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NEW DISCRETE HALANAY INEQUALITIES: STABILITY OF DIFFERENCE EQUATIONS

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ABSTRACT. Some new nonlinear discrete analogue of the continuous Halanay inequality are established. These inequalities can be used as basic tools in the study of the global asymptotic stability of the equilibrium of certain generalized difference equations.

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1. INTRODUCTION

The investigation of stability and instability of nonlinear difference equations with delays has attracted a lot of attention from many researchers [1–6, 10–15] and references sited therein. In [8], Halanay proved an asymptotic formula for the solutions of a differential inequality involving the "maximum" functional and applied it in the stability theory of linear systems with delay. Such an inequality was called *Halanay inequality* in several works [6, 9–12, 14–16], in which some generalizations as well as new applications can be found. In particular, in [5, 12, 14, 16], the authors considered discrete Halanay-type inequalities to study some discrete version of functional differential equations.

In the following results of Liz and Ferreiro [11], authors showed that some discrete versions of these (max) inequalities can be applied to study the global asymptotic stability of generalized difference equations.

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Theorem A *Let $r > 0$ be a natural number, and let $\{x_n\}_{n\geq r}$ be a sequence of real numbers satisfying the inequality*

$$\triangle x_n \leq -ax_n + b\max\{x_n, x_{n-1}, \ldots, x_{n-r}\}, \quad n \geq 0, \tag{1.1}$$

where $\triangle x_n = x_{n+1} - x_n$. If $0 < b < a \leq 1$, then there exists a constant $\lambda_0 \in (0,1)$ such that

$$x_n \leq \max\{0, x_0, x_{-1}, \ldots, x_{-r}\}\lambda_0^n, \quad n \geq 0,$$

Moreover, λ_0 can be chosen as the root in the interval $(0,1)$ of the equation

$$\lambda^{r+1} + (a-1)\lambda^r - b = 0. \tag{1.2}$$

By a simple use of Theorem A, authors also demonstrated the validity of the following statement, viz. Theorem B.

Theorem B *Assume that $0 < a \leq 1$ and that there exists a positive constant $b < a$ such that*

$$|f(n, x_n, \ldots, x_{n-r})| \leq b\|(x_n, \ldots, x_{n-r})\|_\infty, \quad \forall (x_n, \ldots, x_{n-r}) \in \mathbb{R}^{r+1}. \tag{1.3}$$

Then there exists $\lambda_0 \in (0,1)$ such that

$$|x_n| \leq (\max\{|x_i|\})\lambda_0^n, \quad n \geq 0,$$

for every solution $\{x_n\}$ of

$$\triangle x_n = -ax_n + f(n, x_n, x_{n-1}, \ldots, x_{n-r}), \quad a > 0, \tag{1.4}$$

where λ_0 can be calculated in the form established in Theorem A.

The main aim of the present paper is to establish some nonlinear retarded inequalities, which extend the foregoing Theorem A. We shall also derive new global stability conditions for nonlinear difference equations.

2. HALANAY TYPE DISCRETE INEQUALITIES

Let $\mathbb{R}$ denote the set of all real numbers, $\mathbb{R}^+$ the set of positive real numbers, $\mathbb{R}^0$ the set of nonnegative real numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Z}^+$ the set of positive integers, and $\mathbb{Z}^{-r} = \{z \in \mathbb{Z} : z \geq -r\}$. Consider the following nonlinear difference equation

$$\triangle x_n = f(n, x_n, x_{n-1}, \ldots, x_{n-r}), \quad n \in \mathbb{Z}^+, \tag{2.1}$$

where $\triangle x_n = x_{n+1} - x_n$ and $f : \mathbb{N} \times \mathbb{R}^{r+1} \to \mathbb{R}$. The equation (2.1) is a generalized difference equation (see [2; Section 21, 11]). The initial value problem for this equation requires the knowledge of the initial data $\{x_{-r}, x_{-r+1}, \ldots, x_0\}$. This vector is called the initial string in [5]. For every initial string, there exists a unique solution $\{x_n\}_{n\geq \mathbb{Z}^{-r}}$ of (2.1) that can be calculated using the explicit recurrence formula

$$x_{n+1} = x_n + f(n, x_n, x_{n-1}, \ldots, x_{n-r}), \quad n \in \mathbb{Z}^0. \tag{2.2}$$

In this section, we introduce new discrete inequalities which will be used to derive global stability conditions in the next section.

