Communications in Applied Analysis **12** (2008), no. 1, 69–82

SIX FUNCTIONALS FIXED POINT THEOREM

RICHARD AVERY[1], JOHNNY HENDERSON[2], AND DONAL O'REGAN[3]

[1]College of Arts and Sciences, Dakota State University
Madison, South Dakota 57042 USA
E-mail: Rich.Avery@dsu.edu

[2]Department of Mathematics, Baylor University
Waco, Texas 76798 USA
E-mail: Johnny_Henderson@baylor.edu

[3]Department of Mathematics, National University of Ireland
Galway, Ireland
E-mail: donal.oregan@nuigalway.ie

ABSTRACT. The Six Functionals Fixed Point Theorem is a generalization of the Five Functionals Fixed Point Theorem as well as the original triple fixed point theorem of Leggett-Williams. In the Six Functionals Fixed Point Theorem, none of the functional boundaries are required to map above or below the boundary in the functional sense. As an application, the existence of at least three positive solutions to a second order right focal boundary value problem is considered by applying both standard and non-standard choices of functionals. An extension to multivalued maps is provided for completeness.

AMS (MOS) Subject Classification. 47H10.

1. INTRODUCTION

A few years ago, Avery [3] generalized the Leggett-Williams Triple Fixed Point Theorem [10], and that generalization is now commonly called the Five Functionals Fixed Point Theorem. Very recently, Avery, Henderson and O'Regan [6] produced another fixed point theorem (called the Four Functionals Fixed Point Theorem), that also generalized the Leggett-Williams Theorem.

In this paper we apply the techniques of the Four Functionals Fixed Point Theorem to generalize the Five Functionals Fixed Point Theorem. This is a major generalization as none of the functional boundaries are required to map above or below the boundary in the functional sense. That is, $\beta(Ax) \leq \beta(x)$ or $\beta(Ax) \geq \beta(x)$, where A is the operator and β is the functional. The use of a second functional at each boundary replaces these assumptions, and moreover, the choice of the second

Received December 13, 2007

functional can substantially simplify the inequality work in existence of solutions arguments. We conclude with an application and an example of the application that demonstrate how functionals can be chosen in verifying the existence of at least three positive solutions to a second order right focal boundary value problem. In the application of this new Six Functionals Fixed Point Theorem, we utilize both standard and non-standard choices of functionals. Many of the techniques that have been utilized while applying multiple fixed point theorems involving functionals ([2, 3, 4, 5, 7, 10] to mention a few) to verify the existence of solutions can be modified, or applied without modification, when utilizing the Six Functionals Fixed Point Theorem. We will demonstrate the standard choice of functionals ("max" and "min" over an interval) on one of the functional boundaries and nonstandard choices on the remaining functional boundaries in our application. An extension, in which the techniques of Agarwal and O'Regan [1] are applied, is provided for completeness of the theory to generalize this new fixed point theorem to maps which obey an axiomatic index theory; so, in particular, the results apply to all multivalued maps in the literature which have a well-defined fixed point index; see [1, 11, 12] and the references therein.

2. PRELIMINARIES

In this section, we will state the definitions that are used in the remainder of the paper.

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* if it satisfies the following two conditions:

(i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P, -x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y \text{ if and only if } y - x \in P.$$

Definition 2.2. An operator is called *completely continuous* if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. A map α is said to be a *nonnegative continuous concave functional* on a cone P of a real Banach space E if

$$\alpha : P \to [0, \infty)$$

is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly, we say the map β is a *nonnegative continuous convex functional* on a cone P of a real Banach space E if

$$\beta : P \to [0, \infty)$$

is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Let α be a nonnegative continuous concave functional on P, and let β be a nonnegative continuous convex functional on P; then, for positive real numbers r and R we define the sets:

$$Q(\beta, R) = \{x \in P : \beta(x) \leq R\}, \tag{2.1}$$

and

$$Q(\alpha, \beta, r, R) = \{x \in P : r \leq \alpha(x) \text{ and } \beta(x) \leq R\}. \tag{2.2}$$

Definition 2.4. Let D be a subset of a real Banach space E. If $r : E \to D$ is continuous with $r(x) = x$ for all $x \in D$, then D is a *retract* of E, and the map r is a *retraction*. The *convex hull* of a subset D of a real Banach space X is given by

$$conv(D) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in D,\ \lambda_i \in [0,1],\ \sum_{i=1}^{n} \lambda_i = 1, \text{ and } n \in \mathbb{N} \right\}.$$

The following theorem is due to Dugundji and a proof can be found in [8, p. 44].

