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SOME EXISTENCE RESULTS FOR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLINEAR CONDITIONS

BASHIR AHMAD[1] AND S. SIVASUNDARAM[2]

[1]Department of Mathematics, Faculty of Science, King Abdul Aziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia

[2]Department of Mathematics, Embry- Riddle Aeronautical University
Daytona Beach, FL 32114, USA

ABSTRACT. This paper studies some existence and uniqueness results in a Banach space for a fractional integro-differential equation with nonlinear condition

$$\begin{aligned} {}^cD^q x(t) &= f(t, x(t)) + \int_0^t k(t, s, x(s))ds, \ t \in [0, T], \ 0 < q < 1, \\ x(0) &= x_0 - g(x). \end{aligned}$$

The contraction mapping principle and Krasnoselskii's fixed point theorem are employed to establish the results.

Keywords and Phrases. Fractional integro-differential equations, contraction principle, Krasnoselskii's fixed point theorem.

AMS (MOS) Subject Classifications. 34K05, 34A12, 34A40, 45J05

1. INTRODUCTION

Integro-differential equations arise in many engineering and scientific disciplines, often as approximation to partial differential equations, which represent much of the continuum phenomena. Many forms of these equations are possible. Some of the applications are unsteady aerodynamics and aero elastic phenomena, visco elasticity, visco elastic panel in super sonic gas flow, fluid dynamics, electro dynamics of complex medium, many models of population growth, polymer rheology, neural network modeling, sandwich system identification, materials with fading memory, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, heat conduction in materials with memory, theory of lossless transmission lines, theory of population dynamics, compartmental systems, nuclear reactors and mathematical modeling of a hereditary phenomena.

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Recently the application of fractional calculus in physics, continuum mechanics, signal processing, and electromagnetics, with few examples of applications in bioengineering are high lighted in the literatures. The methods of fractional calculus, when defined as a Laplace, Sumudu or Fourier convolution product, are suitable for solving many problems in emerging biomedical research. The electrical properties of nerve cell membranes and the propagation of electrical signals are well characterized by differential equations of fractional order. The fractional derivative accurately describes natural phenomena that occur in such common engineering problems as heat transfer, electrode/electrolyte behavior, and sub-threshold nerve propagation. Application of fractional derivatives to viscoelastic materials establishes, in a natural way, hereditary integrals and the power law stress-strain relationship for modeling biomaterials. Fractional operations by following the original approach of Heaviside, demonstrate the basic operations of fractional calculus on well-behaved functions such as step, ramp, pulse, and sinusoidal of engineering interest, and can easily be applied in electrochemistry, physics, bioengineering, and biophysics.

The differential equations involving Riemann-Liouville differential operators of fractional order occur in the mathematical modelling of several phenomena in the fields of physics, chemistry, engineering, etc. For details, see [3, 4, 9, 11, 12] and the references therein. In consequence, the subject of fractional differential equations is gaining much importance and attention, see for example [1, 5, 6, 7, 10] and the references therein.

The definition of Riemann-Liouville fractional derivative, which did certainly play an important role in the development of theory of fractional derivatives and integrals, could hardly produce the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations. It was Caputo's definition of fractional derivative:

$$ {}^{c}D^{q}x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} x^{(n)}(s)ds, \ n-1 < q < n, $$

which solved this problem. In fact, Caputo'derivative becomes the conventional $n-$th derivative as $q \to n$ and the initial conditions for fractional differential equations retain the same form as that of ordinary differential equations with integer derivatives. Another difference is that the Caputo derivative for a constant is zero while the Riemann-Liouville fractional derivative of a constant is nonzero.

In this paper, we consider a Cauchy problem involving a fractional integro-differential equation with nonlocal condition given by

$$ \begin{cases} {}^{c}D^{q}x(t) = f(t, x(t)) + \int_0^t k(t, \eta, x(\eta))d\eta, \ t \in [0, T], \ T > 0, \ 0 < q < 1, \\ \qquad\qquad x(0) = x_0 - g(x), \end{cases} \tag{1.1} $$

where ${}^cD^q$ denotes Caputo fractional derivative of order q, $f : [0,T] \times X \to X$, $k : [0,T] \times [0,T] \times X \to X$ are jointly continuous, $g : \mathrm{C} \to X$ is continuous. Here, $(X, \|.\|)$ is a Banach space and $\mathrm{C} = \mathrm{C}([0,T], X)$ denotes the Banach space of all continuous functions from $[0,T] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|_{\mathrm{C}}$.