Theorem 2.1. *Let* $a_i, q_i \in \mathbb{R}_0^+$, $h_i \in \mathbb{Z}^0$, $i = 0, \ldots, r-1$; $a_r, q_r \in \mathbb{R}^+$, $h_r \in \mathbb{Z}^+$, *where* $0 = h_0 < h_1 < \cdots < h_r$ *and* $\sum_{i=0}^r q_i < \sum_{i=0}^r a_i \leq 1$. *Also, let* $\{x_n\}_{n\in\mathbb{Z}^{-h_r}}$ *be a sequence of real numbers satisfying the inequality*

$$\triangle x_n \leq \sum_{i=0}^{r} (q_i x_{n-h_i}^p - a_i x_n), \quad n \in \mathbb{Z}^0, \tag{2.3}$$

where $p \leq 1$ *is a constant. Then there exists a constant*$\lambda_0 \in (0,1)$ *such that*

$$x_n \leq \max\{1, x_0, x_{-h_1}, \ldots, x_{-h_r}\}\lambda_0^n, \quad n \in \mathbb{Z}^0. \tag{2.4}$$

Moreover, λ_0 *can be chosen as the root in the interval* $(0,1)$ *of the equation*

$$\lambda^{p(h_r-n)+n+1} + (a-1)\lambda^{p(h_r-n)+n} - \Big[\max_{0\leq i\leq r}\{1, x_{-h_i}\}\Big]^{p-1} \sum_{i=0}^{r} q_i \lambda^{p(h_r-h_i)} = 0, \tag{2.5}$$

where $n \in \mathbb{Z}^0, a = \sum_{i=0}^r a_i$.

Proof. Let $\{y_n\}$ be a solution of the difference equation

$$\triangle y_n = \sum_{i=0}^{r} (q_i y_{n-h_i}^p - a_i y_n), \quad n \in \mathbb{Z}^0. \tag{2.6}$$

Since $(1 - \sum_{i=0}^r a_i) \geq 0, q_i \in \mathbb{R}^0$, it is easy to prove that if $\{x_n\}$ satisfies (2.3) and $x_n \leq y_n$ for $n = -h_r, \ldots, 0$, then $x_n \leq y_n$ for all $n \in \mathbb{Z}^0$.

Now, if $K \geq 1$, $\lambda \in (0,1)$, the sequence $\{y_n\}$ defined by $y_n = K\lambda^n$ is a solution of equation (2.6) if and only if λ is a solution of (2.5). Define a function F by

$$F(\lambda) = \lambda^{p(h_r-n)+n+1} + (a-1)\lambda^{p(h_r-n)+n} - K^{p-1}\sum_{i=0}^{r} q_i \lambda^{p(h_r-h_i)}, \tag{2.7}$$

where $n \in \mathbb{Z}^0$, $a = \sum_{i=0}^r a_i$. F is continuous on $(0,1]$, $\lim_{\lambda\to 0^+} F(\lambda) = -q_r K^{p-1} < 0$, and $F(1) = a - K^{p-1}\sum_{i=1}^r q_i > 0$. Hence, there exists $\lambda_0 \in (0,1)$ such that $F(\lambda_0) = 0$.

Thus, for this λ_0, $\{K\lambda_0^n\}$ is a solution of (2.6) for every $K \geq 1$. Finally, let $K = \max_{0\leq i\leq r}\{1, x_{-h_i}\}$. Clearly, $x_n \leq y_n$ for all $n = -h_r, \ldots, 0$. Hence, using the first part of the proof, we can conclude that $x_n \leq y_n = \{K\lambda_0^n\}$ for all $n \in \mathbb{Z}_0$. □

By the similar argument used in Theorem 2.1 we obtain the following result.

