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ϕ_0-STABILITY OF IMPULSIVE HYBRID SETVALUED DIFFERENTIAL EQUATIONS WITH DELAY BY PERTURBING LYAPUNOV FUNCTIONS

BASHIR AHMAD[1] AND S. SIVASUNDARAM[2]

[1]Department of Mathematics, Faculty of science, King Abdul Aziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia

[2]Department of Mathematics, Embry- Riddle Aeronautical University
Daytona Beach, FL 32114, USA

ABSTRACT. We study ϕ_0-stability of the null solution of impulsive set differential system with delay by means of the perturbing Lyapunov function method. Sufficient conditions for the ϕ_0-stability of the null solution of impulsive set differential equations with delay are presented.

Keywords and Phrases. Impulsive set differential equations with delay, stability, asymptotic stability

AMS (MOS) Subject Classifications. 34K45, 34D20

1. INTRODUCTION

The study of set differential equations has been initiated as an independent subject and several results of interest can be found in [4–5, 10–12, 14]. The interesting feature of the set differential equations is that the results obtained in this new framework become the corresponding results of ordinary differential equations as the Hukuhara derivative and the integral used in formulating the set differential equations reduce to the ordinary vector derivative and integral when the set under consideration is a single valued mapping. Moreover, in the present setup, we have only semilinear complete metric space to work with, instead of complete normed linear space required in the study of the ordinary differential systems. Furthermore, set differential equations, that are generated by multivalued differential inclusions, when the multivalued functions involved do not possess convex values, can be used as a tool for studying multivalued differential inclusions [20]. Set differential equations can also be utilized to investigate fuzzy differential equations [11].

In recent years, a number of research papers has dealt with dynamical systems with impulse effect as a class of general hybrid systems. Examples include the adequate mathematical models for numerous processes and phenomena studied in biology, applied physics, etc. Impulsive dynamical systems are characterized by the

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occurrence of abrupt change in the state of the system which occur at certain time instants over a period of negligible duration. The presence of impulse means that the state trajectory does not preserve the basic properties which are associated with non impulsive dynamical systems. Thus, the theory of impulsive differential equations is quite interesting and has attracted the attention of many scientists, see for instance, [2, 8, 15, 17] and the references therein. Moreover, in certain situations, the future state of the physical problems depends not only on the present state but also on its past history. Thus, introduction of the delay in the governing equations ensures a better modelling of the processes involved [7, 16].

The stability criteria in sense of Lyapunov function is found to be quite elegant to develop the qualitative properties of the null solution of the systems of differential equations. Lakshmikantham and Leela [9] introduced the perturbing Lyapunov function method under weaker conditions to study nonuniform properties of solutions of systems of differential equations. Recently, Soliman [18] discussed the perturbing Lyapunov function method for impulsive differential systems. Akpan and Akinyele [1] introduced the concept of ϕ_0-stability for differential systems.

The purpose of this paper is to extend ϕ_0-stability to impulsive set differential equations with delay. In fact, we apply the perturbing Lyapunov function method [13] to investigate ϕ_0-stability of the null solution of impulsive set differential system with delay.

2. TERMINOLOGY AND PRELIMINARIES

Let E be a Banach space with norm $\|\cdot\|$. Let $P(E)$ denote the class of non-empty compact subsets of E, endowed with the Hausdorff metric

$$\begin{aligned} d_H(A,C) &= \max\{\sup_{x\in A}\inf_{y\in C}\|x-y\|, \sup_{y\in C}\inf_{x\in A}\|x-y\|\} \\ &= \min\{\epsilon>0 | A\subset C+\epsilon B, C\subset A+\epsilon B\}, \end{aligned}$$

and the operations

$$A+C=\{x+y|x\in A, y\in C\},\ \lambda.A=\{\lambda x|x\in A\},$$

where $\|A\| = d_H(A,\{0\})$. $P(E)$ is thus a metric convex cone [19] and the subclass $K = K_c(E)$ of all convex sets in $P(E)$ is a closed convex cone satisfying the following properties:

(a) $\lambda K \subseteq K$, $\lambda \geq 0$, $K+K\subseteq K$;
(b) $K\cap\{-K\}=0$;
(c) $\overline{K}=K$;
(d) $K^o \neq \emptyset$;

where $\overline{K}$ denotes the closure of K in the topology of Hausdorff metric and K^o denotes the interior of K. For the forthcoming analysis, we will restrict ourselves to the finite dimensional space R^n, that is, $E = R^n$. Now, we define the partial ordering in R^n using the approach of [3].

