ON THE SUBMARTINGALE CHARACTERIZATION OF BANACH LATTICES

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ABSTRACT. We propose a submartingale characterization of some Banach lattices, viz. $AL$-spaces and $KB$-spaces.

AMS (MOS) Subject Classification. Primary 46B42. Secondary 46B40, 60B11.

Key words: Banach lattice, submartingale, the Doob’s condition

Let $E$ be a Banach lattice, i.e., see [8] and [14], a vector lattice equipped with monotone ($0 \leq x \leq y$ implies $||x|| \leq ||y||$) and complete norm. As usual, if $x \in E$, then $x^+ = \sup \{x, 0\}$, $x^- = \inf \{x, 0\}$, $|x| = x^+ - x^-$, and by $E_+$ we denote the cone of all positive elements of $E$. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\mathcal{A}_n)$ is an increasing sequence of sub-$\sigma$-algebras of $\mathcal{A}$. Similarly to the real case we will say that the sequence $(X_n, \mathcal{A}_n)$ of $E$-valued integrable random variables is a submartingale if $X_n$ is $\mathcal{A}_n$-measurable and $E(X_{n+1}|\mathcal{A}_n) \geq X_n$ a.e. for $n \in \mathbb{N}$. The classical Doob’s theorem says that the condition $\sup_{n \in \mathbb{N}} E|X_n^+| < \infty$ guarantees the a.s. convergence of the real submartingale $(X_n)$. In the vector case this Doob’s condition can be written as

$$\sup_{n \in \mathbb{N}} E||X_n^+|| < \infty$$

or as

$$\sup_{n \in \mathbb{N}} E X_n^+ \text{ exists in } E.$$  

It is well known, that in general neither is sufficient to assure the almost sure convergence of the submartingale. However, in the separable lattices, every $E$-valued submartingale $(X_n)$ can be (uniquely) written as

$$X_n = M_n + A_n, \quad n \in \mathbb{N},$$

where $(M_n)$ is a martingale and the sequence $(A_n)$ of positive functions is predictable, i.e., $A_n \in L^1(\mathcal{A}_{n-1})$, increasing and a.s. convergent; this is the Doob’s decomposition.
According to [12, Theorem 4.1] the lattice $E$ has the Radon-Nikodym property if and only if each $E$-valued submartingale $(X_n)$ satisfying (1) and

$$\sup_{n \in \mathbb{N}} \mathbb{E}||M_n^-|| < \infty$$

(4)
a.s. converges (cf. also [2]). Other martingale characterizations of order or geometric structure of the underlying Banach lattice may found in [6], [10], [11]. In the present paper we propose such characterizations both of $AL$-spaces and $KB$-spaces. Remind that $E$ is an $AL$-space if

$$||x + y|| = ||x|| + ||y||$$

for $x, y \in E_+$, and $E$ is a $KB$-space if every norm bounded increasing sequence in $E$ converges. Note that any $KB$-space has order continuous norm, hence, in particular, $\sigma$-order continuous, i.e., if $(x_n)$ decreases and $\inf_{n \in \mathbb{N}} x_n = 0$, then $(x_n)$ converges to zero in norm. The most famous characterization of $AL$-spaces was given by Kakutani (see [5]), but following [10] (cf. also [11]) we shall use a quite different characterization. Namely, due to Schlotterbeck [9], $E$ is an $AL$-space if and only if every positive summable sequence in $E$ is absolutely summable. Inspired by J. Szulga [11] and by J. Szulga and W. A. Woyczyński [12] we will prove the following theorems.

**Theorem 1.** A separable Banach lattice $E$ is isomorphic to an $AL$-space if and only if for each $E$-valued submartingale with the Doob’s decomposition (3), condition (1) implies (4).

**Theorem 2.** For a Banach lattice $E$, the following statements are equivalent:

1. $E$ is a $KB$-space;
2. for every sublattice $Y$ of $E$ and for every $Y$-valued submartingale $(X_n)$ condition (1) implies that $\sup_{n \in \mathbb{N}} \mathbb{E}X_n^+$ exists in $Y$;
3. $E$ has $\sigma$-order continuous norm and for each $E$-valued submartingale $(X_n)$ condition (1) implies (2).

