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MONOTONE ITERATIVE SCHEME FOR WEAKLY COUPLED SYSTEM OF FINITE DIFFERENCE REACTION-DIFFUSION EQUATIONS

D. B. DHAIGUDE[1], S. B. KIWNE[2], AND R. M. DHAIGUDE[3]

[1]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad-431 004, INDIA *E-mail:* dnyanraja@gmail.com
[2]Department of Mathematics, Deogiri College, Aurangabad -431 005, INDIA *E-mail:* sbkiwne @ yahoo.co.in
[3]Department of Mathematics, Arts, Science and Commerce College, Badnapur - 431 202, Dist. Jalna.INDIA

ABSTRACT. The purpose of this paper is to develop monotone iteration scheme using the notion of upper and lower solutions for system of nonlinear finite difference equations, which correspond to the weakly coupled system of nonlinear reaction-diffusion equations with nonlinear boundary conditions. Two monotone sequences are constructed for the finite difference equations when both reaction function and boundary function are mixed quasimonotone. It is shown that these sequences converge monotonically to a solution of the finite difference system, which leads to existence-comparison result for the solution of the problem.

1. INTRODUCTION

Various real problems in different fields from science and technology are governed by a weakly coupled system of nonlinear reaction-diffusion equations. The weakly coupled systems are well studied by many researchers for both continuous problems [2, 6, 7] as well as discrete problems [1, 3, 8, 9]. Recently Dhaigude, Kiwne and Dhaigude, [3], have studied such a system by introducing the notion of upper and lower solutions together with associated monotone iterations. Here the system is coupled through reaction and boundary functions. Such systems are also studied by Pao [7], Chandra etc. [2], for continuous problems. Our aim is to extend the results in [3] by developing monotone scheme for system of nonlinear finite difference equations which corresponds to the weakly coupled system of nonlinear reaction-diffusion equations with nonlinear boundary conditions, when both reaction function and boundary function are mixed quasimonotone.

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We organize the paper as follows. Section 2 is devoted for the formation of a system of finite difference equations from corresponding continuous system. In section 3, the notion of upper and lower solutions for discrete problem is introduced. Section 4 deals with development of monotone iterative scheme under mixed quasi-monotone property of functions. It helps us to construct two monotone sequences. The existence-comparison theorem is proved in the last section.

2. THE FINITE DIFFERENCE EQUATIONS

In this section we obtain the discrete version of the inintial boundary value problem (IBVP) for weakly coupled system of nonlinear reaction-diffusion equations with nonlinear boundary conditions :

$$\begin{cases} u_t - D^{(1)}\nabla^2 u + b^{(1)}\nabla u = f^{(1)}(x,t,u,v) & \text{in } D_T \\ v_t - D^{(2)}\nabla^2 v + b^{(2)}\nabla v = f^{(2)}(x,t,u,v) & \text{in } D_T \end{cases} \tag{2.1}$$

$$\begin{cases} \alpha^{(1)}(x)\dfrac{\partial u}{\partial \nu} + \beta^{(1)}(x)u = g^{(1)}(x,t,u,v) & \text{in } S_T \\ \alpha^{(2)}(x)\dfrac{\partial v}{\partial \nu} + \beta^{(2)}(x)v = g^{(2)}(x,t,u,v) & \text{in } S_T \end{cases} \tag{2.2}$$

$$\begin{cases} u(x,0) = \psi^{(1)}(x) & \text{in } \Omega \\ v(x,0) = \psi^{(2)}(x) & \text{in } \Omega \end{cases} \tag{2.3}$$

where Ω is a bounded domain in $R^P(p = 1, 2, \dots)$ with boundary $\partial\Omega, D_T = \Omega \times (0,T], S_T = \partial\Omega \times (0,T]$, $T > 0$. Assume that the functions $f^{(l)}$, $g^{(l)}$ and $\psi^{(l)}$ for $l = 1, 2$ are Holder continuous on their respective domains of their definitions. The boundary function $(g^{(1)}, g^{(2)})$ is in general nonlinear, C^1 function in (u, v) and may depend explicitly on (x, t). The reaction function $(f^{(1)}, f^{(2)})$ is nonlinear in u and v respectively. Note that $D^{(l)}(x,t) > 0$ and $b^{(l)}(x,t)$ for $l = 1, 2$ are diffusion and convection coefficients respectively.

We convert the above continuous IBVP (2.1)–(2.3) into finite difference weakly coupled reaction-diffusion system.Therefore, introduce the following notations. Let $i = (i_1, i_2, \dots i_p)$ be a multiple index with $i = 0, 1, 2, \dots, M_v{+}1$ and $x_i = (x_{i_1}, x_{i_2}, \dots, x_{i_p})$ the arbitrary mesh point in Ω_p where M_v is the total number of interior mesh points in the x_{i_v} coordinate direction. Denote by $\Omega_p, \overline{\Omega}_p, \partial\Omega_p, \Lambda_p$ and S_p the sets of mesh points in $\Omega, \overline{\Omega}, \partial\Omega, \Omega \times (0,T]$ and $\partial\Omega \times (0,T]$ respectively and $\overline{\Lambda}_p$ denote the set of all mesh points in $\overline{\Omega} \times (0,T]$ where $\overline{\Omega}$ is closure of Ω. Assume that the domain $\overline{\Omega}_p = \Omega_p + \partial\Omega_p$ is connected, where $\partial\Omega_p$ is the set of boundary mesh points. Suppose (i, n) represent the mesh point (x_i, t_n).