Definition 1.1 For any $X, Y \in R^n$, if there exists a set $Z \in R^n$ such that $Z \in K(K^o)$ and $X = Y + Z$, then we have $X \geq Y$ $(X > Y)$. Similarly, we can define $X \leq Y$ $(X < Y)$.

Definition 1.2 A cone $K^* = \{y \in R^n : (y, x) \geq 0 \text{ for } x \in K\}$ is defined to be the adjoint cone relative to the cone K. The cone K^* satisfies the conditions (a)-(d) with $K_0^* = K^* - \{0\}$. For further details of abstract cones, see, for instance [1, 6].

Definition 1.3 The set $Z \in K_c(R^n)$ satisfying $X = Y + Z$ is known as the Hukuhara difference of the sets X and Y in $K_c(R^n)$ and is denoted as $X - Y$.

Definition 1.4 A function $g \in C(R^n, R^n)$ is said to be quasi-monotone nondecreasing relative to the cone K if $X \leq Y$ and $(\phi_0, Y - X) = 0$ for some $\phi_0 \in K_0^*$ imply that $(\phi_0, g(Y) - g(X)) \geq 0$.

Definition 1.5 For any interval $I \in R$, the mapping $F : I \to K_c(R^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$, if there exists an element $D_H F(t_0) \in K_c(R^n)$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t_0) - F(t_0 - h)}{h},$$

exist in the topology of $K_c(R^n)$ and each one is equal to $D_H F(t_0)$.

Given any $\tau > 0$, we define $\mathrm{C} = C[[-\tau, 0], K_c(R^n)]$. For any $t \in J_0 = [t_0 - \tau, t_0 + a]$, $a > 0$, $U \in [J_0, K_c(R^n)]$, let U_t denote a translation of the restriction of U to the interval $[t - \tau, t]$, that is, $U_t \in \mathrm{C}$ be defined by $U_t(s) = U((t + s))$, $-\tau \leq s \leq 0$.

Consider the impulsive set differential equation with delay

$$\begin{cases} D_H U(t) = F(t, U_t), & t \neq t_k, \\ U_{t_k^+} = I_k(U_{t_k}), & t = t_k, \\ U_{t_0} = \Theta_0 \in \mathrm{C}, & \end{cases} \tag{2.1}$$

where $F \in PC[R_+ \times \mathrm{C}, K_c(R^n)]$ is piecewise continuous and in particular $F : (t_{k-1}, t_k] \times \mathrm{C} \to K_c(R^n)$ is continuous, $I_k : \mathrm{C} \to \mathrm{C}$ is continuous for each k and $\{t_k\}$ is a sequence of points such that $0 \leq t_0 < t_1 < \cdots t_k < \cdots$ with $\lim_{k \to \infty} t_k = \infty$.

By a solution of (2.1), we mean a piecewise continuous function $U(t) = U(t_0, \Theta_0)(t)$ on $[t_0, \infty)$ which is left continuous on $(t_k, t_{k+1}]$ and is defined by

$$U(t_0,\Theta_0)(t) = \begin{cases} \Theta_0, & t_0 - \tau \le t \le t_0, \\ U_0(t_0,\Theta_0)(t), & t_0 \le t \le t_1, \\ U_1(t_1,\Theta_1)(t), & t_1 < t \le t_2, \\ \vdots & \vdots \\ U_k(t_k,\Theta_k)(t), & t_k < t \le t_{k+1}, \\ \vdots & \vdots \end{cases}$$

where $U_k(t_k, \Theta_k)(t)$ is a solution of the set differential equation with delay

$$D_H U(t) = F(t, U_t), \qquad U_{t_k^+} = \Theta_k, \quad k = 0, 1, 2, \ldots.$$

Definition 1.6 Let $V : R_+ \times K_c(R^n) \times \mathrm{C} \to R_+$. Then V is said to belong to class V_0 if

($\mathbf{a_1}$) $V(t, U, \Theta)$ is continuous in $(t_{k-1}, t_k] \times K_c(R^n) \times \mathrm{C}$ and for each $U \in K_c(R^n)$, $\Theta \in \mathrm{C}$, $k = 1, 2, \ldots$, $\lim_{(t,Y,\Theta)\to(t_k^+,U,\Theta)} V(t, Y, \Theta) = V(t_k^+, U, \Theta)$ exists;

($\mathbf{a_2}$) $V(t, U, \Theta)$ is Lipschitzian in U.