Theorem 2.2. *Let* $a_i, q_i \in \mathbb{R}_0^+$, $h_i \in \mathbb{Z}^0$, $i = 0, \ldots, r-1$; $a_r, q_r \in \mathbb{R}^+$, $h_r \in \mathbb{Z}^+$, *where* $0 = h_0 < h_1 < \cdots < h_r$. *Let* $\alpha_i, \beta_i \in \mathbb{R}^+$, $\sum_{i=0}^r \alpha_i = 1$, *and* $[(1-\delta)\prod_{i=0}^r \beta_i + \delta\sum_{i=0}^r q_i] < \sum_{i=0}^r a_i \leq 1$, *where* $0 \leq \delta \leq 1$ *is a constant. Also, let* $\{x_n\}_{n\in\mathbb{Z}^{-h_r}}$ *be a sequence of nonnegative real numbers satisfying the inequality*

$$\triangle x_n \leq \sum_{i=0}^r (\delta q_i x_{n-h_i} - a_i x_n) + (1-\delta)\prod_{i=0}^r \beta_i x_{n-h_i}^{\alpha_i}, \quad n \in \mathbb{Z}^0. \tag{2.8}$$

Then there exists a constant $\lambda_0 \in (0,1)$ *such that*

$$x_n \leq \max\{0, x_0, x_{-h_1}, \ldots, x_{-h_r}\}\lambda_0^n, \quad n \in \mathbb{Z}^0. \tag{2.9}$$

Moreover, λ_0 *can be chosen as the root in the interval* $(0,1)$ *of the equation*

$$\lambda + (a-1) - (1-\delta)\left(\prod_{i=0}^r \beta_i\right)\lambda^{-\sum_{i=0}^r \alpha_i h_i} - \delta\sum_{i=0}^r q_i\lambda^{-h_i} = 0, \tag{2.10}$$

where $a = \sum_{i=0}^r a_i$.

Proof. Let $\{y_n\}$ be a solution of the difference equation

$$\triangle y_n = \sum_{i=0}^r (\delta q_i y_{n-h_i} - a_i y_n) + (1-\delta)\prod_{i=0}^r \beta_i y_{n-h_i}^{\alpha_i}, \quad n \in \mathbb{Z}^0. \tag{2.11}$$

Since $(1 - \sum_{i=0}^r a_i) \geq 0$, $q_i \in \mathbb{R}^0$, $\beta_i \in \mathbb{R}^+$, it is easy to prove that if $\{x_n\}$ satisfies (2.8) and $x_n \leq y_n$ for $n = -h_r, \ldots, 0$, then $x_n \leq y_n$ for all $n \in \mathbb{Z}^0$.

Now, if $K \geq 0$, $\lambda \in (0,1)$, the sequence $\{y_n\}$ defined by $y_n = K\lambda^n$ is a solution of equation (2.11) if and only if λ is a solution of (2.10). Define a function F by

$$F(\lambda) = \lambda + (a-1) - (1-\delta)\left(\prod_{i=0}^r \beta_i\right)\lambda^{-\sum_{i=0}^r \alpha_i h_i} - \delta\sum_{i=0}^r q_i\lambda^{-h_i}, \tag{2.12}$$

where $a = \sum_{i=0}^r a_i$. F is continuous on $(0,1]$,

$$\lim_{\lambda\to 0^+} F(\lambda) = a - 1(1-\delta)\left(\prod_{i=0}^r \beta_i\right)\lim_{\lambda\to 0^+}\lambda^{-\sum_{i=0}^r \alpha_i h_i} - \delta\sum_{i=0}^r q_i \lim_{\lambda\to 0^+}\lambda^{-h_i} < 0,$$

and $F(1) = a - [(1-\delta)\prod_{i=0}^r \beta_i + \delta\sum_{i=0}^r q_i] > 0$. Hence, there exists $\lambda_0 \in (0,1)$ such that $F(\lambda_0) = 0$.

Thus, for this λ_0, $\{K\lambda_0^n\}$ is a solution of (2.11) for every $K \geq 0$. Finally, let $K = \max_{0\leq i\leq r}\{0, x_{-h_i}\}$. Clearly, $x_n \leq y_n$ for all $n = -h_r, \ldots, 0$. Hence, using the first part of the proof, we can conclude that $x_n \leq y_n = \{K\lambda_0^n\}$ for all $n \in \mathbb{Z}_0$. □

Remark 2.1. In [11], a discrete Halanay-type inequality was given as in Theorem 2.2 where the inequality (2.8) was replaced by

$$\triangle x_n \leq -ax_n + q\max\{x_n, x_{n-1}, \ldots, x_{n-r}\}, \quad n \in \mathbb{Z}^0. \tag{2.13}$$

where $0 < q < p \leq 1$. Note that if a sequence $\{x_n\}_{n\in\mathbb{Z}^{-r}}$ of positive real numbers satises (2.13), then it also satisfies (2.8). On the other hand, let $r = 1$; $a = 6/7$,