$$\begin{aligned} Set \qquad & u_{i,n} \equiv u(x_i,t_n); v_{i,n} \equiv v(x_i,t_n) \\ & D^{(l)}_{i,n} \equiv D^{(l)}(x_i,t_n); b^{(l)}_{i,n} \equiv b^{(l)}(x_i,t_n) \\ & f^{(l)}(u_{i,n},v_{i,n}) \equiv f^{(l)}(x_i,t_n,u(x_i,t_n),v(x_i,t_n)) \\ & g^{(l)}(u_{i,n},v_{i,n}) \equiv g^{(l)}(x_i,t_n,u(x_i,t_n),v(x_i,t_n)) \\ & \psi^{(l)}_i \equiv \psi^{(l)}(x_i) \qquad\qquad l=1,2. \\ & u_{i,0} \equiv u(x_i,0), v_{i,0} \equiv v(x_i,0) \end{aligned} \tag{2.4}$$

Let $k_n = t_n - t_{n-1}$ be n^{th} time increment for $n = 1, 2, \ldots, N$ and h_ν, be the spatial increment in the x_{i_v} coordinate direction. Let $e_v = (0, \ldots, 1, 0, \ldots, 0)$ be the unit vector in R^P where 1 appears in the ν^{th} component and is zero elswhere.

Then the standard first and second order difference operators $\delta^{(\nu)}$ and $\Delta^{(\nu)}$ respectively [1, 4, 5] are given by

$$\begin{aligned} \delta^{(\nu)}u(x_i,t_n) &= 2h_\nu^{-1}[u(x_i+h_\nu e_\nu,t_n) - u(x_i-h_\nu e_\nu,t_n)] \\ \Delta^{(\nu)}u(x_i,t_n) &= h_\nu^{-2}[u(x_i+h_\nu e_\nu,t_n) - 2u(x_i,t_n) + u(x_i-h_\nu e_\nu,t_n)] \end{aligned} \tag{2.5}$$

and usual backward difference approximation for u_t by $k_n^{-1}(u_{i,n} - u_{i,n-1})$.

The implicit finite difference approximation for the equations in parabolic problem (2.1)–(2.3) is given by

$$\begin{aligned} k_n^{-1}(u_{i,n}-u_{i,n-1}) - \sum_{\nu=1}^{p}(D^{(1)}_{i,n}\Delta^{(\nu)}u_{i,n} + b^{(1)}_{i,n}\delta^{(\nu)}u_{i,n}) &= f^{(1)}(u_{i,n},v_{i,n}) \\ k_n^{-1}(v_{i,n}-v_{i,n-1}) - \sum_{\nu=1}^{p}(D^{(2)}_{i,n}\Delta^{(\nu)}u_{i,n} + b^{(2)}_{i,n}\delta^{(\nu)}u_{i,n}) &= f^{(2)}(u_{i,n},v_{i,n}), \quad (i,n) \in \Lambda_p \end{aligned} \tag{2.6}$$

$$\begin{aligned} B^{(1)}[u_{i,n}] &= g^{(1)}(u_{i,n},v_{i,n}), \\ B^{(2)}[v_{i,n}] &= g^{(2)}(u_{i,n},v_{i,n}), \qquad (i,n) \in S_p \end{aligned} \tag{2.7}$$

$$u_{i,0} = \psi_i^1,\ v_{i,0} = \psi_i^2, \qquad i \in \Omega_p \tag{2.8}$$

where

$$\begin{aligned} B^{(1)}[u_{i,n}] &\equiv \alpha^{(1)}(x_i)|x_i-\hat{x}_i|^{-1}[u(x_i,t_n) - u(\hat{x}_i,t_n)] + \beta^{(1)}(x_i)u(x_i,t_n) \\ B^{(2)}[v_{i,n}] &\equiv \alpha^{(2)}(x_i)|x_i-\hat{x}_i|^{-1}[v(x_i,t_n) - v(\hat{x}_i,t_n)] + \beta^{(2)}(x_i)v(x_i,t_n) \end{aligned} \tag{2.9}$$

In the above boundary approximation, $\hat{x}_i$ is a suitable point in Ω_p and $|x_i - \hat{x}_i|$ is the distance between x_i and $\hat{x}_i$. Here boundary surface is assumed to be parallel to the co-ordinate planes.

In problems (2.6)–(2.8) the reaction function $(f^{(1)}, f^{(2)})$ and the boundary function $(g^{(1)}, g^{(2)})$ are assumed to be mixed quasimonotone. We define mixed quasi monotone property of reaction function $(f^{(1)}, f^{(2)})$ and boundary function $(g^{(1)}, g^{(2)})$ as follows:

Definition 2.1. A C^1 functions $(f^{(1)}, f^{(2)}); (g^{(1)}, g^{(2)})$ are said to be mixed quasi-monotone in $J \subset R^2$ if

$$\begin{aligned} &\frac{\partial f^{(1)}}{\partial v} \le 0, \qquad \frac{\partial f^{(2)}}{\partial u} \ge 0 \quad \text{or vice versa} \\ &\frac{\partial g^{(1)}}{\partial v} \le 0, \qquad \frac{\partial g^{(2)}}{\partial u} \ge 0 \quad \text{or vice versa} \end{aligned} \qquad \text{for } (u,v) \in J$$

3. UPPER-LOWER SOLUTIONS

We define upper-lower solutions of the system (2.6)–(2.8).

