end{aligned}$$

So, $\|Tu\| \le \|u\|$. If we set

$$\Omega_1 = \{x \in \mathcal{B} \mid \|x\| < H_1\},$$

then

$$\|Tu\| \le \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.3}$$

Next, from the definitions of f_∞ and g_∞, there exists $\overline{H}_2 > 0$ such that

$$f(x) \ge (f_\infty - \epsilon)x \text{ and } g(x) \ge (g_\infty - \epsilon)x, \quad x \ge \overline{H}_2.$$

Set

$$H_2 = \max\left\{2H_1, \frac{\overline{H}_2}{\gamma}\right\}.$$

and let $u \in \mathcal{P}$ with $\|u\| = H_2$. Then,

$$\min_{t\in[\xi,\eta]} u(t) \ge \gamma\|u\| \ge \overline{H}_2.$$

Consequently we have for $s \in [0,1]$

$$\begin{aligned}
\mu \int_0^1 G(s,r)b(r)g(u(r))dr &\ge \mu \int_\xi^\eta G(s,r)b(r)g(u(r))dr \\
&\ge \mu \int_\xi^\eta G(s,r)b(r)(g_\infty-\epsilon)u(r)dr
\end{aligned}$$

$$\begin{aligned}
&\geq \mu \int_{\xi}^{\eta} \min_{0\leq s\leq 1} G(s,r)b(r)(g_{\infty}-\epsilon)dr\gamma\|u\| \\
&\geq \|u\| \\
&= H_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
Tu(\xi) &= \lambda \int_0^1 G(\xi,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds \\
&\geq \lambda \int_{\xi}^{\eta} G(\xi,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds \\
&\geq \lambda \int_{\xi}^{\eta} G(\xi,s)a(s)(f_{\infty}-\epsilon)\mu\int_0^1 G(s,r)b(r)g(u(r))drds \\
&\geq \lambda\frac{1}{\gamma}\int_{\xi}^{\eta} \min_{0\leq s\leq 1} G(s,\tau)a(\tau)(f_{\infty}-\epsilon)\gamma H_2 d\tau \\
&\geq H_2 \\
&= \|u\|.
\end{aligned}$$

Hence, $\|Tu\| \geq \|u\|$ for $u \in \mathcal{P}$ with $\|u\| = H_2$. So, if we set

$$\Omega_2 = \{x \in \mathcal{B} \mid \|x\| < H_2\},$$

then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.4}$$

Applying Theorem 2.3 to (3.3) and (3.4), we obtain that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. As such, and with v defined by

$$v(t) = \mu\int_0^1 G(t,s)b(s)g(u(s))ds,$$

the pair (u, v) is a desired solution of (1.1), (1.2) for the given λ and μ. The proof is complete. □

Prior to our next result, we define positive numbers L_3 and L_4 by

$$L_3 := \max\left\{\left[\gamma\int_{\xi}^{\eta} \min_{0\leq s\leq 1} G(s,r)a(r)f_0 dr\right]^{-1}, \left[\gamma\int_{\xi}^{\eta} \min_{0\leq s\leq 1} G(s,r)b(r)g_0 dr\right]^{-1}\right\},$$

and

$$L_4 := \min\left\{\left[\int_0^1 \max_{0\leq t\leq 1} G(t,r)a(r)f_{\infty} dr\right]^{-1}, \left[\int_0^1 \max_{0\leq t\leq 1} G(t,r)b(r)g_{\infty} dr\right]^{-1}\right\}.$$

Theorem 3.2. *Assume conditions* (A)–(C) *are satisfied. Then, for each* λ, μ *satisfying*

$$L_3 < \lambda, \mu < L_4, \tag{3.5}$$

there exists a pair (u, v) *satisfying* (1.1), (1.2) *such that* $u(t) > 0$ *and* $v(t) > 0$ *on* $(0, 1)$.