Theorem 2.5. *For Banach spaces X and Y, let $D \subset X$ be closed and let*

$$F : D \to Y$$

be continuous. Then F has a continuous extension

$$\tilde{F} : X \to Y$$

such that

$$\tilde{F}(X) \subset \overline{conv(F(D))}.$$

Corollary 2.6. *Every closed convex set of a Banach space is a retract of the Banach space.*

Note that any cone P of a Banach space E is a retract of E.

3. FIXED POINT INDEX

The following theorem, which establishes the existence and uniqueness of the fixed point index, is from [9, pp. 82–86]; an elementary proof can be found in [8, pp. 58 & 238]. The proof of our main result in the next section will invoke the properties of the fixed point index.

Theorem 3.1. *Let X be a retract of a real Banach space E. Then, for every bounded relatively open subset U of X and every completely continuous operator $A : \overline{U} \to X$ which has no fixed points on ∂U (relative to X), there exists an integer $i(A, U, X)$ satisfying the following conditions:*

($G1$) *Normality:* $i(A,U,X)=1$ *if* $Ax\equiv y_0\in U$ *for any* $x\in\overline{U}$;

($G2$) *Additivity:* $i(A,U,X)=i(A,U_1,X)+i(A,U_2,X)$ *whenever* U_1 *and* U_2 *are disjoint open subsets of* U *such that* A *has no fixed points on* $\overline{U}-(U_1\cup U_2)$;

($G3$) *Homotopy Invariance:* $i(H(t,\cdot),U,X)$ *is independent of* $t\in[0,1]$ *whenever* $H:[0,1]\times\overline{U}\to X$ *is completely continuous and* $H(t,x)\neq x$ *for any* $(t,x)\in[0,1]\times\partial U$;

($G4$) *Permanence:* $i(A,U,X)=i(A,U\cap Y,Y)$ *if* Y *is a retract of* X *and* $A(\overline{U})\subset Y$;

($G5$) *Excision:* $i(A,U,X)=i(A,U_0,X)$ *whenever* U_0 *is an open subset of* U *such that* A *has no fixed points in* $\overline{U}-U_0$;

($G6$) *Solution: If* $i(A,U,X)\neq 0$, *then* A *has at least one fixed point in* U.

Moreover, $i(A,U,X)$ *is uniquely defined.*

4. MAIN RESULT

Theorem 4.1. *Suppose* P *is a cone in a real Banach space* E, α, ψ *and* ζ *are nonnegative continuous concave functionals on* P, β, θ, *and* η *are nonnegative continuous convex functionals on* P, *and there exist nonnegative numbers* l, l', r, r', R *and* R' *such that*

$$A:Q(\beta,R)\to P$$

is a completely continuous operator and

(a) $Q(\beta,R)$ *is a bounded set,*
(b) $Q(\eta,l)$ *and* $Q(\alpha,\beta,r,R)$ *are disjoint subsets of* $Q(\beta,R)$,
(c) $\{x\in P\ :\ \theta(x)<r', r<\alpha(x), R'<\psi(x),\ \text{and}\ \beta(x)<R\}\neq\emptyset$,
(d) $\{x\in P\ :\ l'<\zeta(x)\ \text{and}\ \eta(x)<l\}\neq\emptyset$, *and*
(e) $\{x\in P\ :\ l<\eta(x)\ \text{ and }\ \alpha(x)<r\}\neq\emptyset$.

Let the following properties be satisfied:

(i) $\alpha(Ax)>r$, *for all* $x\in P$ *with* $\alpha(x)=r$, $\beta(x)\le R$, *and* $r'<\theta(Ax)$,
(ii) $\alpha(Ax)>r$, *for all* $x\in P$ *with* $\alpha(x)=r$, $\beta(x)\le R$, *and* $\theta(x)\le r'$,
(iii) $\beta(Ax)<R$, *for all* $x\in P$ *with* $r\le\alpha(x)$, $\beta(x)=R$, *and* $\psi(Ax)<R'$,
(iv) $\beta(Ax)<R$, *for all* $x\in P$ *with* $r\le\alpha(x)$, $\beta(x)=R$, *and* $R'\le\psi(x)$,
(v) $\eta(Ax)<l$, *for all* $x\in P$ *with* $\eta(x)=l$ *and* $\zeta(Ax)<l'$, *and*
(vi) $\eta(Ax)<l$, *for all* $x\in P$ *with* $\eta(x)=l$ *and* $l'\le\zeta(x)$,

then A *has at least three fixed points* x_1, x_2 *and* x_3 *in* $Q(\beta,R)$ *such that*

$$\eta(x_1)\le l,\ r\le\alpha(x_2)\ \text{with}\ \beta(x_2)\le R,\ \text{and}\ \ l<\eta(x_3)\ \text{with}\ \alpha(x_3)<r.$$

Proof. Let

$$W=\{x\in P\ :\ r<\alpha(x)\ \text{ and }\ \beta(x)<R\},$$

$$X=\{x\in P\ :\ \eta(x)<l\},$$

$$Y = \{x \in P \ : \ \beta(x) < R\},$$

and

$$Z = \{x \in P \ : \ l < \eta(x) \ \text{ and } \ \alpha(x) < r\}.$$

Then X, W and Z are open subsets contained in the open set Y of the retract P and we have assumed that X, W are disjoint sets. Thus, the sets X, W, and Z are pairwise disjoint, nonempty, open subsets.

Let $x^* \in \{x \in W \ : \ \theta(x) < r' \text{ and } R' < \psi(x)\}$ (see condition (c)), and let

$$H : [0,1] \times \overline{W} \to P$$

be defined by

$$H(t,x) = (1-t)Ax + tx^*.$$

Clearly, H is continuous and $H\left([0,1] \times \overline{W}\right)$ is relatively compact.

Claim 1: $H(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial W$.

Suppose not; that is, suppose there exists $(t_1, x_1) \in [0,1] \times \partial W$ such that $H(t_1, x_1) = x_1$. Since $x_1 \in \partial W$, we have that $\beta(x_1) = R$ or $\alpha(x_1) = r$.

Case 1 : $\beta(x_1) = R$.

Either $R' \le \psi(Ax_1)$ or $\psi(Ax_1) < R'$. If $\psi(Ax_1) < R'$, then by condition (iii), we have

$$\begin{aligned} \beta(x_1) &= \beta((1-t_1)Ax_1 + t_1x^*) \\ &\le (1-t_1)\beta(Ax_1) + t_1\beta(x^*) \\ &< R, \end{aligned}$$

which is a contradiction. If $R' \le \psi(Ax_1)$, then $R' \le \psi(x_1)$, since

$$\begin{aligned} \psi(x_1) &= \psi((1-t_1)Ax_1 + t_1x^*) \\ &\ge (1-t_1)\psi(Ax_1) + t_1\psi(x^*) \\ &\ge R', \end{aligned}$$

and hence, by condition (iv), we have

$$\begin{aligned} \beta(x_1) &= \beta((1-t_1)Ax_1 + t_1x^*) \\ &\le (1-t_1)\beta(Ax_1) + t_1\beta(x^*) \\ &< R, \end{aligned}$$

which is a contradiction. Thus, $\beta(x_1) \neq R$.

Case 2 : $\alpha(x_1) = r$.

Either $\theta(Ax_1) \le r'$ or $r' < \theta(Ax_1)$. If $r' < \theta(Ax_1)$, then by condition (i), we have

$$\begin{aligned} \alpha(x_1) &= \alpha((1-t_1)Ax_1 + t_1x^*) \\ &\ge (1-t_1)\alpha(Ax_1) + t_1\alpha(x^*) \end{aligned}$$

$$> \; r,$$

which is a contradiction. If $\theta(Ax_1) \leq r'$, then $\alpha(x_1) > r'$, since

$$\begin{aligned}\theta(x_1) &= \theta((1-t_1)Ax_1 + t_1x^*) \\ &\leq (1-t_1)\theta(Ax_1) + t_1\theta(x^*) \\ &\leq r',\end{aligned}$$

and hence, by condition (ii), we have

$$\begin{aligned}\alpha(x_1) &= \alpha((1-t_1)Ax_1 + t_1x^*) \\ &\geq (1-t_1)\alpha(Ax_1) + t_1\alpha(x^*) \\ &> r,\end{aligned}$$

which is a contradiction. Thus, $\alpha(x_1) \neq r$.