Definition 3.1. Two functions $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ and $(\hat{u}_{i,n}, \hat{v}_{i,n})$ in $\overline{\Lambda}_p$ with $(\tilde{u}_{i,n}, \tilde{v}_{i,n}) \ge (\hat{u}_{i,n}, \hat{v}_{i,n})$ are called ordered upper-lower solutions of the system (2.6)–(2.8) if they satisfy the following inequalities

$$\begin{aligned} &k_n^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - L^{(1)}[\tilde{u}_{i,n}] \ge f^{(1)}(\tilde{u}_{i,n} - \hat{v}_{i,n}) \\ &k_n^{-1}(\hat{u}_{i,n} - \hat{u}_{i,n-1}) - L^{(1)}[\hat{u}_{i,n}] \le f^{(1)}(\hat{u}_{i,n} - \tilde{v}_{i,n}) \quad (i,n) \in \Lambda_p \\ &k_n^{-1}(\tilde{v}_{i,n} - \tilde{v}_{i,n-1}) - L^{(2)}[\tilde{v}_{i,n}] \ge f^{(2)}(\tilde{u}_{i,n} - \tilde{v}_{i,n}) \\ &k_n^{-1}(\hat{v}_{i,n} - \hat{v}_{i,n-1}) - L^{(2)}[\hat{v}_{i,n}] \le f^{(2)}(\hat{u}_{i,n} - \tilde{v}_{i,n}) \\ &\qquad \tilde{u}_{i,0} \ge \psi_i^{(1)} \ge \hat{u}_{i,0} \qquad \tilde{v}_{i,0} \ge \psi_i^{(2)} \ge \hat{v}_{i,0} \quad i \in \Omega_p \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &B^{(1)}[\tilde{u}_{i,n}] \ge g^{(1)}(\tilde{u}_{i,n}, \hat{v}_{i,n}) \\ &B^{(1)}[\hat{u}_{i,n}] \ge g^{(1)}(\hat{u}_{i,n}, \tilde{v}_{i,n}) \\ &B^{(2)}[\tilde{v}_{i,n}] \ge g^{(2)}(\tilde{u}_{i,n}, \tilde{v}_{i,n}) \\ &B^{(2)}[\hat{v}_{i,n}] \ge g^{(2)}(\hat{u}_{i,n}, \hat{v}_{i,n}) \qquad (i,n) \in S_p \end{aligned} \tag{3.2}$$

Where

$$L^{(l)}[w_{i,n}] \equiv \sum_{\nu=1}^{p} (D_{i,n}^{(l)} \Delta^{(\nu)} w_{i,n} + b_{i,n}^{(l)} \delta^{(\nu)} w_{i,n}), \quad l = 1, 2.$$

Now we define the sector denoted by $S_{i,n}$ for the pairs $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ and $(\hat{u}_{i,n}, \hat{v}_{i,n})$ of ordered upper-lower solutions of (2.6)–(2.8) as follows:

Definition 3.2. Let $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ and $(\hat{u}_{i,n}, \hat{v}_{i,n})$ be any two functions in $\overline{\Lambda}_p$ with $(\tilde{u}_{i,n}, \tilde{v}_{i,n}) \ge (\hat{u}_{i,n}, \hat{v}_{i,n})$ then we define the sector $S_{i,n}$ as

$$S_{i,n} = \{(u_{i,n}, v_{i,n}) \in \overline{\Lambda}_p : (\hat{u}_{i,n}, \hat{v}_{i,n}) \le (u_{i,n}, v_{i,n}) \le (\tilde{u}_{i,n}, \tilde{v}_{i,n})\} \tag{3.3}$$

where the inequalities are both component wise as well as point wise.

Suppose there exist nonnegative functions $\gamma_{i,n}^{(1)}, \gamma_{i,n}^{(2)}$ such that the function $(f^{(1)}, f^{(2)})$ satisfies the following one sided Lipschitz condition:

$$\begin{aligned} f^{(1)}(u_{i,n}, v_{i,n}) - f^{(1)}(u'_{i,n}, v_{i,n}) &\geq -\gamma_{i,n}^{(1)}(u_{i,n} - u'_{i,n}) \quad \text{when } u_{i,n} \geq u'_{i,n} \\ f^{(2)}(u_{i,n}, v_{i,n}) - f^{(2)}(u_{i,n}, v'_{i,n}) &\geq -\gamma_{i,n}^{(2)}(v_{i,n} - v'_{i,n}) \quad \text{when } v_{i,n} \geq v'_{i,n} \end{aligned} \tag{3.4}$$

for any $(u_{i,n}, v_{i,n}), (u'_{i,n}, v'_{i,n})$ in the sector $S_{i,n}$.

Suppose there exist nonnegative functions $\sigma_{i,n}^{(1)}, \sigma_{i,n}^{(2)}$ such that for any $(u_{i,n}, v_{i,n})$, $(u'_{i,n}, v'_{i,n})$ in the sector $S_{i,n}$, the function $(g^{(1)}, g^{(2)})$ satisfies the following one sided Lipschitz condition:

$$\begin{aligned} g^{(1)}(u_{i,n}, v_{i,n}) - g^{(1)}(u'_{i,n}, v_{i,n}) &\geq -\sigma_{i,n}^{(1)}(u_{i,n} - u'_{i,n}) \quad \text{when } u_{i,n} \geq u'_{i,n} \\ g^{(2)}(u_{i,n}, v_{i,n}) - g^{(2)}(u_{i,n}, v'_{i,n}) &\geq -\sigma_{i,n}^{(2)}(v_{i,n} - v'_{i,n}) \quad \text{when } v_{i,n} \geq v'_{i,n} \end{aligned} \tag{3.5}$$

Let

$$\begin{aligned} F^{(1)}(u_{i,n}, v_{i,n}) &= \gamma_{i,n}^{(1)} u_{i,n} + f^{(1)}(u_{i,n}, v_{i,n}) \\ F^{(2)}(u_{i,n}, v_{i,n}) &= \gamma_{i,n}^{(2)} v_{i,n} + f^{(2)}(u_{i,n}, v_{i,n}) \\ G^{(1)}(u_{i,n}, v_{i,n}) &= \sigma_{i,n}^{(1)} u_{i,n} + g^{(1)}(u_{i,n}, v_{i,n}) \\ G^{(2)}(u_{i,n}, v_{i,n}) &= \sigma_{i,n}^{(2)} v_{i,n} + g^{(2)}(u_{i,n}, v_{i,n}) \end{aligned} \tag{3.6}$$

Now we prove monotone property of $F^{(1)}, G^{(1)}$ for $l = 1, 2$.