In passing, we remark that the application of nonlinear condition $x(0) = x_0 - g(x)$ in physical problems yields better effect than the initial condition $x(0) = x_0$ [2].

As argued in [5] (see also [8]), the cauchy problem is equivalent to the following nonlinear integral equation

$$x(t) = x_0 - g(x) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(s, x(s)) + \int_s^t k(\sigma, s, x(s)) d\sigma] ds, \tag{1.2}$$

where Γ is Gamma function.

Now, we state a known result due to Krasnoselskii which is needed to prove the existence of at least one solution of (1.1).

Theorem 1.1. Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that (i) $Ax + By \in M$ whenever $x, y \in M$ (ii) A is compact and continuous (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.

2. EXISTENCE RESULTS

Theorem 2.1. Assume that

($\mathbf{A_1}$) $\|f(t,x) - f(t,y)\| \leq L_1 \|x - y\|, \forall t \in [0,T],\ x, y \in X$;
($\mathbf{A_2}$) $\|k(t,s,x) - k(t,s,y)\| \leq L_2 \|x - y\|, \forall t, s \in [0,T],\ x, y \in X$;
($\mathbf{A_2}$) $\|g(x) - g(y)\| \leq b\|x - y\|, \forall x, y \in \mathrm{C}$.

Then the Cauchy problem (1.1) has a unique solution provided $b < \frac{1}{2}$, $L_1 \leq \frac{\Gamma(q+1)}{4T^q}$, $L_2 \leq \frac{\Gamma(q+2)}{4qT^{q+1}}$.

Proof. Define $\Theta : \mathrm{C} \to \mathrm{C}$ by

$$(\Theta x)(t) = x_0 - g(x) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(s, x(s)) + \int_s^t k(\sigma, s, x(s)) d\sigma] ds. \tag{2.1}$$

Let us set $\sup_{t \in [0,T]} \|f(t,0)\| = M_1$, $\sup_{t,s \in [0,T]} \|k(t,s,0)\| = M_2$ and $\sup_{x \in \mathrm{C}} \|g(x)\| = G$. Choosing $r \geq 2(\|x_0\| + G + \frac{M_1 T^q}{\Gamma(q+1)} + \frac{q M_2 T^{q+1}}{\Gamma(q+2)})$, it can be shown that $\Theta B_r \subset B_r$, where $B_r = \{x \in \mathrm{C} : \|x\| \leq r\}$. For $x \in \mathrm{C}$, we have

$$\begin{aligned} \|(\Theta x)(t)\| &\leq \|x_0\| + G + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [\|f(s, x(s))\| + \int_s^t \|k(\sigma, s, x(s))\| d\sigma] ds \\ &\leq \|x_0\| + G + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [(\|f(s, x(s)) - f(s, 0)\| + \|f(s, 0)\|) \end{aligned}$$

$$
\begin{aligned}
&+ \int_s^t (\|k(\sigma,s,x(s)) - k(\sigma,s,0)\| + \|k(\sigma,s,0)\|)d\sigma]ds \\
&\leq \|x_0\| + G + \frac{L_1 r + M_1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + \frac{L_2 r + M_2}{\Gamma(q)} \int_0^t (t-s)^q ds \\
&\leq \|x_0\| + G + \frac{(L_1 r + M_1)T^q}{\Gamma(q+1)} + \frac{(L_2 r + M_2) q T^{q+1}}{\Gamma(q+2)} \\
&\leq r.
\end{aligned}
$$

Now, for $x, y \in \mathrm{C}$, we obtain

$$
\begin{aligned}
\|(\Theta x)(t) - (\Theta y)(t)\| &\leq \|g(x) - g(y)\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [\|f(s,x(s)) - f(s,y(s))\| \\
&+ \int_s^t \|k(\sigma,s,x(s)) - k(\sigma,s,y(s))\| d\sigma]ds \\
&\leq \Lambda_{b,L_1,L_2,T,q} \|x-y\|,
\end{aligned}
$$

where $\Lambda_{b,L_1,L_2,T,q} = (b + \frac{L_1 T^q}{\Gamma(q+1)} + \frac{L_2 q T^{q+1}}{\Gamma(q+2)})$ depends only on the parameters involved in the problem. As $\Lambda_{b,L_1,L_2,T,q} < 1$, so the conclusion of the theorem follows by the contraction mapping principle.