Therefore, we have shown that $H(t,x) \neq x$, for all $(t,x) \in [0,1] \times \partial W$, and thus by the homotopy invariance property $(G3)$ and by the normality property $(G1)$ of the fixed point index,

$$i(A, W, P) = i(x^*, W, P) = 1.$$

Also, if we let

$$J : [0,1] \times \overline{Y} \to P$$

be defined by

$$J(t,x) = (1-t)Ax + tx^*,$$

then clearly, J is continuous and $J\left([0,1] \times \overline{Y}\right)$ is relatively compact. Thus, by claim 1, J is free of fixed points on the boundary of Y for all $t \in [0,1]$. Hence, by the homotopy invariance property $(G3)$ and by the normality property $(G1)$ of the fixed point index,

$$i(A, Y, P) = i(x^*, Y, P) = 1.$$

Let $x^{**} \in \{x \in X \ : \ l' < \zeta(x) \text{ and } \eta(x) < l\}$ (see condition (d)), and let

$$K : [0,1] \times \overline{X} \to P$$

be defined by

$$K(t,x) = (1-t)Ax + tx^{**}.$$

Clearly, K is continuous and $K\left([0,1] \times \overline{X}\right)$ is relatively compact.

Claim 2: $K(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial X$.

Suppose not; that is, suppose there exists $(t_2, x_2) \in [0,1] \times \partial X$ such that $K(t_2, x_2) = x_2$. Since $x_2 \in \partial X$, we have that $\eta(x_2) = l$. Either $l' \leq \zeta(Ax_2)$ or $\zeta(Ax_2) < l'$. If $\zeta(Ax_2) < l'$, then by condition (iv), we have

$$\begin{aligned}\eta(x_2) &= \eta((1-t_2)Ax_2 + t_2x^{**}) \\ &\leq (1-t_2)\eta(Ax_2) + t_2\eta(x^{**})\end{aligned}$$

$$< \; l,$$

which is a contradiction. If $l' \le \zeta(Ax_2)$, then $l' \le \zeta(x_2)$, since

$$\begin{aligned} \zeta(x_2) &= \zeta((1-t_2)Ax_2 + t_2x^{**}) \\ &\ge (1-t_2)\zeta(Ax_2) + t_2\zeta(x^{**}) \\ &\ge l', \end{aligned}$$

and hence, by condition (vi), we have

$$\begin{aligned} \eta(x_2) &= \eta((1-t_2)Ax_2 + t_2x^{**}) \\ &\le (1-t_2)\eta(Ax_2) + t_2\eta(x^{**}) \\ &< l, \end{aligned}$$

which is a contradiction. Thus, $\eta(x_2) \neq l$. Therefore, we have shown that $K(t,x) \neq x$, for all $(t,x) \in [0,1] \times \partial X$, and so by the homotopy invariance property $(G3)$ and the normality property $(G1)$ of the fixed point index,

$$i(A, X, P) = i(x^*, X, P) = 1.$$

Also, if $x \in \overline{Y} - (W \cup X \cup Z)$, then either $\eta(x) = l$ or $\alpha(x) = r$ or $\beta(x) = R$ and for all such points we have shown in claims 1 and 2 that $Ax \neq x$. Therefore, by the additivity property $(G2)$ of the fixed point index we have

$$i(A, Y, P) = i(A, W, P) + i(A, X, P) + i(A, Z, P)$$

and hence $i(A, Z, P) = -1$. Therefore, by the solution property $(G6)$ of the fixed point index, the operator A has at least three fixed points

$$x_1 \in X, \; x_2 \in Z, \text{ and } x_3 \in W.$$

$\square$

5. MULTI-VALUED GENERALIZATION

In this section, we provide some background material from fixed point theory related to multi-valued maps.