Definition 1.7 Let $\phi_0 \in K_0^*$. The zero solution of (2.1) is said to be

($\mathbf{b_1}$) ϕ_0-stable if for $\epsilon > 0$ and $t_0 \in R_+$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that $(\phi_0, \Theta_0) < \delta$ implies that $(\phi_0, U(t_0, \Theta_0)(t)) < \epsilon$, $t \ge t_0$;

($\mathbf{b_2}$) uniformly ϕ_0-stable if δ in (b_1) is independent of t_0;

($\mathbf{b_3}$) asymptotically ϕ_0-stable if (b_1) holds and given $\epsilon > 0$, $t_0 \in R_+$, there exist $\delta_0 = \delta_0(t_0) > 0$ and $T(t_0, \epsilon) > 0$ such that $(\phi_0, \Theta_0) < \delta_0$ implies that $(\phi_0, U(t_0, \Theta_0)(t)) < \epsilon$, $t \ge t_0 + T$;

($\mathbf{b_4}$) uniformly asymptotically ϕ_0-stable if (b_2) holds and δ_0, T in (b_3) are independent of t_0.

Remark. For the stability criteria of the null solution of (2.1), one can employ the measure $\|U(t)\| = diam[U(t)]$, $t \ge t_0$. But the $diam[U(t)]$ is nondecreasing in t once the Hukuhara differences are assumed to exist. This problem can be overcome by utilizing the existence of Hukuhara difference in the initial conditions also, which in fact makes it possible to match the behavior of the solution of set differential equations with the corresponding solutions of ordinary differential equations . In order to do so, we suppose that the Hukuhara difference exists for any given initial values $\Phi_0, \Psi_0 \in K_c(R^n)$ so that we set $\Phi_0 - \Psi_0 = \Theta_0$ and consider the stability of the solution $U(t, t_0, \Phi_0 - \Psi_0) = U(t, t_0, \Theta_0)$ of (2.1).

We now define the following spaces:

$$\mathcal{K} = [\nu \in C[R_+, R_+] : \nu(0) = 0 \text{ and } \nu(U) \text{ is strictly increasing}].$$

$$S(\rho) = [U \in K_c(R^n) : \|U\| < \rho], \; S_1(\rho) = [\Theta \in \mathrm{C} : \|\Theta\| < \rho].$$

3. ϕ_0-STABILITY BY THE METHOD OF PERTURBING LYAPUNOV FUNCTIONS

In this section, we discuss the ϕ_0-Stability and asymptotic ϕ_0-Stability of the zero solution of impulsive set valued differential equations with delay (2.1) by means of perturbing Lyapunov functions.

Theorem 3.1. Assume that

($\mathbf{A_1}$) $\phi_0 \in K_0^*$, $V_1 \in PC(R_+ \times S(\rho) \times S_1(\rho), K)$ and there is a $V_{2,\zeta} \in PC(R_+ \times (S(\rho) \cap S^c(\zeta) \times S_1(\rho), K)$ with $\zeta > 0$ and $S^c(\zeta)$ being complement of $S(\zeta)$ such that $V_1(t, U, \Theta) \in V_0$, $V_1(t, 0, \Theta) = 0$ and there exists $\rho_0 > 0$ such that $U_{t_k} \in S_1(\rho_0)$ implies that $I_k(U_{t_k}) \in S_1(\rho)$ for all k and

$$\begin{cases} D^+(\phi_0, V_1(t, U, \Theta)) \leq (\phi_0, g_1(t, V_1(t, U, \Theta))), \qquad t \neq t_k, \\ \quad (\phi_0, V_1(t_k^+, U(t_0, \Theta_0)(t_k^+), U_{t_k^+}(t_0, \Theta_0))) \\ \leq (\phi_0, J_k(V_1(t_k, U(t_0, \Theta_0)(t_k), U_{t_k}(t_0, \Theta_0)))), \quad k = 1, 2, \ldots, \end{cases} \tag{3.1}$$

where $g_1 : R_+ \times K \to R_+$ is continuous with $g_1(t, 0) = 0$ and $J_k : K \to R_+$ is continuous.