$q = q_0 = q_1 = 1/7$, $\alpha_0 = \alpha_1 = \frac{1}{2}$, $\beta_0 = \frac{3}{7}$, $\beta_1 = \frac{1}{\sqrt{2}}$, then we might easily show that the sequence $\{\frac{1}{2^n}\}_{n\in\mathbb{Z}^{-1}}$ satisfies (2.8) but not (2.13). Indeed,

$$\begin{aligned} \triangle x_n &= \frac{1}{2^{n+1}} - \frac{1}{2^n} = -\frac{1}{2^{n+1}}, \\ &< -\frac{6}{7}\frac{1}{2^n} + (1-\delta)\frac{3}{7}\frac{1}{2^n} + \delta\frac{3}{7}\frac{1}{2^n} = -\frac{3}{7}\frac{1}{2^n}, \end{aligned}$$

with $(1-\delta)\prod_{i-0}^{1}\beta_i + \delta\sum_{i=0}^{1} q_i < \frac{6}{7}$. On the other hand,

$$\triangle x_n = -\frac{1}{2^{n+1}} > -\frac{6}{7}\frac{1}{2^n} + \frac{1}{7}\max\left\{\frac{1}{2^n}, \frac{1}{2^{n-1}}\right\} = -\frac{4}{7}\frac{1}{2^n}. \tag{2.14}$$

Therefore, in the case of positive sequences, the discrete inequality (2.8) is less conservative than the discrete Halanay-type inequality given by (2.13).

3. GLOBAL STABILITY OF DIFFERENCE EQUATIONS

We consider the generalized difference equation

$$\triangle x_n = -ax_n + f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}), \tag{3.1}$$

where $n, h_i \in \mathbb{Z}^+$, $i = 1, \dots, r$, $a > 0$.

Although for every initial string $\{x_{-h_r}, x_{-h_r+1}, \dots, x_0\}$, the solution $\{x_n\}$ of (3.1) can be explicitly calculated by a recurrence formula similar to (2.2), it is in general difficult to investigate the asymptotic behavior of the solutions using that formula. The next result gives an asymptotic estimate by a simple use of the discrete Halanay inequality.

Theorem 3.1. *Let $0 < a \le 1$ and there exist $q_i \in \mathbb{R}_0^+$, $h_i \in \mathbb{Z}^0$, $i = 0, \dots, r-1$; $q_r \in \mathbb{R}^+$, $h_r \in \mathbb{Z}^+$, where $0 = h_0 < h_1 < \cdots < h_r$ and $\sum_{i=0}^{r} q_i < a \le 1$ such that*

$$|f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r})| \le \sum_{i=0}^{r} q_i |x_{n-h_i}|^p, \tag{3.2}$$

for all $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$. Then there exists $\lambda_0 \in (0,1)$ such that, for every solution $\{x_n\}$ of the equation (3.1),

$$|x_n| \le \left(\max_{-h_r \le i \le 0}\{1, |x_i|\}\right)\lambda_0^n, \quad n \in \mathbb{Z}^0, \tag{3.3}$$

where λ_0 can be calculated in the form established in Theorem 2.1. As a consequence, the trivial solution of the equation (3.1) is globally asymptotically stable.

Proof. Let $\{x_n\}$ be a solution of the equation (3.1). From [2, Section 11],we know that

$$x_n = x_0(1-a)^n + \sum_{i=0}^{n-1}(1-a)^{n-i-1} f(i, x_i, x_{i-h_1}, \dots, x_{i-h_r}), \quad n \in \mathbb{Z}^0. \tag{3.4}$$

Thus, using the inequality (3.2), we obtain

$$|x_n| \le |x_0|(1-a)^n + \sum_{i=0}^{n-1}\sum_{j=0}^{r}(1-a)^{n-i-1} q_j |x_{i-h_j}|^p, \quad n \in \mathbb{Z}^0. \tag{3.5}$$

Denote $v_n = |x_n|$ for $n = -h_r, \dots, 0$, and

$$v_n = |x_0|(1-a)^n + \sum_{i=0}^{n-1}\sum_{j=0}^{r}(1-a)^{n-i-1} q_j |x_{i-h_j}|^p, \quad n \in \mathbb{Z}^+. \tag{3.6}$$

