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MULTIPLE NONDECREASING POSITIVE SOLUTIONS FOR A SINGULAR THIRD-ORDER THREE-POINT BVP

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ABSTRACT. This paper is concerned with the following nonlinear third-order three-point boundary value problem

$$u'''(t) + a(t)f(u(t)) = 0,\ t \in (0,1),$$
$$u(0) = u''(0) = 0,\ u'(1) - \alpha u(\eta) = 0,$$

where $0 < \eta < 1$, $0 < \alpha < \frac{1}{\eta}$ and $a(t)$ may be singular at $t = 1$. Green's function for the associated linear boundary value problem is constructed and some existence results of multiple nondecreasing positive solutions are obtained by using the well-known Leggett-Williams fixed point theorem.

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1. INTRODUCTION

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three layer beam, electromagnetic waves or gravity driven flows and so on [7]. Recently, third-order boundary value problems (BVPs for short) have received a lot of attention. For example, [5, 6, 9, 12, 13, 17] discussed some third-order two-point BVPs, while [1-3, 14-16] studied some third-order three-point BVPs. In particular, Anderson [1] obtained the existence of positive solutions for the BVP

$$x'''(t) = f(t, x(t)),\ t_1 \le t \le t_3, \tag{1.1}$$

$$x(t_1) = x'(t_2) = 0,\ \gamma x(t_3) + \delta x''(t_3) = 0 \tag{1.2}$$

by using the well-known Guo-Krasnoselskii and Leggett-Williams fixed point theorems [8, 10, 11]. Sun [15] established some criteria on the existence of positive solutions to some third-order differential equation satisfying the following three-point boundary conditions

$$x(0) = x'(\eta) = x''(1) = 0, \tag{1.3}$$

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where $\eta \in [\frac{1}{2}, 1)$. The main tool in [15] was the Guo-Krasnoselskii fixed point theorem [8, 10]. In [3], Green's function for the three-point BVP of focal type

$$x'''(t) = 0,\ t_1 \le t \le t_3, \tag{1.4}$$

$$\alpha x(t_1) - \beta x'(t_1) = 0,\ \gamma x(t_2) + \delta x'(t_2) = 0,\ x''(t_3) = 0 \tag{1.5}$$

was determined and the existence of positive solutions to a related higher-order functional differential equation was discussed by employing the Guo-Krasnoselskii and Avery-Henderson fixed point theorems [4, 8, 10].

Motivated greatly by the above-mentioned excellent works, in this paper we consider the nonlinear third-order three-point BVP

$$u'''(t) + a(t)f(u(t)) = 0,\ t \in (0,1), \tag{1.6}$$

$$u(0) = u''(0) = 0,\ u'(1) - \alpha u(\eta) = 0, \tag{1.7}$$

where $0 < \eta < 1$, $0 < \alpha < \dfrac{1}{\eta}$ and $a(t)$ may be singular at $t = 1$. First, Green's function for the associated linear BVP is constructed, and then, some useful properties of the Green's function are obtained. Finally, existence results of multiple nondecreasing positive solutions for the BVP (1.6)–(1.7) are established.

In order to obtain our main results, we need the following fundamental concepts and the Leggett-Williams fixed point theorem.

Let E be a real Banach space with cone P. A map $\sigma : P \to [0, +\infty)$ is said to be a nonnegative continuous concave functional on P if σ is continuous and

$$\sigma(tx + (1-t)y) \ge t\sigma(x) + (1-t)\sigma(y)$$

for all x, $y \in P$ and $t \in [0,1]$. Let a, b be two numbers such that $0 < a < b$ and σ be a nonnegative continuous concave functional on P. We define the following convex sets

$$P_a = \{x \in P :\ \|x\| < a\} \text{ and } P(\sigma, a, b) = \{x \in P :\ a \le \sigma(x),\ \|x\| \le b\}.$$

Theorem 1.1 (Leggett-Williams fixed point theorem)**.** *Let A: $\overline{P_c} \to \overline{P_c}$ be completely continuous and σ be a nonnegative continuous concave functional on P such that $\sigma(x) \le \|x\|$ for all $x \in \overline{P_c}$. Suppose that there exist $0 < d < a < b \le c$ such that*

(1) $\{x \in P(\sigma, a, b) : \sigma(x) > a\} \ne \emptyset$ and $\sigma(Ax) > a$ for $x \in P(\sigma, a, b)$;

(2) $\|Ax\| < d$ for $\|x\| \le d$;

(3) $\sigma(Ax) > a$ for $x \in P(\sigma, a, c)$ with $\|Ax\| > b$.

