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ON THE SUBMARTINGALE CHARACTERIZATION OF BANACH LATTICES

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ABSTRACT. We propose a submartingale characterization of some Banach lattices, viz. AL-spaces and KB-spaces.

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Let E be a Banach lattice, i.e., see [8] and [14], a vector lattice equipped with monotone ($0 \leq x \leq y$ implies $||x|| \leq ||y||$) and complete norm. As usual, if $x \in E$, then $x^+ = \sup\{x, 0\}$, $x^- = \inf\{x, 0\}$, $|x| = x^+ + x^-$, and by E_+ we denote the cone of all positive elements of E. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\mathcal{A}_n)$ is an increasing sequence of sub-σ-algebras of $\mathcal{A}$. Similarly to the real case we will say that the sequence $(X_n, \mathcal{A}_n)$ of E-valued integrable random variables is a submartingale if X_n is $\mathcal{A}_n$-measurable and $\mathbb{E}(X_{n+1}|\mathcal{A}_n) \geq X_n$ a.e. for $n \in \mathbb{N}$. The clasisical Doob's theorem says that the condition $\sup_{n\in\mathbb{N}} \mathbb{E}X_n^+ < \infty$ guarantees the a.s. convergence of the real submartingale (X_n). In the vector case this Doob's condition can be written as

$$\sup_{n\in\mathbb{N}} \mathbb{E}||X_n^+|| < \infty \tag{1}$$

or as

$$\sup_{n\in\mathbb{N}} \mathbb{E}X_n^+ \text{ exists in } E. \tag{2}$$

It is well known, that in general neither is sufficient to assure the almost sure convergence of the submartingale. However, in the separable lattices, every E-valued submartingale (X_n) can be (uniquely) written as

$$X_n = M_n + A_n, \quad n \in \mathbb{N}, \tag{3}$$

where (M_n) is a martingale and the sequence (A_n) of positive functions is predictable, i.e., $A_n \in L^1(\mathcal{A}_{n-1})$, increasing and a.s. convergent; this is the Doob's decomposition.

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According to [12, Theorem 4.1] the lattice E has the Radon-Nikodym property if and only if each E-valued submartingale (X_n) satisfying (1) and

$$\sup_{n\in\mathbb{N}} \mathbb{E}||M_n^-|| < \infty \tag{4}$$

a.s. converges (cf. also [2]). Other martingale characterizations of order or geometric structure of the underlying Banach lattice may found in [6], [10], [11]. In the present paper we propose such characterizations both of AL-spaces and KB-spaces. Remaind that E is an AL-space if $||x+y|| = ||x|| + ||y||$ for $x, y \in E_+$, and E is a KB-space if every norm bounded increasing sequence in E converges. Note that any KB-space has order continuous norm, hence, in particular, σ-order continuous, i.e., if (x_n) decreases and $\inf_{n\in\mathbb{N}} x_n = 0$, then (x_n) converges to zero in norm. The most famous characterization of AL-spaces was given by Kakutani (see [5]), but following [10] (cf. also [11]) we shall use a quite different characterization. Namely, due to Schlotterbeck [9], E is an AL-space if and only if every positive summable sequence in E is absolutely summable. Inspired by J. Szulga [11] and by J. Szulga and W. A. Woyczyński [12] we will prove the following theorems.

Theorem 1. *A separable Banach lattice E is isomorphic to an AL-space if and only if for each E-valued submartingale with the Doob's decomposition* (3)*, condition* (1) *implies* (4).

Theorem 2. *For a Banach lattice E, the following statements are equivalent:*

(α) *E is a KB-space*;

(β) *for every sublattice Y of E and for every Y-valued submartingale (X_n) condition* (1) *implies that $\sup_{n\in\mathbb{N}}\mathbb{E}X_n^+$ exists in Y*;

(γ) *E has σ-order continuous norm and for each E-valued submartingale (X_n) condition* (1) *implies* (2).