($\mathbf{A_2}$) $V_{2,\zeta}(t, 0, \Theta) = 0$ and $V_{2,\zeta}(t, U, \Theta) \in V_0$ such that

$$b_1(\phi_0, \|U\|) \leq (\phi_0, V_{2,\zeta}(t, U, \Theta)) \leq a_1(\phi_0, \|U\|), \ a_1, b_1 \in \mathcal{K}, \tag{3.2}$$

and

$$\begin{cases} D^+(\phi_0, V_1(t, U, \Theta) + V_{2,\zeta}(t, U, \Theta)) \leq (\phi_0, g_2(t, V_1(t, U, \Theta) + V_{2,\zeta}(t, U, \Theta))), \\ \qquad\qquad t \neq t_k \\ (\phi_0, V_1(t_k^+, U(t_0, \Theta_0)(t_k^+), U_{t_k^+}(t_0, \Theta_0)) + V_{2,\zeta}(t_k^+, U(t_0, \Theta_0)(t_k^+), U_{t_k^+}(t_0, \Theta_0))) \\ \leq (\phi_0, F_k(V_1(t_k, U(t_0, \Theta_0)(t_k), U_{t_k}(t_0, \Theta_0)) + V_{2,\zeta}(t_k, U(t_0, \Theta_0)(t_k), U_{t_k}(t_0, \Theta_0)))), \\ \qquad\qquad k = 1, 2, \ldots, \end{cases} \tag{3.3}$$

where $g_2 : R_+ \times K \to R_+$ is continuous with $g_2(t, 0) = 0$ and $F_k : K \to R_+$ is continuous.

($\mathbf{A_3}$) The zero solution of the problem

$$\begin{cases} w' = g_1(t, w), & t \neq t_k, \\ w(t_k^+) = J_k(w(t_k)), & t = t_k, \ k = 1, 2, \ldots, \\ w(t_0) = w_0 \geq 0, \end{cases} \tag{3.4}$$

is ϕ_0-table and the zero solution of

$$\begin{cases} v' = g_1(t, v), & t \neq t_k, \\ v(t_k^+) = F_k(v(t_k)), & t = t_k, \ k = 1, 2, \ldots, \\ v(t_0) = v_0 \geq 0, \end{cases} \tag{3.5}$$

is uniformly ϕ_0-stable.

Then the zero solution of (2.1) is ϕ_0-stable.

Proof. Let $0 < \epsilon < \rho$, $t_0 \in R_+$ and $b_1(\epsilon) > 0$. Since the zero of (3.5) is uniformly ϕ_0-stable, there is a $\delta' = \delta'(\epsilon)$ such that $(\phi_0, v_0) < \delta'$ implies that $(\phi_0, v(t; t_0, v_0) < b_1(\epsilon)$, $t \geq t_0$, where $V(t; t_0, v_0)$ is any solution of (3.5). Now, we choose $\delta_2 = \delta_2 > 0$ such that $0 < \delta_2 < \epsilon$ and

$$a_1(\delta_2) < \delta'/2. \tag{3.6}$$

Now, in view of the fact that the zero solution of (3.4) is ϕ_0-table, for $\delta'/2 > 0$ and $t_0 \in R_+$, there exists a $\delta_3 = \delta_3(t_0, \epsilon)$ such that

$$(\phi_0, w_0) < \delta_3 \text{ implies that } (\phi_0, w(t; t_0, w_0)) < \delta'/2, \ \ t \geq t_0, \tag{3.7}$$

where $w(t; t_0, w_0)$ is any solution of (3.4). Fix $w_0 = V_1(t_0, \Theta_0, \Theta)$ and choose some $\delta_1 > 0$ such that $(\phi_0, \Theta_0) < \delta_1$ implies that $(\phi_0, V_1(t_0, \Theta_0, \Theta)) < \delta_3$.

Let $\delta = \min(\delta_1, \delta_2)$ so that $(\phi_0, \Theta_0) < \delta$ implies that $(\phi_0, U(t_0, \Theta_0)(t)) < \epsilon$, $t \geq t_0$, where $U(t_0, \Theta_0)(t)$ is any solution of (2.1). Suppose this is not true, then there would exist a solution $U(t_0, \Theta_0)(t)$ of (2.1) with $(\phi_0, \Theta_0) < \delta$ and t_1, t_2 satisfying $t_k < t_1 < t_2 \leq t_{k+1}$ for some k such that

$$\begin{cases} (\phi_0, U(t_0, \Theta_0)(t)) < \epsilon, \quad t_k \geq t \geq t_0, \\ \text{and } (\phi_0, U(t_0, \Theta_0)(t_2)) \geq \epsilon, \quad (\phi_0, U(t_0, \Theta_0)(t_1)) = \delta_2, \end{cases} \tag{3.8}$$