Proof. Let λ, μ be as in (3.5) and let $\epsilon > 0$ be chosen such that

$$\max\left\{\left[\gamma\int_\xi^\eta \min_{0\le s\le 1} G(s,r)a(r)(f_0-\epsilon)dr\right]^{-1},\right.$$
$$\left.\left[\gamma\int_\xi^\eta \min_{0\le s\le 1} G(s,r)b(r)(g_0-\epsilon)dr\right]^{-1}\right\} \le \lambda,\mu$$

and

$$\lambda,\mu \le \min\left\{\left[\int_0^1 \max_{0\le t\le 1} G(t,r)a(r)(f_\infty+\epsilon)dr\right]^{-1},\right.$$
$$\left.\left[\int_0^1 \max_{0\le t\le 1} G(t,r)b(r)(g_\infty+\epsilon)dr\right]^{-1}\right\}.$$

Let T be the cone preserving, completely continuous operator that was defined by (3.2).

From the definitions of f_0 and g_0, there exists $\overline{H_3} > 0$ such that

$$f(x) \ge (f_0-\epsilon)x \text{ and } g(x) \ge (g_0-\epsilon)x, \quad 0 < x \le \overline{H_3}.$$

Also, from the continuity of g at 0 it follows that $g(0) = 0$ and we may consider an $H_3 \in (0, \overline{H_3})$ such that

$$\mu g(x) \le \frac{\overline{H_3}}{\int_0^1 \max_{0\le s\le 1} G(s,r)b(r)dr}, \quad 0 \le x \le H_3.$$

Choose $u \in \mathcal{P}$ with $\|u\| = H_3$. Then

$$\begin{aligned}
\mu\int_0^1 G(s,r)b(r)g(u(r))dr &\le \mu\int_0^1 \max_{0\le s\le 1} G(s,r)b(r)g(u(r))dr\\
&\le \frac{\int_0^1 \max_{0\le s\le 1} G(s,r)b(r)\overline{H_3}dr}{\int_0^1 \max_{0\le s\le 1} G(s,r)b(r)dr}\\
&\le \overline{H_3}.
\end{aligned}$$

Hence,

$$\begin{aligned}
Tu(\xi) &= \lambda\int_0^1 G(\xi,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds\\
&\ge \lambda\int_\xi^\eta G(\xi,s)a(s)f\left(\mu\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds\\
&\ge \lambda\int_\xi^\eta G(\xi,s)a(s)(f_0-\epsilon)\mu\int_0^1 G(s,r)b(r)g(u(r))drds\\
&\ge \lambda\int_\xi^\eta G(\xi,s)a(s)(f_0-\epsilon)\mu\gamma\int_\xi^\eta \min_{0\le s\le 1} G(s,r)b(r)(g_0-\epsilon)\|u\|drds\\
&\ge \lambda\frac{1}{\gamma}\int_\xi^\eta \min_{0\le s\le 1} G(s,\tau)a(s)(f_0-\epsilon)\gamma\|u\|d\tau
\end{aligned}$$

$$\geq \|u\|,$$

and so, $\|Tu\| \geq \|u\|$ for $u \in \mathcal{P}$ with $\|u\| = H_3$. If we put

$$\Omega_3 = \{x \in \mathcal{B} \mid \|x\| < H_3\},$$

then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_3. \tag{3.6}$$

Next, in view of the definitions of f_∞ and g_∞, there exists $\overline{H}_4 > 0$ such that

$$f(x) \leq (f_\infty + \epsilon)x \text{ and } g(x) \leq (g_\infty + \epsilon)x, \quad x \geq \overline{H}_4.$$

Clearly, since g_∞ is assumed to be a positive real number, it follows that g is unbounded at ∞, and so, there exists $\widetilde{H_4} > \max\{2H_3, \overline{H}_4\}$ such that $g(x) \leq g(\widetilde{H_4})$, for $0 < x \leq \widetilde{H_4}$.