Then A has at least three fixed points x_1, x_2 and x_3 in $\overline{P_c}$ satisfying

$$\|x_1\| < d,\ a < \sigma(x_2),\ \|x_3\| > d \text{ and } \sigma(x_3) < a.$$

2. PRELIMINARIES

Throughout this paper, we assume that $0 < \eta < 1$, $0 < \alpha < \dfrac{1}{\eta}$ and the Banach space $E = C[0,1]$ is equipped with the norm $\|u\| = \max_{0\le t\le 1} |u(t)|$.

Lemma 2.1. *For any $y \in E$, the BVP*

$$u'''(t) + y(t) = 0,\ t \in (0,1), \tag{2.1}$$

$$u(0) = u''(0) = 0,\ u'(1) - \alpha u(\eta) = 0 \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s) = \frac{1}{2(1-\alpha\eta)}\begin{cases} 2t - (1-\alpha\eta)(t^2+s^2) - \alpha t(\eta^2+s^2), & s \le \min\{\eta,t\}, \\ 2t - t^2 - s^2 + \alpha\eta(t-s)^2, & \eta \le s \le t, \\ 2t - 2ts(1-\alpha\eta) - \alpha t(\eta^2+s^2), & t \le s \le \eta, \\ 2t(1-s), & s \ge \max\{\eta,t\} \end{cases}$$

is called the Green's function.

Proof. Since the proof is obvious by a simple calculation, we omit it. □

For convenience, we denote

$$g(s) = \frac{1+\alpha\eta}{1-\alpha\eta}(1-s),\ s \in [0,1].$$

Lemma 2.2. *For any $(t,s) \in [0,1]\times[0,1]$,*

$$0 \le G(t,s) \le g(s)$$

and

$$0 \le G_t(t,s) \le g(s).$$

Proof. First, we will show that $G(t,s) \ge 0$ for any $(t,s) \in [0,1]\times[0,1]$. Since it is obvious in other cases, we only need to prove the cases when $s \le \min\{\eta,t\}$ and $t \le s \le \eta$. If $s \le \min\{\eta,t\}$, then

$$\begin{aligned} G(t,s) &= \frac{1}{2(1-\alpha\eta)}\left[2t - (1-\alpha\eta)\left(t^2+s^2\right) - \alpha t\left(\eta^2+s^2\right)\right] \\ &\ge \frac{1}{2(1-\alpha\eta)}\left[2t - (1-\alpha\eta)(t+s) - \alpha t(\eta+s)\right] \\ &= \frac{1}{2(1-\alpha\eta)}\left[t - s + \alpha s(\eta - t)\right] \\ &\ge \frac{1}{2(1-\alpha\eta)}\left[t - s + \alpha s(s-t)\right] \\ &\ge \frac{1}{2(1-\alpha\eta)}(t-s)(1-\alpha\eta) \ge 0. \end{aligned}$$

If $t \le s \le \eta$, then

$$\begin{aligned} G(t,s) &= \frac{1}{2(1-\alpha\eta)}\left[2t - 2ts(1-\alpha\eta) - \alpha t\left(\eta^2 + s^2\right)\right] \\ &\ge \frac{1}{2(1-\alpha\eta)}\left[2t - 2ts(1-\alpha\eta) - \alpha t(\eta + s)\right] \\ &\ge \frac{1}{2(1-\alpha\eta)}\left[2t - 2ts(1-\alpha\eta) - 2\alpha\eta t\right] \\ &= \frac{1}{1-\alpha\eta} t(1-s)(1-\alpha\eta) \ge 0. \end{aligned}$$

Therefore, for any $(t,s) \in [0,1] \times [0,1]$, we have

$$G(t,s) \ge 0. \tag{2.3}$$

Next, for any fixed $s \in [0,1]$, it is easy to know that

$$G_t(t,s) = \frac{1}{2(1-\alpha\eta)} \begin{cases} 2 - 2(1-\alpha\eta)t - \alpha(\eta^2 + s^2), & s \le \min\{\eta, t\}, \\ 2 - 2t + 2\alpha\eta(t-s), & \eta \le s \le t, \\ 2 - 2s(1-\alpha\eta) - \alpha(\eta^2 + s^2), & t \le s \le \eta, \\ 2(1-s), & \max\{\eta, t\} \le s. \end{cases}$$

If $s \le \min\{\eta, t\}$, then

$$\begin{aligned} G_t(t,s) &= \frac{1}{2(1-\alpha\eta)}\left[2 - 2(1-\alpha\eta)t - \alpha(\eta^2 + s^2)\right] \\ &\le \frac{1}{2(1-\alpha\eta)}\left[2 - 2(1-\alpha\eta)s - 2\alpha\eta s\right] \\ &= \frac{1}{1-\alpha\eta}(1-s) \le g(s). \end{aligned}$$