and $U(t_0, \Theta_0)(t) \in \overline{S(\epsilon)}$ on $[t_1, t_2]$. Now, we take $\delta_2 = \zeta$ by requiring that $V_{2,\zeta}$ satisfies (A_2). For $t \in [t_1, t_2]$, we set

$$m(t) = V_1(t, U(t_0, \Theta_0)(t), U_t(t_0, \Theta_0)) + V_{2,\zeta}(t, U(t_0, \Theta_0)(t), U_t(t_0, \Theta_0)), \tag{3.9}$$

and

$$(\phi_0, m(t_1)) \leq (\phi_0, r_2(t_1; t_0, v_0)), \tag{3.10}$$

where $r_2(t_1; t_0, v_0)$ is the maximal solution of (3.5). In view of (3.3) together with (3.9) and (3.10), it follows that

$$(\phi_0, m(t)) \leq (\phi_0, r_2(t; t_0, v_0)), t \in [t_1, t_2].$$

Also, we have

$$(\phi_0, V_1(t_1, U(t_0, \Theta_0)(t_1), U_{t_1}(t_0, \Theta_0))) \leq (\phi_0, r_1(t_1; t_0, w_0)),$$

which, in view of (3.7), yields

$$(\phi_0, V_1(t_1, U(t_0, \Theta_0)(t_1), U_{t_1}(t_0, \Theta_0))) \leq \delta'/2,$$

where $r_1(t; t_0, w_0)$ is the maximal solution of (3.4). Using (A_2), (3.6) and (3.8), we get

$$(\phi_0, V_{2,\zeta}(t_1, U(t_0, \Theta_0)(t_1), U_{t_1}(t_0, \Theta_0))) \leq a_1(\phi_0, U(t_0, \Theta_0)(t_1)) = a_1(\delta_2) < \delta'/2.$$

Hence, by virtue of (A_2) and the fact that $V_1 \in V_0$, we have

$$\begin{aligned} b_1(\epsilon) &\leq b_1(\phi_0, U(t_0,\Theta_0)(t_2)) \\ &\leq (\phi_0, V_{2,\zeta}(t_2, U(t_0,\Theta_0)(t_2), U_{t_2}(t_0,\Theta_0))) \\ &\leq (\phi_0, r_2(t_2; t_0, v_0)) < b_1(\epsilon), \end{aligned}$$

which is a contradiction. This completes the proof.

Theorem 3.2. Assume that

($\mathbf{B_1}$) The assumptions (A_2)–(A_3) of Theorem 3.1 hold with the exception that the zero solution of (3.4) is ϕ_0-table;

($\mathbf{B_2}$) The zero solution of (3.4) is uniformly ϕ_0-table;

($\mathbf{B_3}$) $V_1(t,U,\Theta) \in V_0$, $V_1(t,0,\Theta) = 0$ and

$$\begin{cases} D^+(\phi_0, V_1(t,U,\Theta)) + (\phi_0, h(t,U,\Theta)) \leq (\phi_0, g_1(t, V_1(t,U,\Theta))), & t \neq t_k, \\ \quad (\phi_0, V_1(t_k^+, U(t_0,\Theta_0)(t_k^+), U_{t_k^+}(t_0,\Theta_0))) + (\phi_0, \int_{t_0}^{t_k} p(s,U,\Theta_0)ds) & \\ \quad \leq (\phi_0, J_k(V_1(t_k, U(t_0,\Theta_0)(t_k), U_{t_k}(t_0,\Theta_0)))), & k = 1,2,\ldots, \end{cases} \tag{3.11}$$

where $g_1 : R_+ \times K \to R_+$ is continuous and $g_1(t,w)$ is nondecreasing in w with $g_1(t,0) = 0$ and $J_k : K \to R_+$ is continuous and $J_k(w)$ is nondecreasing in w, $p : R_+ \times S(\rho) \times S_1(\rho) \to R_+$ is continuous, integrable and locally Lipschitzian in U and $(\phi_0, h(t,U,\Theta)) \geq b_2(\phi_0, U)$, $b_2 \in \mathcal{K}$.

Then the zero solution of (2.1) is asymptotically ϕ_0-table.