Let X be a closed, convex subset of some Banach space $E = (E, \|\cdot\|)$. Suppose, for every open subset U of X and every upper semicontinuous map $A : \overline{U^X} \to 2^X$ (here 2^X denotes the family of nonempty subsets of X), which satisfies property (B) (to be specified later), with $x \notin Ax$ for $x \in \partial_X U$ (here $\overline{U^X}$ and $\partial_X U$ denote the closure and boundary of U in X, respectively), there exists an integer, denoted by $i_X(A, U)$, satisfying the following properties:

(P1). If $x_0 \in U$, then $i_X(\hat{x}_0, U) = 1$ (here $\hat{x}_0$ denotes the map whose constant value is x_0);

(P2). For every pair of disjoint open subsets U_1, U_2 of U, such that A has no fixed points on $\overline{U^X}\setminus(U_1\cup U_2)$,

$$i_X(A,U)=i_X(A,U_1)+i_X(A,U_2);$$

(P3). For every upper semicontinuous map $H:[0,1]\times\overline{U^X}\to 2^X$, which satisfies property (B), and $x\notin H(t,x)$ for $(t,x)\in[0,1]\times\partial_X U$,

$$i_X(H(1,\cdot),U)=i_X(H(0,\cdot),U);$$

(P4). If Y is a closed convex subset of X and $A(\overline{U^X})\subseteq Y$, then

$$i_X(A,U)=i_Y(A,U\cap Y).$$

Also, assume the family

$$\left\{ i_X(A,U)\ :\ \begin{array}{l} X \text{ a closed, convex subset of a Banach space } E, \\ U \text{ open in } X, \text{ and } A:\overline{U^X}\to 2^X \text{ is an upper semicontinuous} \\ \text{map that satisfies property (B) with } x\notin Ax \text{ on } \partial_X U \end{array} \right\}$$

is uniquely determined by the properties (P1)–(P4).

We note that property (B) is any property on the map so that the fixed point index is well-defined. Usually in applications, property (B) will mean that the map is compact with convex compact values. Other examples of maps with a well defined fixed point index (e.g., property (B) could mean that the map is countably condensing with convex compact values) can be found in the literature.

If the above hold, notice also that

(P5). For every open subset V of U, such that A has no fixed points on $\overline{U^X}\setminus V$,

$$i_X(A,U)=i_X(A,V);$$

and

(P6). If $i_X(A,U)\neq 0$, then A has at least one fixed point in U.

The proof of the following generalization of Theorem 4.1 to multi-valued maps is essentially the same as the proof of Theorem 4.1 following the techniques applied in [2] and is therefore omitted.

Theorem 5.1. *Let $E=(E,\|\cdot\|)$ be a Banach space and X a closed, convex subset of E. Suppose for every open subset U of X and every upper semicontinuous map $A:\overline{U^X}\to 2^X$, which satisfies property (B) with $x\notin Ax$, for $x\in\partial_X U$, there exists an integer $i_X(A,U)$ satisfying (P1)–(P4). In addition, assume the family*

$$\left\{ i_X(A,U)\ :\ \begin{array}{l} X \textit{ a closed, convex subset of a Banach space } E, \\ U \textit{ open in } X, \textit{ and } A:\overline{U^X}\to 2^X \textit{ is an upper semicontinuous} \\ \textit{map that satisfies property (B) with } x\notin Ax \textit{ on } \partial_X U \end{array} \right\}$$

is uniquely determined by the properties (P1)–(P4). Let $P \subset E$ be a cone in E and suppose that α, ψ and ζ are nonnegative continuous concave functionals on P, β, θ and η are nonnegative continuous convex functionals on P, and there exist nonnegative numbers l, l',r, r', R and R' such that

$$F : Q(\beta, R) \to 2^P$$

is an upper semicontinuous map which satisfies property (B) and

(a) *$Q(\beta, R)$ is a bounded set,*
(b) *$Q(\eta, l)$ and $Q(\alpha, \beta, r, R)$ are disjoint subsets of $Q(\beta, R)$,*
(c) *there exists $x^* \in \{x \in P \ : \ \theta(x) < r', r < \alpha(x), R' < \psi(x), \text{ and } \beta(x) < R\}$, such that the mapping $H : [0, 1] \times Q(\beta, R) \to 2^P$, given by $H(t, x) = (1-t)\, F x + tx^*$, satisfies property (B),*
(d) *there exists $x^{**} \in \{x \in P \ : \ l' < \zeta(x) \text{ and } \eta(x) < l\}$, such that the mapping $K : [0, 1] \times Q(\beta, R) \to 2^P$, given by $K(t, x) = (1-t)\, F x + tx^{**}$, satisfies property (B), and*
(e) *$\{x \in P \ : \ l < \eta(x) \ \text{ and } \ \alpha(x) < r\} \neq \emptyset$.*