Lemma 3.3. *Suppose that $(u_{i,n}, v_{i,n})$ and $(u'_{i,n}, v'_{i,n})$ are any two functions in the sector $S_{i,n}$ such that $(u_{i,n}, v_{i,n}) \geq (u'_{i,n}, v'_{i,n})$ and suppose that the Lipschitz conditions* (3.4) *and* (3.5) *hold; then*

$$\begin{aligned} (i) \qquad & F^{(1)}(u_{i,n}, v'_{i,n}) \geq F^{(1)}(u'_{i,n}, v_{i,n}) \\ & F^{(2)}(u_{i,n}, v_{i,n}) \geq F^{(2)}(u'_{i,n}, v'_{i,n}) \\ (ii) \qquad & G^{(1)}(u_{i,n}, v'_{i,n}) \geq G^{(1)}(u'_{i,n}, v_{i,n}) \\ & G^{(2)}(u_{i,n}, v_{i,n}) \geq G^{(2)}(u'_{i,n}, v'_{i,n}) \end{aligned}$$

PROOF: We prove

$$F^{(1)}(u_{i,n}, v'_{i,n}) \geq F^{(1)}(u'_{i,n}, v_{i,n})$$

Using (3.6), we have

$$\begin{aligned} F^{(1)}(u_{i,n}, v'_{i,n}) - F^{(1)}(u'_{i,n}, v'_{i,n}) &= \gamma_{i,n}^{(1)}(u_{i,n} - u'_{i,n}) + f^{(1)}(u_{i,n}, v'_{i,n}) - f^{(1)}(u'_{i,n} - v_{i,n}) \\ &= [\gamma_{i,n}^{(1)}(u_{i,n} - u'_{i,n}) + f^{(1)}(u_{i,n}, v'_{i,n}) - f^{(1)}(u'_{i,n} - v'_{i,n})] \\ &\quad + [f^{(1)}(u'_{i,n}, v'_{i,n}) - f^{(1)}(u'_{i,n} - v_{i,n})] \end{aligned}$$

Inequality follows from Lipschitz condition (3.4) and mixed quasimonotone property for the function $f^{(1)}$.

Similarly, we can prove

$$F^{(2)}(u_{i,n}, v_{i,n}) \geq F^{(2)}(u'_{i,n}, v'_{i,n})$$

using Lipschitz condition (3.4) and mixed quasimonotone property for the function $f^{(2)}$. This proves (i).

Now we prove (ii). Using (3.6), we have

$$G^{(2)}(u_{i,n}, v_{i,n}) - G^{(2)}(u'_{i,n}, v'_{i,n}) = \sigma^{(2)}_{i,n}(v_{i,n} - v'_{i,n}) + g^{(2)}(u_{i,n}, v_{i,n}) - g^{(2)}(u'_{i,n} - v'_{i,n})$$

The result follows from Lipschitz condition (3.5) and mixed quasimonotone property for the function $g^{(2)}$.

Similarly, we can prove $G^{(1)}(u_{i,n}, v'_{i,n}) \geq G^{(1)}(u'_{i,n}, v_{i,n})$,using Lipschitz condition (3.5) and mixed quasimonotone property for the function $g^{(1)}$. The proof of the Lemma is completed.

Lemma 3.4 (Positivity Lemma, Pao, [8]). *Suppose that $u_{i,n}$ satisfies*

(i) $k_n^{-1}(u_{i,n} - u_{i,n-1}) - \left(\sum_{\nu=1}^{p} (D_{i,n}\Delta^\nu u_{i,n} + b_{i,n}\delta^\nu u_{i,n})\right) + c_{i,n}u_{i,n} \geq 0 \quad (i,n) \in \Lambda_p$

(ii) $\alpha(x_i, t_n)|x_i - \hat{x}_i|[u(x_i, t_n) - u(\hat{x}_i, t_n)] + \beta(x_i, t_n)u(x_i, t_n) \geq 0 \quad (i,n) \in S_p$

(iii) $u_{i,0} \geq \quad 0 \in \Omega_p$

where $c_{i,n} \geq 0$, then $u_{i,n} \geq 0$ in $\overline{\Lambda}_p$

4. MONOTONE ITERATIVE SCHEME

We choose suitable initial iteration$(u^{(0)}_{i,n}, v^{(0)}_{i,n})$ as either $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ or $(\hat{u}_{i,n}, \hat{v}_{i,n})$ and construct a sequence of iterations $\{u^{(m)}_{i,n}, v^{(m)}_{i,n}\}$ simultaneously from the following iteration processes I and II.

Iteration Process I :

$$\begin{aligned}
k_n^{-1}(\overline{u}^{(m)}_{i,n} - \overline{u}^{(m)}_{i,n-1}) - L^{(1)}[\overline{u}^{(m)}_{i,n}] + \gamma^{(1)}_{i,n}\overline{u}^{(m)}_{i,n} &= F^{(1)}(\overline{u}^{(m-1)}_{i,n}, \underline{v}^{(m-1)}_{i,n}) \\
k_n^{-1}(\overline{v}^{(m)}_{i,n} - \overline{v}^{(m)}_{i,n-1}) - L^{(2)}[\overline{v}^{(m)}_{i,n}] + \gamma^{(2)}_{i,n}\overline{v}^{(m)}_{i,n} &= F^{(2)}(\overline{u}^{(m)}_{i,n}, \overline{v}^{(m-1)}_{i,n}) \\
B^{(1)}[\overline{u}^{(m)}_{i,n}] + \sigma^{(1)}_{i,n}\overline{u}^{(m)}_{i,n} &= G^{(1)}(\overline{u}^{(m-1)}_{i,n}, \underline{v}^{(m-1)}_{i,n}) \\
B^{(2)}[\overline{v}^{(m)}_{i,n}] + \sigma^{(2)}_{i,n}\overline{v}^{(m)}_{i,n} &= G^{(2)}(\overline{u}^{(m)}_{i,n}, \overline{v}^{(m-1)}_{i,n}) \qquad (4.1) \\
\overline{u}^{(m)}_{i,0} &= \psi^{(1)}_i \\
\overline{v}^{(m)}_{i,0} &= \psi^{(2)}_i
\end{aligned}$$