If $\eta \le s \le t$, then

$$G_t(t,s) = \frac{1}{1-\alpha\eta}\left[1 - t + \alpha\eta(t-s)\right] \le \frac{1}{1-\alpha\eta}\left[1 - s + \alpha\eta(1-s)\right] = g(s).$$

If $t \le s \le \eta$, then

$$\begin{aligned} G_t(t,s) &= \frac{1}{2(1-\alpha\eta)}\left[2 - 2s(1-\alpha\eta) - \alpha(\eta^2 + s^2)\right] \\ &\le \frac{1}{2(1-\alpha\eta)}\left[2 - 2(1-\alpha\eta)s - 2\alpha\eta s\right] \\ &= \frac{1}{1-\alpha\eta}(1-s) \le g(s). \end{aligned}$$

If $\max\{\eta, t\} \le s$, then

$$G_t(t,s) = \frac{1}{1-\alpha\eta}(1-s) \le g(s).$$

Therefore,

$$G_t(t,s) \le g(s),\ (t,s) \in [0,1] \times [0,1]. \tag{2.4}$$

And so, for any $(t, s) \in [0,1] \times [0,1]$, we have

$$G(t,s) = \int_0^t G_\tau(\tau, s)d\tau \le \int_0^t g(s)d\tau = tg(s) \le g(s). \tag{2.5}$$

Moreover, it is easy to verify that $G_t(t,s) \ge 0$, which together with (2.3), (2.4) and (2.5) complete the proof. □

In the remainder of this paper, we always assume that the following two conditions are fulfilled:

(A1) $f \in C([0,\infty), [0,\infty))$;

(A2) $a \in C([0,1), [0,\infty))$ satisfies

$$0 < \int_\eta^1 (1-s)a(s)ds \text{ and } \int_0^1 (1-s)a(s)ds < +\infty.$$

Denote

$$P = \{u \in E : \ u(t) \ge 0 \text{ for } t \in [0,1]\}.$$

Then it is obvious that P is a cone in E. For $u \in P$, we define

$$(Au)(t) = \int_0^1 G(t,s)a(s)f(u(s))ds, \ t \in [0,1].$$

Lemma 2.3. *$A : P \to P$ is completely continuous.*

Proof. Suppose that $u \in P$. By the continuity of f, we may let $M = \sup_{x\in[0,\ \|u\|]} f(x)$. Then it follows from Lemma 2.1, Lemma 2.2 and (A2) that

$$\begin{aligned} 0 \le (Au)(t) &= \int_0^1 G(t,s)a(s)f(u(s))ds \le \int_0^1 g(s)a(s)f(u(s))ds \\ &= \frac{1+\alpha\eta}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(u(s))ds \\ &\le \frac{M(1+\alpha\eta)}{1-\alpha\eta}\int_0^1 (1-s)a(s)ds < +\infty, \ t \in [0,1], \end{aligned}$$

which shows that A is well defined and $A(P) \subset P$.

Now we prove that A is a compact operator. Suppose that $D \subset P$ is an arbitrary bounded set. Then there is $M_1 > 0$ such that $\|u\| \le M_1$ for all $u \in D$. Let $M_2 = \sup_{x\in[0,M_1]} f(x)$. With the similar arguments as above, for all $u \in D$, we have

$$0 \le (Au)(t) = \int_0^1 G(t,s)a(s)f(u(s))ds \le \frac{M_2(1+\alpha\eta)}{1-\alpha\eta}\int_0^1 (1-s)a(s)ds, \ t \in [0,1],$$

which shows that $A(D)$ is uniformly bounded. At the same time, for all $u \in D$, in view of Lemma 2.2, we get

$$0 \le (Au)'(t) = \int_0^1 G_t(t,s)a(s)f(u(s))ds \le \frac{M_2(1+\alpha\eta)}{1-\alpha\eta}\int_0^1 (1-s)a(s)ds, \ t \in [0,1],$$

which implies that $A(D)$ is equicontinuous. Therefore, it follows from Arzela-Ascoli theorem that A is compact.