Then we have $|x_n| \le v_n$ and hence,

$$\triangle v_n = -av_n + \sum_{i=0}^{r} q_i |x_{n-h_i}|^p \le av_n + \sum_{i=0}^{r} q_i v_{n-h_i}^p, \quad n \in \mathbb{Z}^0. \tag{3.7}$$

Consequently, Theorem 2.1 ensures the validity of the following inequality

$$|x_n| \le v_n \le \Big(\max_{-h_r \le i \le 0}\{1, v_i\}\Big)\lambda_0^n = \Big(\max_{-h_r \le i \le 0}\{1, |x_i|\}\Big)\lambda_0^n, \quad n \in \mathbb{Z}^0, \tag{3.8}$$

where λ_0 is chosen as in Theorem 2.1. This completes the proof of the theorem. □

Similarly, by using Theorem 2.2 instead of Theorem 2.1, we obtain the following result.

Theorem 3.2. *Assume that* $0 < a \le 1$. *Let* $q_i \in \mathbb{R}_0^+$, $h_i \in \mathbb{Z}^0$, $i = 0, \dots, r-1$; $a_r, q_r \in \mathbb{R}^+$, $h_r \in \mathbb{Z}^+$, *where* $0 = h_0 < h_1 < \dots < h_r$. *Let* $\alpha_i, \beta_i \in \mathbb{R}^+$, $\sum_{i=0}^{r}\alpha_i = 1$ *and* $[(1-\delta)\prod_{i=0}^{r}\beta_i + \delta\sum_{i=0}^{r} q_i] < a \le 1$, *where* $0 \le \delta \le 1$ *is a constant. If*

$$|f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r})| \le \sum_{i=0}^{r}\delta q_i |x_{n-h_i}| + (1-\delta)\prod_{i=0}^{r}\beta_i |x_{n-h_i}|^{\alpha_i}, \tag{3.9}$$

for all $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$ *then there exists* $\lambda_0 \in (0,1)$ *such that, for every solution* $\{x_n\}$ *of equation* (3.1),

$$|x_n| \le \Big(\max_{-h_r \le i \le 0}\{|x_i|\}\Big)\lambda_0^n, \quad n \in \mathbb{Z}^0, \tag{3.10}$$

where λ_0 *can be calculated in the form established in Theorem* 2.2. *As a consequence, the trivial solution of the equation* (3.1) *is globally asymptotically stable.*

Remark 3.2. The equation (3.1) covers a variety of difference equations. For instance, we can mention the equation

$$\triangle x_n = -ax_n + f(x_{n-k}), \quad a > 0 \tag{3.11}$$

investigated recently in [7, 11]. Theorem 3.2 ensures that if there exists $q, \beta \in \mathbb{R}^+, 0 \le \delta \le 1$ such that $|f(x)| \le (\delta q|x| + (1-\delta)\beta|x|)$ for all x, and $[\delta q + (1-\delta)\beta] < a < 1$, then all solutions of (3.11) converge to zero.

On the other hand, the condition (3.9) is satisfied by some linear and nonlinear generalized difference equations. We can mention the equation

$$x_{n+1} = \sum_{i=n-r}^{n} \delta q_i x_i + (1-\delta) \prod_{i=n-r}^{n} \beta_i x_i^{\alpha_i}, \quad q_i, \beta_i \in \mathbb{R} \tag{3.12}$$

for which Theorem 3.2 gives the global asymptotic stability of the equilibrium if

$$\sup_{n\in\mathbb{N}} \left(\sum_{i=n-r}^{n} \delta |q_i| + (1-\delta) \prod_{i=n-r}^{n} |\beta_i| \right) < 1, \tag{3.13}$$

since the equation (3.12) can be rewritten in the form

$$\triangle x_n = -x_n + f(x_n, \dots, x_{n-r}), \tag{3.14}$$

with $f(x_n, \dots, x_{n-r}) = \sum_{i=n-r}^{n} \delta q_i x_i + (1-\delta) \prod_{i=n-r}^{n} \beta_i x_i^{\alpha_i}$. For example, if $p > r+1$, then the inequality (3.13) is satisfied by the equation $x_{n+1} = (1/p)(x_n + \cdots + x_{n-r})$ with $\delta = 1$.

For more general results on the asymptotic behavior of generalized difference systems, refer [2, Section 9].

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