Note that Theorem 1 and [12, Theorem 4.1] imply the following well known submartingale convergence theorem.

**Corollary 3.** If $E$ is isomorphic to $l_1$, then every $E$-valued submartingale satisfying (1) converges a.s. to an integrable function.

**Remark 4.** In [12, Theorem 4.1] one can find another martingale-type condition equivalent to the Radon-Nikodym property of the separable Banach lattice, viz.

$$\sup_{n \in \mathbb{N}} \mathbb{E}||X_n^+||^p < \infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} \mathbb{E}||M_n^-||^p < \infty$$

for a given $p \in (1, \infty)$ implies the convergence of $(X_n)$ in $L^p(E)$. Note however, that the condition $\sup_{n \in \mathbb{N}} \mathbb{E}||X_n^+||^p < \infty$ in general does not imply (even in the scalar case) that $\sup_{n \in \mathbb{N}} \mathbb{E}||M_n^-||^p < \infty$. To see this it is enough to take (see [11]) $X_n = \sum_{k=1}^n \left( \frac{1}{k^p} - k1_{S_k} \right)$, where $\{S_k\}$ is a family of independent events with $P(S_k) = \frac{1}{k^{p+1}}$. 


Proof of Theorem 1. Fix $n \in \mathbb{N}$ and assume that $(X_n, A_n)$ is an $E$-valued submartingale satisfying (1). Since

$$M_0 = X_0, \quad M_n = X_0 + \sum_{k=1}^{n} (X_k - \mathbb{E}(X_k \mid A_{k-1})),$$

it follows that

$$\mathbb{E}(M_n - X_0) = \sum_{k=1}^{n} (\mathbb{E}X_k - \mathbb{E}(\mathbb{E}(X_k \mid A_{k-1}))) = 0,$$

whence

$$\mathbb{E}(M_n - X_0)^+ = \mathbb{E}(M_n - X_0)^-.$$

(5)

Suppose first that $E$ is an AL-space. Then it is easily shown that for each integrable function $\Phi : \Omega \to E_+$ we have

$$\mathbb{E}\|\Phi\| = \|\mathbb{E}\Phi\|.$$

From this and (5) we get

$$\mathbb{E}\|(M_n - X_0)^+\| = \mathbb{E}\|(M_n - X_0)^-\|.$$

(6)

On the other hand,

$$M_n^+ \leq X_n^+, \quad (M_n - X_0)^+ \leq M_n^+ + X_0^-, \quad M_n^- \leq (X_0 - M_n)^+ + X_0^-.$$

Therefore

$$\|M_n^+\| \leq \|X_n^+\|, \quad \|(M_n - X_0)^+\| \leq \|M_n^+\| + \|X_0^-\|,$$

$$\|M_n^-\| \leq \|(X_0 - M_n)^+\| + \|X_0^-\|.$$

Hence, according to (6), we have

$$\mathbb{E}\|M_n^-\| \leq \mathbb{E}\|(M_n - X_0)^-\| + \mathbb{E}\|X_0^-\|$$

$$= \mathbb{E}\|(M_n - X_0)^+\| + \mathbb{E}\|X_0^-\| \leq \mathbb{E}\|X_n^+\| + 2\mathbb{E}\|X_0^-\|.$$

This gives (4).

Now, if $T$ is a lattice isomorphism of $E$ onto an AL-space, then $(T \circ X_n)$ is a submartingale with the Doob’s decomposition

$$T \circ X_n = T \circ M_n + T \circ A_n$$

for which

$$\mathbb{E}\|(T \circ X_n)^+\| \leq \|T\|\mathbb{E}\|X_n^+\|.$$

Consequently $\sup_{n \in \mathbb{N}} \mathbb{E}\|(T \circ X_n)^+\| < \infty$ and by the first part of our proof we obtain

$$\sup_{n \in \mathbb{N}} \mathbb{E}\|M_n^-\| \leq \|T^{-1}\| \sup_{n \in \mathbb{N}} \mathbb{E}\|(T \circ M_n)^-\| < \infty.$$