when $m = 1, 2, \ldots$

Iteration Process II :

$$\begin{aligned} k_n^{-1}(\underline{u}_{i,n}^{(m)} - \underline{u}_{i,n-1}^{(m)}) - L^{(1)}[\underline{u}_{i,n}^{(m)}] + \gamma_{i,n}^{(1)}\underline{u}_{i,n}^{(m)} &= F^{(1)}(\underline{u}_{i,n}^{(m-1)}, \overline{v}_{i,n}^{(m-1)}) \\ k_n^{-1}(\underline{v}_{i,n}^{(m)} - \underline{v}_{i,n-1}^{(m)}) - L^{(2)}[\underline{v}_{i,n}^{(m)}] + \gamma_{i,n}^{(2)}\underline{v}_{i,n}^{(m)} &= F^{(2)}(\underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m-1)}) \\ B^{(1)}[\underline{u}_{i,n}^{(m)}] + \sigma_{i,n}^{(1)}\underline{u}_{i,n}^{(m)} &= G^{(1)}(\underline{u}_{i,n}^{(m-1)}, \overline{v}_{i,n}^{(m-1)}) \\ B^{(2)}[\underline{v}_{i,n}^{(m)}] + \sigma_{i,n}^{(2)}\underline{v}_{i,n}^{(m)} &= G^{(2)}(\underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m-1)}) \\ \underline{u}_{i,0}^{(m)} &= \psi_i^{(1)} \\ \underline{v}_{i,0}^{(m)} &= \psi_i^{(2)} \end{aligned} \tag{4.2}$$

when $m = 1, 2, \ldots$

The construction of sequence of iterations $\{u_{i,n}^{(m)}, v_{i,n}^{(m)}\}$ is possible if we split up the iteration process I & II given above into the following four sub-iteration processes.

Set

$$\begin{aligned} \mathscr{L}^{(1)}[u_{i,n}^{(m)}] &\equiv k_n^{-1}(u_{i,n}^{(m)}, u_{i,n-1}^{(m)}) - L^{(1)}[u_{i,n}^{(m)}] + \gamma_{i,n}^{(1)}u_{i,n}^{(m)} \\ \mathscr{L}^{(2)}[v_{i,n}^{(m)}] &\equiv k_n^{-1}(v_{i,n}^{(m)}, v_{i,n-1}^{(m)}) - L^{(2)}[v_{i,n}^{(m)}] + \gamma_{i,n}^{(2)}v_{i,n}^{(m)} \\ \mathscr{B}^{(1)}[u_{i,n}^{(m)}] &\equiv B^{(1)}[u_{i,n}^{(m)}] + \sigma_{i,n}^{(1)}u_{i,n}^{(m)} \\ B^{(2)}[v_{i,n}^{(m)}] &\equiv B^{2}[v_{i,n}^{(m)}] + \sigma_{i,n}^{(2)}v_{i,n}^{(m)} \end{aligned} \tag{4.3}$$

Sub-Iteration Process I (A) :

$$\begin{aligned} \mathscr{L}^{(1)}[\overline{u}_{i,n}^{(m)}] &= \gamma_{i,n}^{(1)}\overline{u}_{i,n}^{(m-1)} + f^{(1)}(\overline{u}_{i,n}^{(m-1)}, \underline{v}_{i,n}^{(m-1)}) \\ \mathscr{B}^{(1)}[\overline{u}_{i,n}^{(m)}] &= \sigma_{i,n}^{(1)}\overline{u}_{i,n}^{(m-1)} + g^{(1)}(\overline{u}_{i,n}^{(m-1)}, \underline{v}_{i,n}^{(m-1)}) \\ \overline{u}_{i,0}^{(m)} &= \psi_i^{(1)} \end{aligned}$$

where $m = 1, 2, \ldots$

Sub-Iteration Process I (B) :

$$\begin{aligned} \mathscr{L}^{(2)}[\overline{v}_{i,n}^{(m)}] &= \gamma_{i,n}^{(2)}\overline{v}_{i,n}^{(m-1)} + f^{(2)}(\overline{u}_{i,n}^{(m-1)}, \overline{v}_{i,n}^{(m-1)}) \\ \mathscr{B}^{(2)}[\overline{u}_{i,n}^{(m)}] &= \sigma_{i,n}^{(2)}\overline{v}_{i,n}^{(m-1)} + g^{(2)}(\overline{u}_{i,n}^{(m)}, \overline{v}_{i,n}^{(m-1)}) \\ \overline{v}_{i,0}^{(m)} &= \psi_i^{(2)} \end{aligned}$$

where $m = 1, 2, \ldots$

Sub-Iteration Process II (A) :

$$\begin{aligned} \mathscr{L}^{(1)}[\underline{u}_{i,n}^{(m)}] &= \gamma_{i,n}^{(1)}\underline{u}_{i,n}^{(m-1)} + f^{(1)}(\underline{u}_{i,n}^{(m-1)}, \overline{v}_{i,n}^{(m-1)}) \\ \mathscr{B}^{(1)}[\underline{u}_{i,n}^{(m)}] &= \sigma_{i,n}^{(1)}\underline{u}_{i,n}^{(m-1)} + g^{(1)}(\underline{u}_{i,n}^{(m-1)}, \overline{v}_{i,n}^{(m-1)}) \\ \underline{u}_{i,0}^{(m)} &= \psi_i^{(1)} \end{aligned}$$