Set

$$f^*(t) = \sup_{0\leq s\leq t} f(s), \quad g^*(t) = \sup_{0\leq s\leq t} g(s), \quad \text{for} \quad t \geq 0.$$

Clearly f^* anf g^* are nondecreasing real valued function for which it holds

$$\lim_{x\to\infty} \frac{f^*(x)}{x} = f_\infty, \quad \lim_{x\to\infty} \frac{g^*(x)}{x} = g_\infty.$$

Hence, there exists $H_4 > \overline{H}_4$ such that $f^*(x) \leq f^*(H_4)$, $g^*(x) \leq g^*(H_4)$ for $0 < x \leq H_4$.

Choosing $u \in \mathcal{P}$ with $\|u\| = H_4$, we have

$$\begin{aligned}
Tu(t) &= \lambda \int_0^1 G(t,s)a(s)f\left(\mu \int_0^1 G(s,r)b(r)g(u(r))dr\right)ds \\
&\leq \lambda \int_0^1 \max_{0\leq t\leq 1} G(t,s)a(s)f^*\left(\mu \int_0^1 \max_{0\leq s\leq 1} G(s,r)b(r)g(u(r))dr\right)ds \\
&\leq \lambda \int_0^1 \max_{0\leq t\leq 1} G(t,s)a(s)f^*\left(\mu \int_0^1 \max_{0\leq s\leq 1} G(s,r)b(r)g^*(u(r))dr\right)ds \\
&\leq \lambda \int_0^1 \max_{0\leq t\leq 1} G(t,s)a(s)f^*\left(\mu \int_0^1 \max_{0\leq s\leq 1} G(s,r)b(r)g^*(H_4)dr\right)ds \\
&\leq \lambda \int_0^1 \max_{0\leq t\leq 1} G(t,s)a(s)f^*\left(\mu \int_0^1 \max_{0\leq s\leq 1} G(s,r)b(r)(g_\infty + \epsilon)H_4dr\right)ds \\
&\leq \lambda \int_0^1 \max_{0\leq t\leq 1} G(t,s)a(s)f^*(H_4)ds \\
&\leq \lambda \int_0^1 \max_{0\leq t\leq 1} G(t,s)a(s)ds(f_\infty + \epsilon)H_4 \\
&\leq H_4 \\
&= \|u\|,
\end{aligned}$$

and so $\|Tu\| \le \|u\|$. For this case, if we let

$$\Omega_4 = \{x \in \mathcal{B} \mid \|x\| < H_4\},$$

then

$$\|Tu\| \le \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_4. \tag{3.7}$$

Application of part (ii) of Theorem 2.3 yields a fixed point u of T belonging to $\mathcal{P} \cap (\overline{\Omega}_4 \setminus \Omega_3)$, which in turn yields a pair (u, v) satisfying (1.1), (1.2) for the chosen value of λ and μ. The proof is complete. □

4. APPLICATIONS AND EXAMPLES

Consider the BVP consisting of the fourth order ordinary differential equation

$$u^{(4)}(t) + 2\phi(t)u'''(t) - \{\phi'(t) - \phi(t)\}\, u''(t) - \rho\psi(t)g\,[u(t)] = 0 \tag{4.1}$$

along with the boundary conditions

$$\begin{aligned} u(0) &= \alpha u(\xi), \quad u(1) = \beta u(\eta), \\ u''(0) &= Au''(\xi), \quad u''(1) = Bu''(\eta), \end{aligned} \tag{4.2}$$

where $0 < \xi < \eta < 1, 0 \le \alpha, \beta < 1$, $A = \alpha e^{-\int_0^\xi \phi(s)ds}$ and $B = \beta e^{-\int_\eta^1 \phi(s)ds}$.

