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OSCILLATION OF SUPERLINEAR AND SUBLINEAR NEUTRAL DELAY DYNAMIC EQUATIONS

S. H. SAKER

Department of Mathematics, College of Science, King Saud University
Riyadh 11451 Saudi Arabia
Email: saker@mans.edu.eg

ABSTRACT. The paper is concerned with oscillation of certain class of second-order nonlinear neutral delay dynamic equations on time scales. We will use a unified approach on time scales and employing the Riccati techniques to establish some new criteria for oscillation. The results represent further improvements on those given for the superlinear neutral dynamic equations and in the sublinear case the results are essentially new. An example is considered to illustrate the main results.

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1. INTRODUCTION

In this paper, we are concerned with oscillation of the second-order nonlinear neutral delay dynamic equation

$$\left[a(t)\left(\left[y(t)+p(t)y(\tau(t))\right]^{\Delta}\right)^{\gamma}\right]^{\Delta}+q(t)y^{\gamma}(\delta(t))=0, \tag{1.1}$$

on a time scale $\mathbb{T}$, where $\gamma > 0$ is a quotient of odd positive integers,

(h$_1$) $\tau(t) : \mathbb{T} \to \mathbb{T}$, $\delta(t) : \mathbb{T} \to \mathbb{T}$, $\tau(t) \leq t$, $\delta(t) \leq t$ for all $t \in \mathbb{T}$ and $\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \tau(t) = \infty$;

(h$_2$) $\int_{t_0}^{\infty}\left(\frac{1}{a(t)}\right)^{\frac{1}{\gamma}}\Delta t = \infty$, $a^{\Delta}(t) \geq 0$, and $0 \leq p(t) < 1$.

Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. Throughout this paper these assumptions will be supposed to hold. Let $\tau^*(t) = \min\{\tau(t),\ \delta(t)\}$ and let $T_0 = \min\{\tau^*(t) : t \geq 0\}$ and $\tau^*_{-1}(t) = \sup\{s \geq 0 : \tau^*(s) \leq t\}$ for $t \geq T_0$. Clearly $\tau^*_{-1}(t) \geq t$ for $t \geq T_0$, $\tau^*_{-1}(t)$ is nondecreasing and coincides with the inverse of $\tau^*(t)$ when the latter exists.

By a solution of (1.1) we mean a nontrivial real-valued function $y(t)$ which has the properties $[y(t) + p(t)y(\tau(t))] \in C^1_{rd}[\tau^*_{-1}(t_0), \infty)$, and $a(t)\left[y(t) + p(t)y(\tau(t))\right]^{\Delta} \in$

On leave from Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

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$C^1_{rd}[\tau^*_{-1}(t_0), \infty)$. Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_y, \infty)$ and satisfy $\sup\{|y(t)| : t > t_1\} > 0$ for any $t_1 \geq t_y$. A solution $y(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

The study of dynamic equations on time scales, which goes back to Stefan Hilger [7], is an area of mathematics that has recently received a lot of attention. It has been created to unify the study of differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. Several authors have expounded on various aspects of this new theory, see the survey paper by Agarwal, Bohner, O'Regan, and Peterson [1] and the references cited therein. The three most popular examples of calculus on time scales are differential calculus, difference calculus and quantum calculus (see Kac and Cheung [9]), i.e, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$, where $q > 1$. There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A recent cover story article in New Scientist [16] discusses several possible applications. The books on the subject of time scales by Bohner and Peterson [5] summarizes and organizes much of time scale calculus and some applications.

Recently there has been much research activity concerning the oscillation and nonoscillation of solutions of neutral delay dynamic equations on time scales. The oscillation results not only unify the oscillation results of differential and difference equations but also involve the oscillation conditions for different types of equations on different time scales which the oscillation behavior of the solutions is not known before.