Note that Theorem 1 and [12, Theorem 4.1] imply the following well known submartingale convergence theorem.

Corollary 3. *If E is isomorphic to l_1, then every E-valued submartingale satisfying* (1) *converges a.s. to an integrable function.*

Remark 4. In [12, Theorem 4.1] one can find another martingale-type condition equivalent to the Radon-Nikodym property of the separable Banach lattice, viz.

$$\sup_{n\in\mathbb{N}} \mathbb{E}||X_n^+||^p < \infty \quad \text{and} \quad \sup_{n\in\mathbb{N}} \mathbb{E}||M_n^-||^p < \infty$$

for a given $p \in (1,\infty)$ implies the convergence of (X_n) in $L^p(E)$. Note however, that the condition $\sup_{n\in\mathbb{N}} \mathbb{E}||X_n^+||^p < \infty$ in general does not imply (even in the scalar case) that $\sup_{n\in\mathbb{N}} \mathbb{E}||M_n^-||^p < \infty$. To see this it is enough to take (see [11]) $X_n = \sum_{k=1}^n \left(\frac{1}{k^p} - k\mathbf{1}_{S_k}\right)$, where $\{S_k\}$ is a family of independent events with $P(S_k) = \frac{1}{k^{p+1}}$.

Proof of Theorem 1. Fix $n \in \mathbb{N}$ and assume that $(X_n, \mathcal{A}_n)$ is an E-valued submartingale satisfying (1). Since

$$M_0 = X_0, \quad M_n = X_0 + \sum_{k=1}^{n} \big(X_k - \mathbb{E}(X_k|\mathcal{A}_{k-1})\big),$$

it follows that

$$\mathbb{E}(M_n - X_0) = \sum_{k=1}^{n} \big(\mathbb{E}X_k - \mathbb{E}(\mathbb{E}(X_k|\mathcal{A}_{k-1}))\big) = 0,$$

whence

$$\mathbb{E}(M_n - X_0)^+ = \mathbb{E}(M_n - X_0)^-. \tag{5}$$

Suppose first that E is an AL-space. Then it is easily shown that for each integrable function $\Phi : \Omega \to E_+$ we have

$$\mathbb{E}||\Phi|| = ||\mathbb{E}\Phi||.$$

From this and (5) we get

$$\mathbb{E}||(M_n - X_0)^+|| = \mathbb{E}||(M_n - X_0)^-||. \tag{6}$$

On the other hand,

$$M_n^+ \leq X_n^+, \quad (M_n - X_0)^+ \leq M_n^+ + X_0^-, \quad M_n^- \leq (X_0 - M_n)^+ + X_0^-.$$

Therefore

$$||M_n^+|| \leq ||X_n^+||, \quad ||(M_n - X_0)^+|| \leq ||M_n^+|| + ||X_0^-||,$$
$$||M_n^-|| \leq ||(X_0 - M_n)^+|| + ||X_0^-||.$$

Hence, according to (6), we have

$$\begin{aligned}\mathbb{E}||M_n^-|| &\leq \mathbb{E}||(M_n - X_0)^-|| + \mathbb{E}||X_0^-|| \\ &= \mathbb{E}||(M_n - X_0)^+|| + \mathbb{E}||X_0^-|| \leq \mathbb{E}||X_n^+|| + 2\mathbb{E}||X_0^-||.\end{aligned}$$

This gives (4).