Finally, we prove that A is continuous. Suppose that u_m, $u \in P$ and $\|u_m - u\| \to 0$ $(m \to \infty)$. Then there exists $M_3 > 0$ such that $\|u_m - u\| \leq M_3$ for all m. Let $M_4 = M_3 + \|u\|$. Then it is easy to see that $\|u_m\| \leq M_4$ for all m. By Lemma 2.2, for all m, we obtain

$$\begin{aligned} G(t,s)a(s)f(u_m(s)) &\leq g(s)a(s)f(u_m(s)) \\ &\leq \frac{M_5(1+\alpha\eta)}{1-\alpha\eta}(1-s)a(s), \quad (t,s) \in [0,1] \times [0,1), \end{aligned}$$

where $M_5 = \sup_{x\in[0,M_4]} f(x)$. According to Lebesgue Dominated Convergence theorem, we know that

$$\begin{aligned} \lim_{m\to\infty}(Au_m)(t) &= \lim_{m\to\infty}\int_0^1 G(t,s)a(s)f(u_m(s))ds = \int_0^1 \lim_{m\to\infty} G(t,s)a(s)f(u_m(s))ds \\ &= \int_0^1 G(t,s)a(s)f(\lim_{m\to\infty} u_m(s))ds = \int_0^1 G(t,s)a(s)f(u(s))ds \\ &= (Au)(t),\ t \in [0,1], \end{aligned}$$

which shows that A is continuous. To sum up, $A : P \to P$ is completely continuous. □

Lemma 2.4. *For any $u \in P$, Au is nondecreasing on $[0,1]$ and $\min_{t\in[\eta,1]}(Au)(t) \geq \eta\,\|Au\|$.*

Proof. Suppose $u \in P$. Then it is easy to know from the definition of $(Au)(t)$ that

$$(Au)'''(t) + a(t)f(u(t)) = 0,\ t \in (0,1)$$

and

$$(Au)(0) = (Au)''(0) = 0,\ (Au)'(1) - \alpha(Au)(\eta) = 0.$$

Since $(Au)'''(t) = -a(t)f(u(t)) \leq 0$ for $t \in (0,1)$, $(Au)''$ is non-increasing on [0,1]. So, $(Au)''(t) \leq (Au)''(0) = 0$ for $t \in (0,1)$. This shows that $(Au)'$ is non-increasing on [0,1], which implies that $(Au)'(t) \geq (Au)'(1) = \alpha(Au)(\eta) \geq 0$ for $t \in (0,1)$. Therefore, Au is nondecreasing on $[0,1]$.

On the other hand, by combining the fact $(Au)''(t) \leq 0$ for $t \in (0,1)$, we obtain that Au is concave down on $[0,1]$. Furthermore, since $(Au)'(t) \geq 0$ for $t \in (0,1)$, we know that $\min_{t\in[\eta,1]}(Au)(t) = (Au)(\eta)$ and $\|Au\| = (Au)(1)$. So, it is clear that

$$\min_{t\in[\eta,1]}(Au)(t) \geq \eta\,\|Au\|\,.$$

□

3. MAIN RESULTS

Since it is obvious that fixed points of the operator A are solutions of the BVP (1.6)–(1.7), in this section we will apply the Leggett-Williams fixed point theorem to the operator A to obtain the existence of multiple nondecreasing positive solutions for the BVP (1.6)–(1.7). For convenience, we let

$$C = \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)ds \text{ and } D = \frac{1+\alpha\eta}{1-\alpha\eta}\int_0^1 (1-s)a(s)ds.$$

Theorem 3.1. *Assume that there exist numbers d_0, d_1 and c with $0 < d_0 < d_1 < \frac{d_1}{\eta} < c$ such that*

$$f(u) < \frac{d_0}{D} \text{ for } u \in [0, d_0], \tag{3.1}$$

$$f(u) > \frac{d_1}{C} \text{ for } u \in \left[d_1, \frac{d_1}{\eta}\right] \tag{3.2}$$

and

$$f(u) < \frac{c}{D} \text{ for } u \in [0, c]. \tag{3.3}$$

Then the BVP (1.6)–(1.7) *has at least three nondecreasing positive solutions.*

Proof. For $u \in P$, we define

$$\sigma(u) = \min_{t\in[\eta,1]} u(t).$$

Then it is easy to check that σ is a nonnegative continuous concave functional on P with $\sigma(u) \le \|u\|$ for $u \in P$.

We first assert that if there exists a positive number r such that $f(u) < \frac{r}{D}$ for $u \in [0, r]$, then $A : \overline{P_r} \to P_r$.

Indeed, if $u \in \overline{P_r}$, then for $t \in [0, 1]$, by Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}(Au)(t) &= \int_0^1 G(t,s)a(s)f(u(s))ds \le \int_0^1 g(s)a(s)f(u(s))ds \\ &< \frac{r}{D}\frac{1+\alpha\eta}{1-\alpha\eta}\int_0^1 (1-s)a(s)ds = r.\end{aligned}$$

Thus, $\|Au\| < r$, that is, $Au \in P_r$. Hence, we have shown that if (3.1) and (3.3) hold, then A maps $\overline{P_{d_0}}$ into P_{d_0} and $\overline{P_c}$ into P_c.