We assume that $0 < \alpha < 1/(1-\xi)$, $0 \le \beta < 1/\eta$ and that

(A$_1$) $g \in C([0,\infty), [0,\infty))$;
(B$_1$) $a_0 > 0, \phi \in C([0,1],)$, $\psi \in C([0,1], [0,\infty))$, and ψ does not vanish identically on any subinterval of $[0,1]$;
(C$_1$) the limits

$$g_0 := \lim_{x\to 0^+} \frac{g(x)}{x} \quad \text{and} \quad g_\infty := \lim_{x\to\infty} \frac{g(x)}{x}$$

exist as positive real numbers.

Set

$$a(t) = a_0 \exp\left[\int_0^t \phi(s)ds\right], \quad t \in [0,1]$$

$$b(t) = \frac{1}{pa_0}\psi(t) \exp\left[-\int_0^t \phi(s)ds\right], \quad t \in [0,1]$$

and let

$$v(t) = -\frac{u''(t)}{\lambda pa(t)}, \quad t \in [0,1],$$

where $\lambda \in (0,1)$.

It is not difficult to verify that the BVP (4.1), (4.2) may be written in the form

$$\begin{aligned} u''(t) + \lambda pa(t)v(t) = 0, \quad 0 < t < 1, \\ v''(t) + \mu b(t)g(u(t)) = 0, \quad 0 < t < 1, \end{aligned} \tag{4.3}$$

$$\begin{aligned} u(0) &= \alpha u(\xi), \quad u(1) = \beta u(\eta), \\ v(0) &= \alpha v(\xi), \quad v(1) = \beta v(\eta). \end{aligned} \tag{4.4}$$

where we have set $\mu = \frac{\rho}{\lambda}$ with $\mu \in (0,1)$. Moreover, it is easy to see that if (A_1), (B_1) and (C_1) hold, then (A), (B) and (C) are satisfied with $f(u) = pu$. Thus, we may apply Theorem 3.2 to the BVP (4.1), (4.2) to obtain the following result.

Theorem 4.1. *Assume conditions* (A_1), (B_1), *and* (C_1) *are satisfied. Let* $\widehat{L}_1$ *and* $\widehat{L}_2$ *be defined by*

$$\widehat{L}_1 := \max\left\{ \left[pa_0 \int_\xi^\eta \min_{0\le s\le 1} G(s,r) e^{\int_0^r \phi(s)ds} dr\right]^{-1}, \right.$$

$$\left. \left[\frac{g_\infty}{pa_0} \int_\xi^\eta \min_{0\le s\le 1} Gs,r)\psi(r) e^{-\int_0^r \phi(s)ds} dr\right]^{-1} \right\},$$

$$\widehat{L}_2 := \min\left\{ \left[pa_0 \int_0^1 G(r,r) e^{\int_0^r \phi(s)ds} dr\right]^{-1}, \left[\frac{g_0}{pa_0} \int_0^1 G(r,r)\psi(r) e^{-\int_0^r \phi(s)ds} dr\right]^{-1} \right\},$$

and set $\widehat{L}_2^* = \min\left\{1, \widehat{L}_2\right\}$. *If*

$$1 < \min\left\{ pa_0 \int_\xi^\eta \min_{0\le s\le 1} G(s,r) e^{\int_0^r \phi(s)ds} dr, \frac{g_\infty}{pa_0} \int_\xi^\eta \min_{0\le s\le 1} G(s,r)\psi(r) e^{-\int_0^r \phi(s)ds} dr \right\},$$

then there exists a pair (u,v) *satisfying* (4.1), (4.2) *such that* $u(t) > 0$ *and* $v(t) > 0$ *on* $(0,1)$.

Example 4.2. As an example of Theorem 4.1 we may take $\phi(t) = q$ and consider the fourth order equation

$$u^{(4)}(t) + 2qu'''(t) + qu''(t) - \rho\psi(t) g\left[u(t)\right] = 0, \tag{4.5}$$

where the function $g \in C([0,\infty),[0,\infty))$ satisfies (C_1).