In 2004 Agarwal et al. [2] considered the second-order nonlinear neutral delay dynamic equation

$$\left[a(t)([y(t) + p(t)y(t-\tau)]^{\Delta})^{\gamma}\right]^{\Delta} + f(t, y(t-\delta)) = 0, \tag{1.2}$$

on a time scale $\mathbb{T}$; here $\gamma > 0$ is a quotient of odd positive integers, τ and δ are positive constants such that the delay functions $\tau(t) := t - \tau < t$ and $\delta(t) := t - \delta < t$ satisfy $\tau(t) : \mathbb{T} \to \mathbb{T}$ and $\delta(t) : \mathbb{T} \to \mathbb{T}$ for all $t \in \mathbb{T}$, $a(t)$ and $p(t)$ are real valued rd-continuous positive functions defined on $\mathbb{T}$, and the following conditions are satisfied:

(A$_1$) $\int_{t_0}^{\infty} \left(\frac{1}{a(t)}\right)^{\frac{1}{\gamma}} \Delta t = \infty$, $0 \leq p(t) < 1$,

(A$_2$) $f(t, u) : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous function such that $uf(t, u) > 0$ for all $u \neq 0$ and there exists a positive rd-continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t, u)| \geq q(t)\,|u^{\gamma}|$.

In [2] the authors considered the case when $\gamma > 0$ is an odd positive integer and proved that the oscillation of (1.2) is equivalent to the oscillation of a first order delay dynamic inequality and established some sufficient conditions for oscillation. Also they considered the case when $\gamma \geq 1$ and established some sufficient conditions for oscillation by employing the Riccati technique. The results were applied only in discrete time scales, i.e., when the graininess function $\mu(t) \neq 0$.

In 2006 Saker [13] considered (1.2) where $\gamma \geq 1$ is an odd positive integer, $(A_1)-(A_2)$ hold and established some new sufficient conditions for oscillation of (1.2) by employed the Riccati transformation technique. However the results established in [2, 13] can be applied only on the time scales $\mathbb{T} = R$, $\mathbb{T} = \mathbb{N}$, $\mathbb{T} = h\mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}} = \{t : t = q^k, k \in \mathbb{N}, q > 1\}$, and cannot be applied on the time scales $\mathbb{T} = \mathbb{N}^2 = \{t^2 : t \in \mathbb{N}\}$, $\mathbb{T}_2 = \{\sqrt{n} : n \in \mathbb{N}_0\}$, $\mathbb{T}_3 = \{\sqrt[3]{n} : n \in \mathbb{N}_0\}$, and $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}_0\}$ where $t_n\}$ is the set of harmonic numbers. This follows from the fact that when $t \in T$, the functions $t - \tau$ and $t - \delta$ may be not belong to the time scales $\mathbb{T} = \mathbb{N}^2$, $\mathbb{T} = \mathbb{T}_2$, $\mathbb{T} = \mathbb{T}_3$ and $\mathbb{T} = \mathbb{T}_n$.

In 2006 also Saker [12] considered the second-order superlinear neutral delay dynamic equation of Emden-Fowler type

$$\left[a(t)(y(t) + r(t)y(\tau(t)))^{\Delta}\right]^{\Delta} + p(t)\left|y(\delta(t))\right|^{\gamma} signy(\delta(t)) = 0,$$

on a time scale $\mathbb{T}$; where $\gamma > 1$, $a(t)$, $r(t)$, $\tau(t)$, $p(t)$ and $\delta(t)$ real-valued positive functions defined on $\mathbb{T}$ and established some oscillation results which improved the oscillation results for that has been established for superlinear neutral delay differential equations.

In 2006 Şahiner [11] considered the general equation

$$\left[a(t)\left(\left[y(t) + p(t)y(\tau(t))\right]^{\Delta}\right)^{\gamma}\right]^{\Delta} + f(t, y(\delta(t))) = 0, \tag{1.3}$$

on a time scale $\mathbb{T}$ and followed the argument in [2] by reducing the oscillation of (1.3) to oscillation of a first order delay dynamic inequality and established some sufficient conditions for oscillation, when the following conditions are satisfied:

(B$_1$) δ, τ are positive rd-continuous functions, $\delta, \tau : \mathbb{T} \to \mathbb{T}$,
(B$_2$) $\int_{t_0}^{\infty} \left(\frac{1}{a(t)}\right)^{\frac{1}{\gamma}} \Delta t = \infty$, $\gamma \geq 1$, and $0 \leq p(t) < 1$;
(B$_3$) $f(t, u) : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $uf(t, u) > 0$ for all $u \neq 0$ and there exists a positive rd-continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t, u)| \geq q(t)\left|u^{\gamma}\right|$.