Next, we assert that $\{u \in P(\sigma, d_1, d_1/\eta) : \sigma(u) > d_1\} \ne \emptyset$ and $\sigma(Au) > d_1$ for all $u \in P(\sigma, d_1, d_1/\eta)$.

In fact, the constant function

$$\frac{d_1 + d_1/\eta}{2} \in \{u \in P(\sigma, d_1, d_1/\eta) : \sigma(u) > d_1\}.$$

On the other hand, for $u \in P(\sigma, d_1, d_1/\eta)$, we have

$$d_1/\eta \ge \|u\| \ge u(t) \ge \min_{t\in[\eta,1]} u(t) = \sigma(u) \ge d_1$$

for all $t \in [\eta, 1]$. Thus, in view of (3.2) and the proof of Lemma 2.4, we get

$$\begin{aligned}
\sigma(Au) &= (Au)(\eta) \\
&= -\frac{1}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)^2 a(s)f(u(s))ds + \frac{\eta}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(u(s))ds \\
&\geq -\frac{1}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)a(s)f(u(s))ds + \frac{\eta}{1-\alpha\eta}\int_0^1 a(s)f(u(s))ds \\
&\quad -\frac{\eta}{1-\alpha\eta}\int_0^1 sa(s)f(u(s))ds \\
&\geq \frac{\eta}{2(1-\alpha\eta)}\int_\eta^1 a(s)f(u(s))ds + \frac{\eta}{2(1-\alpha\eta)}\int_0^1 a(s)f(u(s))ds \\
&\quad +\frac{\eta}{2\,(1-\alpha\eta)}\int_1^\eta sa(s)f(u(s))ds + \frac{\eta}{2\,(1-\alpha\eta)}\int_1^0 sa(s)f(u(s))ds \\
&= \frac{\eta}{2(1-\alpha\eta)}\int_\eta^1 (1-s)a(s)f(u(s))ds + \frac{\eta}{2\,(1-\alpha\eta)}\int_0^1 (1-s)a(s)f(u(s))ds \\
&\geq \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)f(u(s))ds \\
&> \frac{d_1}{C}\frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)ds \\
&= d_1
\end{aligned}$$

as required.

Finally, we assert that if $u \in P(\sigma, d_1, c)$ and $\|Au\| > d_1/\eta$, then $\sigma(Au) > d_1$.

To see this, we suppose that $u \in P(\sigma, d_1, c)$ and $\|Au\| > d_1/\eta$, then it follows from Lemma 2.4 that

$$\sigma(Au) = \min_{t\in[\eta,1]}(Au)(t) \geq \eta\,\|Au\| > \eta\frac{d_1}{\eta} = d_1.$$

To sum up, all the hypotheses of the Leggett-Williams fixed point theorem are satisfied. Hence A has at least three fixed points u, v and w, that is, the BVP (1.6)–(1.7) has at least three positive solutions u, v and w satisfying

$$\|u\| < d_0,\ d_1 < \min_{t\in[\eta,1]} v(t),\ \|w\| > d_0 \text{ and } \min_{t\in[\eta,1]} w(t) < d_1.$$

Furthermore, we know from Lemma 2.4 that u, v and w are also nondecreasing. □

Example 3.2. We consider the following BVP

$$u'''(t) + \frac{1}{1-t}f(u(t)) = 0,\ t \in (0,1), \tag{3.4}$$

$$u(0) = u''(0) = 0,\ u'(1) - u(\frac{1}{2}) = 0, \tag{3.5}$$

where

$$f(u) = \begin{cases} \frac{u^2}{4}, & u \in [0,1], \\ \frac{19}{4}u - \frac{9}{2}, & u \in [1,2], \\ u^2 + 1, & u \in [2,4], \\ -\frac{3}{20}u + \frac{88}{5}, & u \in [4,64], \\ u^{\frac{1}{2}}, & u \in [64,\infty). \end{cases}$$

A simple calculation shows that

$$C = \frac{1}{2} \text{ and } D = 3.$$

If we choose

$$d_0 = 1,\ d_1 = 2 \text{ and } c = 64,$$

then the conditions (3.1)–(3.3) are satisfied. Therefore, it follows from Theorem 3.1 that the BVP (3.4)–(3.5) has at least three nondecreasing positive solutions.

Remark 3.3. It is easy to extend the main results of this paper to the corresponding BVP of dynamic equations on time scales.

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