We set

$$\begin{aligned} a(t) &= a_0 e^{qt}, \quad t \in [0,1], \\ b(t) &= \frac{1}{pa_0}\psi(t) e^{-qt}, \quad t \ge 0. \end{aligned}$$

We have the following result.

Corollary 4.3. *Assume that* $a_0 > 0$, $\psi \in C([0,1],[0,\infty))$, *and* ψ *does not vanish identically on any subinterval of* $[0,1]$. *Let* L_1 *and* L_2 *be defined by*

$$L_1 := \max\left\{ \left[pa_0 \int_\xi^\eta \min_{0\le s\le 1} G(s,r) e^{qr} dr\right]^{-1}, \left[\frac{g_\infty}{pa_0} \int_\xi^\eta \min_{0\le s\le 1} G(s,r)\psi(r) e^{-qr} dr\right]^{-1} \right\},$$

and

$$L_2 := \min\left\{ \left[pa_0 \int_0^1 G(r,r) e^{qr} dr\right]^{-1}, \left[\frac{g_0}{pa_0} \int_0^1 G(r,r)\psi(r) e^{-qr} dr\right]^{-1} \right\}.$$

Then there exists a pair (u, v) *satisfying the BVP* (4.5), (4.2) *such that* $u(t) > 0$ *and* $v(t) > 0$ *on* $(0, 1)$.

Remark 4.4. Some interesting applications of Theorems 3.1 and 3.2 may be given for systems where the functions f and/or g are linear combinations of $\sin u$, u, ue^{-u}, i.e.

$$\begin{aligned} f(u) &= p_1 \sin u + q_1 u + k_1 u e^{-u} \\ g(u) &= p_2 \sin u + q_2 u + k_2 u e^{-u} \end{aligned}$$

where p_i, q_i, k_i for $i = 1, 2$ are suitable real constants or bounded real valued continuous functions.

Example 4.5. Let us now present an example illustrating the first of our main results, Theorem 3.1. For the sake of simplicity we consider the BVP (1.1), (1.2) with $a(t) = t = b(t)$, $\alpha = \beta = \frac{1}{2}$ and $\xi = \frac{1}{3}$, $\eta = \frac{2}{3}$, i.e., the BVP

$$\begin{aligned} &u''(t) + \lambda t f(v(t)) = 0, \quad 0 < t < 1, \\ &v''(t) + \mu t g(u(t)) = 0, \quad 0 < t < 1, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &u(0) = \tfrac{1}{2} u\left(\tfrac{1}{3}\right), \quad u(1) = \tfrac{1}{2} u\left(\tfrac{2}{3}\right), \\ &v(0) = \tfrac{1}{2} v\left(\tfrac{1}{3}\right), \quad v(1) = \tfrac{1}{2} v\left(\tfrac{2}{3}\right), \end{aligned} \tag{4.7}$$

where $f, g \in C([0, \infty), [0, \infty))$ satisfy condition (C).

By simple calculations we find

$$\begin{aligned} \gamma &= \frac{1}{2}, \\ \delta &= \frac{5}{12}, \\ \int_\xi^\eta \min_{0 \le s \le 1} G(s, r) a(r) f_\infty dr &= \frac{1}{40} f_\infty, \\ \int_0^1 \max_{0 \le t \le 1} G(t, r) a(r) f_0 dr &= f_0 \frac{79}{54}, \\ L_1 &= 80 \frac{1}{\min\{f_\infty, g_\infty\}}, \\ L_2 &= \frac{54}{79} \frac{1}{\max\{f_0, g_0\}}. \end{aligned}$$

Taking into consideration the above calculations, asssuming that

$$80 \max\{f_0, g_0\} < \frac{54}{79} \min\{f_\infty, g_\infty\},$$

from Theorem 3.1 we obtain that for each (λ, μ) satisfying $L_1 < \lambda, \mu < L_2$ there exists a pair (u, v) satisfying (4.6), (4.7) such that $u(t) > 0$ and $v(t) > 0$ on $(0, 1)$.

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