However one can easily see that the two examples presented in [11] to illustrate the main results are valid only when $\mathbb{T} = \mathbb{R}$ and cannot be applied when $\mathbb{T} = \mathbb{N}$ since the delay functions that are considered in his paper are given by $t/2$, $\sqrt{t}$ and $t/64$ which are not in $C_{rd}(\mathbb{T}, \mathbb{T})$ for a general time scale $\mathbb{T}$. Also the results cannot give a sharp sufficient condition for oscillation of (1.3) when $q(t) = \gamma/t^2$.

In 2006 Wu et al. [17] considered also (1.3) on a time scale $\mathbb{T}$. They followed the argument in [2] by using the Riccati transformation technique and the Chain rule

$$(u \circ \nu)^{\Delta}(t) = (u^{\bar{\Delta}} \circ \nu)\nu^{\Delta},$$

where $\bar{\Delta}$ is the delta derivative defined on $\bar{\mathbb{T}}$ and $\nu(t)$ is strictly increasing, and established some sufficient conditions for oscillation of (1.3), when the following conditions are satisfied:

(C_1) $\delta : \mathbb{R} \to \mathbb{R}$ is continuous $\delta : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\bar{\mathbb{T}} = \delta(\mathbb{T}) \subset \mathbb{T}$ is a time scale;
(C_2) $(\delta \circ \sigma)(t) = (\sigma \circ \delta)(t)$;
(C_3) $\int_{t_0}^{\infty} \left(\frac{1}{a(t)}\right)^{\frac{1}{\gamma}} \Delta t = \infty$, $\gamma \geq 1$, and $0 \leq p(t) < 1$;
(C_4) $f(t, u) : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $uf(t, u) > 0$ for all $u \neq 0$ and there exists a positive rd-continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t, u)| \geq q(t)\,|u^{\gamma}|$.

We note that the results in [17], which are based on the Chain rule, can only be applied if $\bar{\mathbb{T}}$ is a time scale and if $\tau(t) \leq t$ and $\delta(t) \geq \tau(\delta(t))$. The condition (C_2) also is a restrictive condition, since on the time scale $\mathbb{T} = q^{\mathbb{N}}$ by choosing $\delta(t) = t - q^{n_0}$ one can easily see that $\delta(\sigma(t)) = \delta(qt) = qt - q^{n_0} \neq \sigma(\delta(t)) = q(t - q^{n_0}) = qt - q^{n_0+1}$, so the results in [17] cannot be applied on the time scale $\mathbb{T} = q^{\mathbb{N}}$ when $\delta(t) = t - q^{n_0}$. Also in the proof of the main results in [17] in Lemma 2.5, the authors used the Chain rule $(f(g(t)))^{\Delta} = f^{\Delta}(g(t))g^{\Delta}(t)$ which is not true on general time scales. Of course trivially $(x \circ \tau)^{\Delta} = (x^{\Delta} \circ \tau)\tau^{\Delta}$ if δ is a constant with $\tau(t) = t - \delta \in \mathbb{T}$ for $t \in \mathbb{T}$.