where $m = 1, 2, \ldots$

Sub-Iteration Process II (B) :

$$\begin{aligned}
\mathscr{L}^{(2)}[\underline{v}_{i,n}^{(m)}] &= \gamma_{i,n}^{(2)}\underline{v}_{i,n}^{(m-1)} + f^{(2)}(\underline{u}_{i,n}^{(m-1)}, \underline{v}_{i,n}^{(m-1)}) \\
\mathscr{B}^{(2)}[\underline{u}_{i,n}^{(m)}] &= \sigma_{i,n}^{(2)}\underline{v}_{i,n}^{(m-1)} + g^{(2)}(\underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m-1)}) \\
\underline{v}_{i,0}^{(m)} &= \psi_i^{(2)}
\end{aligned}$$

where $m = 1, 2, \ldots$

We start with initial iteration as $(\overline{u}_{i,n}^{(0)}, \overline{v}_{i,n}^{(0)}) = (\tilde{u}_{i,n}, \tilde{v}_{i,n})$ and $(\underline{u}_{i,n}^{(0)}, \underline{v}_{i,n}^{(0)}) = (\hat{u}_{i,n}, \hat{v}_{i,n})$ and obtain the sequences denoted by $\{\overline{u}_{i,n}^{(m)}, \overline{v}_{i,n}^{(m)}\}$ and $\{\underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}\}$ respectively, from the sub-iteration processes I A, I B , II A and II B simultaneously.

Lemma 4.1. *Let $(f^{(1)}, f^{(2)})$ and $(g^{(1)}, g^{(2)})$ be mixed quasimonotone C^1- functions in $S_{i,n}$. Then the sequences $\{\overline{u}_{i,n}^{(m)}, \overline{v}_{i,n}^{(m)}\}$ and $\{\underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}\}$ obtained from the iteration process* (4.1), (4.2) *simultaneously by using the initial iterations $(\overline{u}_{i,n}^{(0)}, \overline{v}_{i,n}^{(0)}) = (\tilde{u}_{i,n}, \tilde{v}_{i,n})$ and $(\underline{u}_{i,n}^{(0)}, \underline{v}_{i,n}^{(0)}) = (\hat{u}_{i,n}, \hat{v}_{i,n})$ respectively possess the monotone properly*

$$(\underline{u}_{i,n}^{(m-1)}, \underline{v}_{i,n}^{(m-1)}) \leq (\underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}) \leq (\overline{u}_{i,n}^{(m)}, \overline{v}_{i,n}^{(m)}) \leq (\overline{u}_{i,n}^{(m-1)}, \overline{v}_{i,n}^{(m-1)}), \quad (i,n) \in \overline{\Lambda}_p$$

where $m = 1, 2, \ldots$

PROOF: Let $w_{i,n} = \overline{u}_{i,n}^{(0)} - \overline{u}_{i,n}^{(1)} = \tilde{u}_{i,n} - \overline{u}_{i,n}^{(1)}$
$z_{i,n} = \overline{v}_{i,n}^{(0)} - \overline{v}_{i,n}^{(1)} = \tilde{v}_{i,n} - \overline{v}_{i,n}^{(1)}$

Then by (3.1), (3.2), (3.6) and iteration process (4.1)

$$\begin{aligned}
\mathscr{L}^{(1)}[w_{i,n}] &= \mathscr{L}^{(1)}[\tilde{u}_{i,n}] - F^{(1)}(\overline{u}_{i,n}^{(0)}, \underline{v}_{i,n}^{(0)}) \\
&= k_n^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - L^{(1)}[\tilde{u}_{i,n}] - f^{(1)}(\tilde{u}_{i,n}, \hat{v}_{i,n}) \geq 0 \\
\mathscr{B}^{(1)}[w_{i,n}] &= \mathscr{B}^{(1)}[\tilde{u}_{i,n}] - G^{(1)}(\overline{u}_{i,n}^{(0)}, \underline{v}_{i,n}^{(0)}) \\
&= B^{(1)}[\tilde{u}_{i,n}] - g^{(1)}(\tilde{u}_{i,n}, \hat{v}_{i,n}) \geq 0.
\end{aligned}$$

By (3.1) and iteration process (4.1)

$$w_{i,0} = \overline{u}_{i,0}^{(0)} - \overline{u}_{i,0}^{(1)} = \tilde{u}_{i,0} - \psi_i^{(1)} \geq 0$$