For the converse, basing on J. Szulga’s idea [11], let $(x_n)$ be a summable sequence of positive elements of $E$. On account of the above mentioned characterization theorem of U. Schlotterbeck it is enough to prove that the series $\sum_{n=1}^{\infty} x_n$ absolutely
converges. Due to [11, Lemma 3] there exists a sequence \((\xi_n)\) of positive independent random variables such that \(\mathbb{E}\xi_n = 1\) and for some positive constant \(c\) we have

\[
\sum_{k=1}^{n} ||x_k|| \leq c \mathbb{E}\left|\sum_{k=1}^{n} x_k \xi_k\right|
\]

for every \(n \in \mathbb{N}\). Clearly

\[
M_n = \sum_{k=1}^{n} x_k (1 - \xi_k)
\]

is a martingale and \(M_n^+ \leq \sum_{k=1}^{n} x_k\). Hence \((M_n)\) is \(L^1\)-bounded. From this and

\[
\mathbb{E}\left|\sum_{k=1}^{n} x_k \xi_k\right| \leq \mathbb{E}|M_n| + \left|\sum_{k=1}^{\infty} x_k\right|
\]

we see that

\[
\sup_{n \in \mathbb{N}} \mathbb{E}\left|\sum_{k=1}^{n} x_k \xi_k\right| < \infty,
\]

and by (7) the series \(\sum_{n=1}^{\infty} x_n\) is absolutely convergent as desired. \(\square\)

Proof of Theorem 2. Recall first that for every normed lattice \(E\) the following simple fact holds

if an increasing sequence \((x_n)\) of \(E\) converges to \(x\), then \(\sup_{n \in \mathbb{N}} x_n = x\). \(\quad (8)\)

\((\alpha) \Rightarrow (\beta) \land (\gamma)\): Assume that \(Y\) is a sublattice of \(E\) and let \((X_n)\) be an \(Y\)-valued submartingale satisfying (1). Since \((X_n^+)\) is a positive submartingale, the sequence \((\mathbb{E}X_n^+)\) increases, and being also bounded, converges in \(Y\). By (8) we get that \(\sup_{n \in \mathbb{N}} \mathbb{E}X_n^+\) exists in \(Y\).

\((\gamma) \lor (\beta) \Rightarrow (\alpha)\): Suppose \(E\) is not a \(KB\)-space. Then by the Tzafriri theorem (see [13], cf. also [8, 5.15 Proposition], \(c_0\) is (lattice) embeddable in \(E\), i.e., there exists a sublattice \(Y\) of \(E\) and a lattice isomorphism \(T\) of \(c_0\) onto \(Y\). Let \((Z_n)\) be an arbitrary martingale with value in \(c_0\) such that \((Z_n^+)\) is \(L^1\)-bounded but \((\mathbb{E}Z_n^+)\) is not order bounded. (We can take, e.g., \((\sum_{k=1}^{n} r_k e_k, \sigma(\{r_1, \ldots, r_n\}))\), where \(r_k\) are the Rademacher functions on \((0,1]\), and \(e_k\) is the vector from \(c_0\) with 1 on the \(k\)-place and with 0 everywhere else; cf. [7, pp. 110–111].) Clearly \(X_n := T \circ Z_n\) is a \(Y\)-valued submartingale satisfying (1).

Assume \((\gamma)\). Then \(x := \sup_{n \in \mathbb{N}} \mathbb{E}X_n^+\) exists in \(E\) and \((\mathbb{E}X_n^+)\) converges to \(x\). Consequently \(x \in T(c_0)\) and \((\mathbb{E}Z_n^+)\) converges to \(T^{-1}(x)\) in \(c_0\). Applying (8) we get the boundedness of \((\mathbb{E}X_n^+)\), a contradiction.

In the case of \((\beta)\) we see that \(x := \sup_{n \in \mathbb{N}} \mathbb{E}X_n^+\) exists in \(Y\). Since \(Y\) has \(\sigma\)-order continuous norm (see [1, Exercise 1, p. 245], cf. also [4, p. 94], [3]), it follows that the sequence \((\mathbb{E}X_n^+)\) converges to \(x\), and we continue as above in the case of \((\gamma)\). \(\square\)
ACKNOWLEDGMENTS

The research was supported by the Silesian University Mathematics Department (Iterative Functional Equations and Real Analysis program).

REFERENCES