Let the following properties be satisfied:

(H1). *If $x \in P$ with $\alpha(x) = r$, $\beta(x) \le R$, and $r' < \theta(y)$ for some $y \in F x$, then $\alpha(y) > r$;*

(H2). *If $x \in P$ with $\alpha(x) = r$, $\beta(x) \le R$, and $\theta(x) \le r'$, then $\alpha(y) > r$ for all $y \in F x$;*

(H3). *If $x \in P$ with $r \le \alpha(x)$, $\beta(x) = R$ and $\psi(y) < R'$ for some $y \in F x$, then $\beta(y) < R$;*

(H4). *If $x \in P$ with $r \le \alpha(x)$, $\beta(x) = R$ and $R' \le \psi(x)$ then $\beta(y) < R$ for all $y \in F x$.*

(H5). *If $x \in P$ with $\eta(x) = l$ and $\zeta(y) < l'$ for some $y \in F x$, then $\eta(y) < l$;*

(H6). *If $x \in P$ with $\eta(x) = l$ and $l' \le \zeta(x)$ then $\eta(y) < l$ for all $y \in F x$.*

Then F has at least three fixed points x_1, x_2, and x_3 in $Q(\beta, R)$.

6. APPLICATION

A standard technique to verify the existence of solutions, by applying a fixed point theorem to a boundary value problem, is to assume the nonlinearity is bounded by a constant on intervals in order to verify certain inequalities, in which case, choosing the minimum of a function over an interval (concave functional) and the maximum of a function over an interval (convex functional) often simplify the arguments. In this application, we will not only demonstrate the standard technique, but we will also choose alternative functionals involving integrals with an upper bound that is linear.

Consider the second-order nonlinear focal boundary-value problem

$$x''(t) + f(x(t)) = 0, \quad t \in (0,1), \tag{6.1}$$

$$x(0) = 0 = x'(1), \tag{6.2}$$

where $f : \mathbb{R} \to [0,\infty)$ is continuous. If x is a fixed point of the operator A defined by

$$Ax(t) := \int_0^1 G(t,s) f(x(s)) ds,$$

where

$$G(t,s) = \begin{cases} t & : \ t \leq s, \\ s & : \ s \leq t, \end{cases}$$

is the Green's function for the operator L defined by

$$Lx(t) := -x'',$$

with right-focal boundary conditions

$$x(0) = 0 = x'(1),$$

then it is well known that x is a solution of the boundary value problem (6.1), (6.2). Throughout this section of the paper we will use the facts that $G(t,s)$ is nonnegative, and for each fixed $s \in [0,1]$, the Green's function is nondecreasing in t.

Define the cone $P \subset E = C[0,1]$ by

$$P := \{x \in E \ : \ \text{x is nonnegative, nondecreasing and concave}\}.$$

Define the concave functionals α and ψ by

$$\alpha(x) := \min_{t \in [1/4,1]} x(t) = x(1/4),$$

$$\psi(x) := \int_{1/4}^1 \frac{s^2}{2} x(s) ds$$

and the convex functionals θ and β by

$$\theta(x) := \max_{t \in [0,1]} x(t) = x(1),$$

$$\beta(x) := \int_0^1 x(s) ds.$$

In the following theorem, we demonstrate how to apply the Six Functionals Fixed Point Theorem, Theorem 4.1, to prove the existence of at least three positive solutions to (6.1), (6.2).

Theorem 6.1. *Suppose there exist positive real numbers M_0, B_0, M, B, r and R, with $\frac{320B}{3(128-131M)} \leq R$, $\frac{320B_0}{(128-131M_0)} \leq r$, $\frac{16r}{3} \leq Mr + B$ and $r < \frac{3R}{128} < 4r$, and a continuous function $f : [0, 2R] \to [0,\infty)$, such that,*

(a) $f(z) < M_0 z + B_0$ for all $z \in [0, 2r/3]$,

(*b*) $f(z) < Mz + B$ *for all* $z \in [0, 2R]$, *and*

(*c*) $f(z) > \frac{16r}{3}$ *for all* $z \in [r, 4r]$.

Then, the operator A *has at least three positive solutions* x_1, x_2 *and* x_3 *in* $Q(\beta, R)$.