Applying Lemma 3.4, we get $w_{i,n} \geq 0$, which gives

$$\tilde{u}_{i,n} \geq \overline{u}_{i,n}^{(1)}$$

By (3.1), (3.2), (3.6) and iteration process (4.1),

$$\begin{aligned}
\mathscr{L}^{(2)}[z_{i,n}] &= \mathscr{L}^{(2)}[\tilde{v}_{i,n}] - F^{(2)}(\overline{u}^{(1)}_{i,n}, \overline{v}^{(0)}_{i,n}) \\
&= k_n^{-1}(\tilde{v}_{i,n} - \tilde{v}_{i,n-1}) - L^{(2)}[\tilde{v}_{i,n}] - f^{(2)}(\overline{u}^{(1)}_{i,n}, \tilde{v}_{i,n}) \\
&\geq f^{(2)}(\tilde{u}_{i,n}, \tilde{v}_{i,n}) - f^{(2)}(\overline{u}^{(1)}_{i,n}, \tilde{v}_{i,n}) \\
&\geq 0 \quad (\because\ f^{(2)} \text{ is quasimonotone nondecreasing and } \tilde{u}_{i,n} \geq \overline{u}^{(1)}_{i,n}\ ). \\
\mathscr{B}^{(2)}[z_{i,n}] &= \mathscr{B}^{(2)}[\tilde{v}_{i,n}] - G^{(2)}(\overline{u}^{(1)}_{i,n}, \overline{v}^{(0)}_{i,n}) \\
&= B^{(2)}[\tilde{v}_{i,n}] - g^{(2)}(\overline{u}^{(1)}_{i,n}, \tilde{v}_{i,n}) \\
&\geq g^{(2)}(\tilde{u}_{i,n}, \tilde{v}_{i,n}) - g^{(2)}(\overline{u}^{(1)}_{i,n}, \tilde{v}_{i,n}) \\
&\geq 0 \quad (\because\ g^{(2)} \text{ is quasimonotone nondecreasing and } \tilde{u}_{i,n} \geq \overline{u}^{(1)}_{i,n}).
\end{aligned}$$

By (3.1) and iteration process (4.1)

$$z_{i,n} = \overline{v}^{(0)}_{i,0} - \overline{v}^{(1)}_{i,0} = \tilde{v}_{i,0} - \psi^{(2)}_i \geq 0$$

By Lemma 3.4, we get $z_{i,n} \geq 0$. So $\tilde{v}_{i,0} \geq \overline{v}^{(1)}_{i,0}$

Similarly letting $\quad w_{i,n} = \underline{u}^{(1)}_{i,n} - \underline{u}^{(0)}_{i,n} = \overline{u}^{(1)}_{i,n} - \hat{u}_{i,n}$

$z_{i,n} = \underline{v}^{(1)}_{i,n} - \underline{v}^{(0)}_{i,n} = \underline{v}^{(1)}_{i,n} - \hat{v}_{i,n}$

Then by (3.1), (3.2), (3.6) and iteration process (4.2),similar arguments lead to

$$\underline{u}^{(1)}_{i,n} \geq \hat{u}_{i,n} \quad \text{and} \quad \underline{v}^{(1)}_{i,n} \geq \hat{v}_{i,n}$$

Now let $w^{(1)}_{i,n} = \overline{u}^{(1)}_{i,n} - \underline{u}^{(1)}_{i,n} \qquad z^{(1)}_{i,n} = \overline{v}^{(1)}_{i,n} - \underline{v}^{(1)}_{i,n}$

Then by iteration processes (4.1), (4.2) and Lemma 3.3

$$\begin{aligned}
\mathscr{L}^{(1)}[w^{(1)}_{i,n}] &= \mathscr{L}^{(1)}[\overline{u}^{(1)}_{i,n}] - \mathscr{L}^{(1)}[\underline{u}^{(1)}_{i,n}] \\
&= F^{(1)}(\overline{u}^{(0)}_{i,n}, \underline{v}^{(0)}_{i,n}) - F^{(1)}(\underline{u}^{(0)}_{i,n}, \overline{v}^{(0)}_{i,n}) \geq 0 \\
\mathscr{B}^{(1)}[w^{(1)}_{i,n}] &= \mathscr{B}^{(1)}[\overline{u}^{(1)}_{i,n}] - \mathscr{B}^{(1)}[\underline{u}^{(1)}_{i,n}] \\
&= G^{(1)}(\overline{u}^{(0)}_{i,n}, \underline{v}^{(0)}_{i,n}) - G^{(1)}(\underline{u}^{(0)}_{i,n}, \overline{v}^{(0)}_{i,n}) \geq 0
\end{aligned}$$

By iteration process (4.1) and (4.2)

$$w^{(1)}_{i,0} = \overline{u}^{(1)}_{i,0} - \underline{u}^{(1)}_{i,0} = \psi^{(1)}_i - \psi^{(1)}_i = 0$$

By Lemma 3.4, $w^{(1)}_{i,n} \geq 0$. It leads to

$$\overline{u}^{(1)}_{i,n} \geq \underline{u}^{(1)}_{i,n}$$

By iteration process (4.1), (4.2) and Lemma 3.3

$$\begin{aligned}\mathscr{L}^{(2)}[z_{i,n}^{(1)}] &= \mathscr{L}^{(2)}[\overline{v}_{i,n}^{(1)}] - \mathscr{L}^{(2)}[\underline{v}_{i,n}^{(1)}] \\ &= F^{(2)}(\overline{u}_{i,n}^{(1)}, \overline{v}_{i,n}^{(0)}) - F^{(2)}(\underline{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(0)}) \geq 0 \\ \mathscr{B}^{(2)}[z_{i,n}^{(1)}] &= \mathscr{B}^{(2)}[\overline{v}_{i,n}^{(1)}] - \mathscr{B}^{(2)}[\underline{v}_{i,n}^{(1)}] \\ &= G^{(2)}(\overline{u}_{i,n}^{(1)}, \overline{v}_{i,n}^{(0)}) - G^{(2)}(\underline{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(0)}) \geq 0\end{aligned}$$

By iteration process (4.1) and (4.2)

$$z_{i,0}^{(1)} = \overline{v}_{i,0}^{(1)} - \underline{v}_{i,0}^{(1)} = \psi_i^{(2)} - \psi_i^{(2)} = 0$$

Suppose the monotone property holds for some integer m. Define

$$w_{i,n}^{(m)} = \overline{u}_{i,n}^{(m)} - \overline{u}_{i,n}^{(m+1)}, \quad z_{i,n}^{(m)} = \overline{v}_{i,n}^{(m)} - \overline{v}_{i,n}^{(m+1)}$$