Proof. In view of the assumptions (B_1)–(B_2), we let $0 < \epsilon = \sigma$ such that $(\phi_0, \Theta_0) < \delta(t_0,\sigma)$ implies that $(\phi_0, U(t_0,\Theta_0)(t)) < \sigma$, $t \geq t_0$, where $U(t_0,\Theta_0)(t)$ is any solution of (2.1). Select $T = T(\epsilon) = \delta'/2b_2(\delta(\epsilon))$ such that $t_0 + T \neq t_k$, $k = 1,2,\ldots$. We claim that there is a $t^* \in [t_0, t_0+T]$ such that $(\phi_0, p(t^*, U, \Theta_0)) \leq b_2(\delta(\epsilon))$ for any solution of (2.1) provided that $(\phi_0,\Theta_0) < \delta(t_0,\sigma)$. If it is not true, then $\forall t \in [t_0, t_0+T]$, $(\phi_0, p(t,U,\Theta)) \geq b_2(\delta(\epsilon))$.

For $t \in [t_0, t_1]$, we have

$$\begin{aligned} \left(\phi_0, \int_{t_0}^{t_1} p(s,U,\Theta)ds\right) + (\phi_0, V_1(t,U,\Theta)) &\leq (\phi_0, r_1(t; t_0, V_1(t,U,\Theta))) \\ &= (\phi_0, r_1(t; t_0, w_0)). \end{aligned}$$

Now, for $t \in (t_{k-1}, t_k]$,

$$\begin{aligned} &(\phi_0, V_1(t_k^+, U(t_0,\Theta_0)(t_k), U_{t_k^+}(t_0,\Theta_0))) + \left(\phi_0, \int_{t_0}^{t_k} p(s,U,\Theta_0)ds\right) \\ &\leq (\phi_0, J_k(r_1(t_k; t_0, w_0))) \leq (\phi_0, J_k(r_1^{k-1}(t_k; t_{k-1}, w_{k-1}^+))) = (\phi_0, w_k^+), \end{aligned}$$

where

$$r_1(t;t_0,w_0)=\begin{cases} r_1^0(t;t_0,w_0), & t_0\le t\le t_1,\\ r_1^1(t;t_0,w_1^+), & t_1<t\le t_2,\\ \vdots & \vdots\\ r_1^{k-1}(t;t_0,w_{k-1}^+), & t_{k-1}<t\le t_k,\\ \vdots & \vdots \end{cases}$$

Suppose that

$$\begin{aligned}&(\phi_0, V_1(t_1^+, U(t_0,\Theta_0)(t_1^+), U_{t_1^+}(t_0,\Theta_0)))\\ &\quad+\left(\phi_0,\int_{t_0}^{t_1} p(s,U,\Theta_0)ds\right)\le(\phi_0,w_1^+),\qquad t_1<t\le t_2.\end{aligned}$$

In view of (B_3), we have

$$(\phi_0,V_1(t,U,\Theta))+\left(\phi_0,\int_{t_0}^{t} p(s,U,\Theta)ds\right)\le(\phi_0,r_1(t;t_0,V_1(t,U,\Theta))),\quad t_1<t\le t_2.$$

Hence, by induction, we get

$$(\phi_0,V_1(t,U,\Theta))+\left(\phi_0,\int_{t_0}^{t} p(s,U,\Theta)ds\right)\le(\phi_0,r_1(t;t_0,V_1(t,U,\Theta))),\qquad t\ge t_0.$$

Thus it follows that

$$\begin{aligned}0&\le(\phi_0,V_1(t_0+T,U,\Theta))+\left(\phi_0,\int_{t_0}^{t_0+T}p(s,U,\Theta)ds\right)\le(\phi_0,r_1(t_0+T;t_0,w_0))\\ &\le(\phi_0,r_1(t_0+T;t_0,w_0))-\left(\phi_0,\int_{t_0}^{t_0+T}b_2(\delta(\epsilon))ds\right)\\ &<(\phi_0,\delta'/2)-(\phi_0,b_2(\delta(\epsilon))T)<0,\end{aligned}$$

which leads to a contradiction.

This implies that there is a $t^*\in[t_0,t_0+T]$ such that $(\phi_0,p(t^*,U,\Theta_0))\le b_2(\delta(\epsilon))$ for any solution of (1.1) provided that $(\phi_0,\Theta_0)<\delta(t_0,\sigma)$. Also, $b_2(\delta(\epsilon))>(\phi_0,p(t^*,U,\Theta_0))\ge b_2(\phi_0,U(t_0,\Theta_0)(t))$, then $(\phi_0,U(t_0,\Theta_0)(t))<\delta(\epsilon)$ for some $t^*\in[t_0,t_0+T]$. We assert that $(\phi_0,U(t_0,\Theta_0)(t))<\epsilon$, $t\ge t_0+T$. Suppose that our assertion is not true, which turns out to be a contradiction using the procedure employed in Theorem 3.1. Hence the proof is complete.

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