Proof. Let $R' = \frac{3R}{128}$, $r' = 4r$, $l = \frac{r}{3}$ and $l' = \frac{3l}{128}$. By the properties of G and f we have that

$$A : Q(\beta, R) \to P$$

is completely continuous. Applying a standard calculus argument, we have that the set $Q(\beta, R)$ is bounded, since if $x \in Q(\beta, R)$, then x is concave, and hence

$$\frac{x(1) - x(0)}{2} \leq \int_0^1 x(s)ds \leq R.$$

Also, it can easily be shown that

$$\frac{R' + 4r}{2} \in \{x \in P \ : \ \theta(x) < r', r < \alpha(x), R' < \psi(x), \text{ and } \beta(x) < R\},$$

$$\frac{l}{4} \in \{x \in P \ : \ l' < \zeta(x) \text{ and } \eta(x) < l\}, \text{ and}$$

$$\frac{r + l}{2} \in \{x \in P \ : \ l < \eta(x) \ \text{ and } \ \alpha(x) < r\},$$

and hence the sets are nonempty. Moreover, if $x \in Q(\eta, l)$, then we have

$$\frac{x(1) - x(0)}{2} \leq \int_0^1 x(s)ds \leq l,$$

and hence

$$\alpha(x) = x(1/4) \leq x(1) \leq 2l < r;$$

thus, $x \notin Q(\alpha, \beta, r, R)$. Therefore, the set conditions (*a*), (*b*), (*c*), (*d*) and (*e*) of Theorem 4.1 are met. Now we verify the functional conditions.

Claim 1: $\alpha(Ax) > r$ for all $x \in Q(\alpha, \beta, r, R)$ with $\alpha(x) = r$ and $r' < \theta(Ax)$.

Let $x \in Q(\alpha, \beta, r, R)$, with $\alpha(x) = r$ and $r' < \theta(Ax)$. Then since $4G(1/4, s) \geq G(1, s)$, for all $s \in [0, 1]$, we have

$$\alpha(Ax) = \int_0^1 G(1/4, s)x(s)ds \geq \frac{\int_0^1 G(1, s)x(s)ds}{4} = \theta(Ax) > \frac{r'}{4} = r.$$

Claim 2: $\alpha(Ax) > r$, for all $x \in \{x \in Q(\alpha, \beta, r, R) \ : \ \theta(x) \leq r'\}$ with $\alpha(x) = r$.

Let $x \in \{x \in Q(\alpha, \beta, r, R) \ : \ \theta(x) \leq r'\}$, with $\alpha(x) = r$. Thus, $r \leq x(s) \leq 4r$, for $s \in [1/4, 1]$, and hence $f(x(s)) \geq \frac{16r}{3}$, for $s \in [1/4, 1]$. Therefore,

$$\begin{aligned} \alpha(Ax) &= \int_0^1 G(1/4, s) \ f(x(s)) \ ds \\ &\geq \int_{1/4}^1 G(1/4, s) \ f(x(s)) \ ds \end{aligned}$$

$$> \int_{1/4}^{1} G(1/4,s) \left(\frac{16r}{3}\right) ds = r.$$

Claim 3: $\beta(Ax) < R$, for all $x \in Q(\alpha,\beta,r,R)$, with $\beta(x) = R$ and $\psi(Ax) < R'$.

Since

$$R' > \psi(Ax) = \int_{1/4}^{1} \frac{s^2}{2} Ax(s)ds \geq \frac{1}{32}\int_{1/4}^{1} Ax(s)ds$$

and

$$3\int_0^{1/4} Ax(s)ds \leq \int_{1/4}^{1} Ax(s)ds,$$

we have

$$32R' > \int_{1/4}^{1} Ax(s)ds$$

and

$$\frac{32}{3}R' > \left(\frac{1}{3}\right)\int_{1/4}^{1} Ax(s)ds \geq \int_0^{1/4} Ax(s)ds.$$

Therefore,

$$R = \frac{128R'}{3} > \int_0^{1/4} Ax(s)ds + \int_{1/4}^{1} Ax(s)ds = \int_0^1 Ax(s)ds = \beta(Ax).$$

Let $\eta = \beta$ and $\zeta = \psi$, then the same argument as in claim 3 can be used to verify that $\eta(Ax) < l$, for all $x \in Q(\eta,l)$, with $\eta(x) = l$ and $\zeta(Ax) < l'$ by simply replacing R by l and R' by l' in the arguments.