Now, if T is a lattice isomorphism of E onto an AL-space, then $(T \circ X_n)$ is a submartingale with the Doob's decomposition

$$T \circ X_n = T \circ M_n + T \circ A_n$$

for which

$$\mathbb{E}||(T \circ X_n)^+|| \leq ||T||\, \mathbb{E}||X_n^+||.$$

Consequently $\sup_{n\in\mathbb{N}} \mathbb{E}||(T \circ X_n)^+|| < \infty$ and by the first part of our proof we obtain

$$\sup_{n\in\mathbb{N}} \mathbb{E}||M_n^-|| \leq ||T^{-1}|| \sup_{n\in\mathbb{N}} \mathbb{E}||(T \circ M_n)^-|| < \infty.$$

For the converse, basing on J. Szulga's idea [11], let (x_n) be a summable sequence of positive elements of E. On account of the above mentioned characterization theorem of U. Schlotterbeck it is enough to prove that the series $\sum_{n=1}^{\infty} x_n$ absolutely converges. Due to [11, Lemma 3] there exists a sequence (ξ_n) of positive independent random variables such that $\mathbb{E}\xi_n = 1$ and for some positive constant c we have

$$\sum_{k=1}^{n} ||x_k|| \leq c\,\mathbb{E}\Big|\Big|\sum_{k=1}^{n} x_k\xi_k\Big|\Big| \tag{7}$$

for every $n \in \mathbb{N}$. Clearly

$$M_n = \sum_{k=1}^{n} x_k(1-\xi_k)$$

is a martingale and $M_n^+ \leq \sum_{k=1}^{n} x_k$. Hence (M_n) is L^1-bounded. From this and

$$\mathbb{E}\Big|\Big|\sum_{k=1}^{n} x_k\xi_k\Big|\Big| \leq \mathbb{E}||M_n|| + \Big|\Big|\sum_{k=1}^{\infty} x_k\Big|\Big|$$

we see that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big|\Big|\sum_{k=1}^{n} x_k\xi_k\Big|\Big| < \infty,$$

and by (7) the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent as desired. □

Proof of Theorem 2. Recall first that for every normed lattice E the following simple fact holds

$$\text{if an increasing sequence } (x_n) \text{ of } E \text{ converges to } x, \text{ then} \sup_{n\in\mathbb{N}} x_n = x. \tag{8}$$

$(\alpha) \Rightarrow (\beta) \wedge (\gamma)$: Assume that Y is a sublattice of E and let (X_n) be an Y-valued submartingale satisfying (1). Since (X_n^+) is a positive submartingale, the sequence $(\mathbb{E}X_n^+)$ increases, and being also bounded, converges in Y. By (8) we get that $\sup_{n\in\mathbb{N}} \mathbb{E}X_n^+$ exists in Y.

$(\gamma) \vee (\beta) \Rightarrow (\alpha)$: Suppose E is not a KB-space. Then by the Tzafriri theorem (see [13], cf. also [8, 5.15 Proposition], c_0 is (lattice) embeddable in E, i.e., there exists a sublattice Y of E and a lattice isomorphism T of c_0 onto Y. Let (Z_n) be an arbitrary martingale with value in c_0 such that (Z_n^+) is L^1-bounded but $(\mathbb{E}Z_n^+)$ is not order bounded. (We can take, e.g., $\big(\sum_{k=1}^{n} r_k e_k,\, \sigma(\{r_1,\ldots,r_n\})\big)$, where r_k are the Rademacher functions on (0,1], and e_k is the vector from c_0 with 1 on the k-place and with 0 everywhere else; cf. [7, pp. 110–111].) Clearly $X_n := T \circ Z_n$ is a Y-valued submartingale satisfying (1).

Assume (γ). Then $x := \sup_{n\in\mathbb{N}} \mathbb{E}X_n^+$ exists in E and $(\mathbb{E}X_n^+)$ converges to x. Consequently $x \in T(c_0)$ and $(\mathbb{E}Z_n^+)$ converges to $T^{-1}(x)$ in c_0. Applying (8) we get the boundedness of $(\mathbb{E}X_n^+)$, a contradiction.

In the case of (β) we see that $x := \sup_{n\in\mathbb{N}} \mathbb{E}X_n^+$ exists in Y. Since Y has σ-order continuous norm (see [1, Exercise 1, p. 245], cf. also [4, p. 94], [3]), it follows that the sequence $(\mathbb{E}X_n^+)$ converges to x, and we continue as above in the case of (γ). □

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