Then by iteration process (4.1) and Lemma 3.3

$$\begin{aligned}\mathscr{L}^{(1)}[w_{i,n}^{(m)}] &= F^{(1)}(\overline{u}_{i,n}^{(m-1)}, \underline{v}_{i,n}^{(m-1)}) - F^{(1)}(\overline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}) \geq 0 \\ \mathscr{B}^{(1)}[w_{i,n}^{(m)}] &= G^{(1)}(\overline{u}_{i,n}^{(m-1)}, \underline{v}_{i,n}^{(m-1)}) - G^{(1)}(\overline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}) \geq 0\end{aligned}$$

By iteration process (4.1)

$$w_{i,0}^{(m)} = \overline{u}_{i,0}^{(m)} - \overline{u}_{i,0}^{(m+1)} = \psi_i^{(1)} - \psi_i^{(1)} = 0$$

Lemma 3.4 implies that $w_{i,n}^{(m)} \geq 0$. So $\overline{u}_{i,n}^{(m)} \geq \overline{u}_{i,n}^{(m+1)}$

By iteration process (4.1) and Lemma 3.3

$$\begin{aligned}\mathscr{L}^{(2)}[z_{i,n}^{(m)}] &= F^{(2)}(\overline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m-1)}) - F^{(2)}(\overline{u}_{i,n}^{(m+1)}, \overline{v}_{i,n}^{(m)}) \geq 0 \\ \mathscr{B}^{(2)}[z_{i,n}^{(m)}] &= G^{(2)}(\overline{u}_{i,n}^{(m)}, \overline{v}_{i,n}^{(m-1)}) - G^{(2)}(\overline{u}_{i,n}^{(m+1)}, \overline{v}_{i,n}^{(m)}) \geq 0\end{aligned}$$

By iteration process (4.1)

$$z_{i,0}^{(m)} = \overline{v}_{i,0}^{(m)} - \overline{v}_{i,0}^{(m+1)} = \psi_i^{(2)} - \psi_i^{(2)} = 0$$

Lemma 3.4 implies that $z_{i,n}^{(m)} \geq 0$. Therefore $\overline{v}_{i,n}^{(m)} \geq \overline{v}_{i,n}^{(m+1)}$.

Similar arguments yield

$$\begin{aligned} &\underline{u}_{i,n}^{(m+1)} \geq \underline{u}_{i,n}^{(m)}, \qquad \underline{v}_{i,n}^{(m+1)} \geq \underline{v}_{i,n}^{(m)} \\ \textit{and} \qquad &\overline{u}_{i,n}^{(m+1)} \geq \underline{u}_{i,n}^{(m+1)}, \qquad \overline{v}_{i,n}^{(m+1)} \geq \underline{v}_{i,n}^{(m+1)}\end{aligned}$$

Thus monotone property follows from the principle of induction, for all m. This completes the proof.

Clearly

$$\begin{aligned} &\lim_{m\to\infty} (\overline{u}_{i,n}^{(m)}, \overline{v}_{i,n}^{(m)}) = (\overline{u}_{i,n}, \overline{v}_{i,n}), \\ &\lim_{m\to\infty} (\underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}) = (\underline{u}_{i,n}, \underline{v}_{i,n}), \\ &\lim_{m\to\infty} (\overline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}) = (\overline{u}_{i,n}, \underline{v}_{i,n}), \\ \textit{and} \quad &\lim_{m\to\infty} (\underline{u}_{i,n}^{(m)}, \overline{v}_{i,n}^{(m)}) = (\underline{u}_{i,n}, \overline{v}_{i,n}), \end{aligned} \tag{4.4}$$

exist in$\overline{\Lambda}_p$. As $m \to \infty$, in the iteration process (4.1) and (4.2) implies that $(\overline{u}_{i,n}, \overline{v}_{i,n})$ and $(\underline{u}_{i,n}, \underline{v}_{i,n})$ are solutions of the discrete problem (2.6)–(2.8).

5. EXISTENCE-COMPARISON THEOREM

Now, we can prove existence-comparison theorem for the solution of the discrete problem (2.6)–(2.8).

Theorem 5.1. *Let $(f^{(1)}, f^{(2)})$ and $(g^{(1)}, g^{(2)})$ be mixed quasimonotone C^1-functions in $S_{i,n}$. Let $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ and $(\hat{u}_{i,n}, \hat{v}_{i,n})$ be ordered upper-lower solutions of the discrete problem* (2.6)–(2.8)*. Then the sequences $\{\overline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}\}$ and $\{\underline{u}_{i,n}^{(m)}, \overline{v}_{i,n}^{(m)}\}$ obtained from the iteration processes* (4.1) *and* (4.2) *simultaneously with initial iterations $(\overline{u}_{i,n}^{(0)}, \overline{v}_{i,n}^{(0)}) = (\tilde{u}_{i,n}, \tilde{v}_{i,n})$ and $(\underline{u}_{i,n}^{(0)}, \underline{v}_{i,n}^{(0)}) = (\hat{u}_{i,n}, \hat{v}_{i,n})$ converge monotonically to their respective solutions $(\overline{u}_{i,n}, \overline{v}_{i,n}), (\underline{u}_{i,n}, \underline{v}_{i,n})$ of the discrete problem* (2.6)–(2.8)*. Moreover*

$$(\hat{u}_{i,n}, \hat{v}_{i,n}) \le (\underline{u}_{i,n}, \underline{v}_{i,n}) \le (\overline{u}_{i,n}, \overline{v}_{i,n}) \le (\tilde{u}_{i,n}, \tilde{v}_{i,n}) \quad (i,n) \in \overline{\Lambda}_p. \tag{5.1}$$

PROOF : By results in Lemma 4.1, it is observed that the limits $(\overline{u}_{i,n}, \overline{v}_{i,n})$ and $(\underline{u}_{i,n}, \underline{v}_{i,n})$ in (4.4) exist and are solutions of the discrete problem (2.6)–(2.8). These solutions satisfy the relation 5.1. This completes the proof.

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