Claim 4: $\beta(Ax) < R$, for all $x \in \{x \in Q(\alpha,\beta,r,R) \ : \ R' \leq \psi(x)\}$, with $\beta(x) = R$.

Let $x \in \{x \in Q(\alpha,\beta,r,R) \ : \ R' \leq \psi(x)\}$ with $\beta(x) = R$. Thus, $\psi(x) \geq R' = \frac{3R}{128}$, and hence

$$\begin{aligned}
\beta(Ax) &= \int_0^1 \int_0^1 G(t,s)\, f(x(s))\, ds\, dt \\
&= \int_0^1 \int_0^t s\, f(x(s))\, ds\, dt + \int_0^1 \int_t^1 t\, f(x(s))\, ds\, dt \\
&= \int_0^1 \int_s^1 s\, f(x(s))\, dt\, ds + \int_0^1 \int_0^s t\, f(x(s))\, dt\, ds \\
&= \int_0^1 \left(1 - \frac{s^2}{2}\right) f(x(s))\, ds \\
&< \int_0^1 \left(1 - \frac{s^2}{2}\right) (Mx(s) + B)\, ds \\
&= \frac{5B}{6} + M\beta(x) - M\psi(x) \\
&\leq \frac{5B}{6} + MR - MR' \leq R.
\end{aligned}$$

Note, the same argument as in claim 4 can be used to verify that $\eta(Ax) < l$, for all $x \in \{x \in Q(\eta,l) \ : \ l' \leq \zeta(x)\}$, with $\eta(x) = l$ by simply replacing R by l, R' by l',

M by M_0 and B by B_0 in the arguments. Therefore, the hypotheses of Theorem 4.1 have been satisfied; thus the operator A has at least three positive solutions. □

Example 6.2. The boundary value problem

$$x'' + \frac{13 + 10\ln(x^4+1)}{3 + 5e^{-20(x-.5)^3}} = 0, \tag{6.3}$$

$$x(0) = 0 = x'(1), \tag{6.4}$$

has at least three positive solutions which can easily be verified using a computer algebra system by invoking Theorem 6.1, with

$$r = 1,\ R = 100, M = \frac{48}{131}, M_0 = \frac{8}{131}, B_0 = \frac{8}{3} \text{ and } B = 75.$$

REFERENCES

[1] R. P. Agarwal and D. O'Regan, A generalization of the Petryshyn-Leggett-Williams fixed point theorem with applications to integral inclusions, *Applied Mathematics and Computation* **123** (2001), 263–274.

[2] R. P. Agarwal, R. I. Avery, J. Henderson and D. O'Regan, The five functionals fixed point theorem generalized to multivalued maps, *Journal of Nonlinear and Convex Analysis* **4** (2003), 455–462.

[3] R. I. Avery, A generalization of the Leggett-Williams fixed point theorem, *MSR Hot-Line* **3** (1999), 9–14.

[4] R. I. Avery and J. Henderson, An extension of the five functionals fixed point theorem, *International Journal of Differential Equations and Applications* **1** (2000), 275–290.

[5] R. I. Avery and J. Henderson, Two positive fixed points of nonlinear operators on ordered Banach spaces, *Communications on Applied Nonlinear Analysis* **8** (2001), 27–36.

[6] R. I. Avery, J. Henderson and D. O'Regan, Four functionals fixed point theorem, submitted.

[7] R. I. Avery and A. C. Peterson, Three positive fixed points of nonlinear operators on ordered Banach spaces, *Computers & Mathematics with Applications (Advances in Difference Equations III)* **42** (2001), 313–322.

[8] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.

[9] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[10] R. W. Leggett and L. R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.* **28** (1979), 673–688.

[11] D. O'Regan, Integral inclusions of upper semicontinuous or lower semicontinuous type, *Proc. Amer. Math. Soc.* **124** (1996), 2391–2399.

[12] G. V. Smirnov, *Introduction to the Theory of Differential Inclusions*, Graduate Studies in Mathematics, Vol. 41, American Mathematical Society, Providence, 2002.

[13] W. V. Petryshyn, Multiple positive solutions of multivalued condensing mappings with some applications, *J. Math. Anal. Appl.* **124** (1987), 237–253.