Theorem 2.2. Assume that

($\mathbf{B_1}$) $\|f(t,x)\| \leq \mu(t), \forall (t,x) \in [0,T] \times X$, $\|k(t,s,x)\| \leq \sigma(t), \forall (t,s,x) \in [0,T] \times [0,T] \times X$, where $\mu, \sigma \in L^1([0,T], R^+)$;

($\mathbf{B_2}$) $\|g(x) - g(y)\| \leq b\|x-y\|, \forall x, y \in \mathrm{C}, b < 1$.

Then the Cauchy problem (1.1) has at least one solution on $[0,T]$.

Proof. Fix $r \geq (\|x_0\| + G + \frac{\|\mu\|_{L^1} T^q}{\Gamma(q+1)} + \frac{q\|\sigma\|_{L^1} T^{q+1}}{\Gamma(q+2)})$ and consider $B_r = \{x \in \mathrm{C} : \|x\| \leq r\}$. We define the operators Φ and Ψ on B_r as

$$
(\Phi x)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(s,x(s)) + \int_s^t k(\sigma,s,x(s)) d\sigma]ds,
$$

$$
(\Psi x)(t) = x_0 - g(x).
$$

For $x, y \in B_r$, we find that

$$
\|\Phi x + \Psi y\| \leq \|x_0\| + G + \frac{\|\mu\|_{L^1} T^q}{\Gamma(q+1)} + \frac{q\|\sigma\|_{L^1} T^{q+1}}{\Gamma(q+2)}) \leq r.
$$

Thus, $\Phi x + \Psi y \in B_r$. It follows from the assumption (B_2) that Ψ is a contraction mapping. Continuity of f and k demanded in (1.1) implies that the operator Φ is continuous. Also, Φ is uniformly bounded on B_r as

$$
\|\Phi x\| \leq (\frac{\|\mu\|_{L^1} T^q}{\Gamma(q+1)} + \frac{q\|\sigma\|_{L^1} T^{q+1}}{\Gamma(q+2)}).
$$

Now we prove the compactness of the operator Φ. Since f and k are respectively bounded on the compact sets $\Omega_1 = [0,T] \times X$ and $\Omega_2 = [0,T] \times [0,T] \times X$, therefore, we define $\sup_{(t,x) \in \Omega_1} \|f(t,x)\| = C_1$, $\sup_{(t,s,x) \in \Omega_2} \|k(t,s,x)\| = C_2$.

For $t_1, t_2 \in [0, T]$, $x \in B_r$, we have

$$
\begin{aligned}
& \|(\Phi x)(t_1) - (\Phi x)(t_2)\| \\
= \;& \frac{1}{\Gamma(q)} \| \int_0^{t_1} (t_1 - s)^{q-1} [f(s, x(s)) + \int_s^{t_1} k(\sigma, s, x(s)) d\sigma] ds \\
- \;& \int_0^{t_2} (t_2 - s)^{q-1} [f(s, x(s)) + \int_s^{t_2} k(\sigma, s, x(s)) d\sigma] ds \| \\
= \;& \frac{1}{\Gamma(q)} \| \int_{t_2}^{t_1} (t_1 - s)^{q-1} [f(s, x(s)) + \int_s^{t_1} k(\sigma, s, x(s)) d\sigma] ds \\
- \;& \int_0^{t_2} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] f(s, x(s)) ds \\
- \;& \int_0^{t_2} [(t_2 - s)^{q-1} \int_{t_2}^{s} k(\sigma, s, x(s)) d\sigma - (t_1 - s)^{q-1} \int_{t_1}^{s} k(\sigma, s, x(s)) d\sigma] ds \| \\
\leq \;& \frac{C_1}{\Gamma(q+1)} |2(t_1 - t_2)^q + t_2^q - t_1^q| + \frac{qC_2}{\Gamma(q+2)} |2(t_1 - t_2)^{q+1} + t_2^{q+1} - t_1^{q+1}| \\
\leq \;& \frac{2C_1}{\Gamma(q+1)} |t_1 - t_2|^q + \frac{2qC_2}{\Gamma(q+2)} |t_1 - t_2|^{q+1},
\end{aligned}
$$

which is independent of x. So Φ is relatively compact on B_r. Hence, By Arzela Ascoli Theorem, Φ is compact on B_r. Thus all the assumptions of Theorem 1.1 are satisfied. Consequently, the conclusion of Theorem 1.1 applies and the Cauchy problem (1.1) has at least one solution.

Remark. By taking $k \equiv 0$ in (1.1), the results of reference [10] appear as a special case of our results. Moreover, the results for an initial value problem involving fractional integro-differential equations can be obtained if we take $g \equiv 0$ in (1.1).

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