Agarwal, O'Regan and Saker [3] considered the general nonlinear neutral delay dynamic equation (1.3) where $\gamma \geq 1$ is an odd positive integer,

(D_1) $\tau(t) : \mathbb{T} \to \mathbb{T}$, $\delta(t) : \mathbb{T} \to \mathbb{T}$, $\tau(t) \leq t$, $\delta(t) \leq t$ for all $t \in \mathbb{T}$ and $\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \tau(t) = \infty$;
(D_2) $\int_{t_0}^{\infty} \left(\frac{1}{a(t)}\right)^{\frac{1}{\gamma}} \Delta t = \infty$, $a^{\Delta}(t) \geq 0$, $0 \leq p(t) < 1$;
(D_3) $f(t, u) : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous function such that $uf(t, u) > 0$ for all $u \neq 0$ and there exists a positive rd-continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t, u)| \geq q(t)\,|u^{\gamma}|$,

and employed the Riccati technique and established some new oscillation criteria which can be applied on any time scale $\mathbb{T}$ and improved the results established in [2], [11] and [17].

Recently Agarwal, O'Regan and Saker [4] considered the second-order nonlinear neutral delay dynamic equation

$$\left[a(t)\left[y(t)+p(t)y(\tau(t))\right]^{\Delta}\right]^{\Delta}+q(t)f(y(\delta(t)))=0, \tag{1.4}$$

on a time scale $\mathbb{T}$ where the delay functions $\tau(t)\leq t$ and $\delta(t)\leq t$ satisfy $\tau(t):\mathbb{T}\to\mathbb{T}$ and $\delta(t):\mathbb{T}\to\mathbb{T}$ for all $t\in\mathbb{T}$ and $\lim_{t\to\infty}\delta(t)=\lim_{t\to\infty}\tau(t)=\infty$, $a(t)$, $p(t)$ and $q(t)$ are real valued rd-continuous positive functions defined on $\mathbb{T}$, and $0\leq p(t)<1$, $f(u):\mathbb{R}\to\mathbb{R}$ is continuous function such that $uf(u)>0$ and $f(u)/u\geq K>0$ for all $u\neq 0$ and employed a different technique which is the generalized Riccati transformation technique and established some new criteria for oscillation and studied the asymptotic behavior of the nonoscillatory solutions.

In oscillation theory of differential equations there are two important conditions established by Hille and Nehari since more than 50 years. In 1948 Hille [8] considered the second-order differential equation

$$x^{''}(t)+p(t)x(t)=0, \tag{1.5}$$

and proved that if

$$\liminf_{t\to\infty} t\int_t^{\infty} p(s)ds>\frac{1}{4}, \tag{1.6}$$

then every solution of (1.5) oscillates. In 1957 Nehari [10] also considered (1.5) and proved that if

$$\liminf_{t\to\infty}\frac{1}{t}\int_{t_0}^{t} s^2p(s)ds>\frac{1}{4}, \tag{1.7}$$

then every solution oscillates. We note that the inequalities (1.6) and (1.7) are sharp and cannot be weakened. Indeed, let $p(t)=1/4t^2$ for $t\geq 1$, then, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_{t_0}^{t} s^2p(s)ds=\liminf_{t\to\infty} t\int_t^{\infty} p(s)ds=\frac{1}{4}, \tag{1.8}$$

and the second-order Euler differential equation

$$x^{''}(t)+\frac{1}{4t^2}x(t)=0,\ t\geq 1, \tag{1.9}$$

has a nonoscillatory solution $x(t)=\sqrt{t}$.

Saker in [14, 15] considered the linear equation

$$\left[y(t)+p(t)y(\tau(t))\right]^{\Delta\Delta}+q(t)y(\delta(t))=0, \tag{1.10}$$

by employed the Riccati transformation technique and established some new oscillation criteria of Hille and Nehari types and also established some alternative oscillation criteria when the Hille and Nehari types criteria fail to apply.

We note that in all the above results the condition $\gamma \geq 1$ is required. In this paper the study is free of this restriction and contain the case when $0 < \gamma < 1$.

The main aim of this paper is to extend the oscillation results that has been established by Saker [14, 15] for linear neutral dynamic equations to the equation (1.1). Our results improve the results established by Agarwal et al. [2], Şahiner [11] and Wu et al. [17], since our results are sharp because the results of Hille and Nehari are sharp results. The main results are proved in the next section which is organized as follows: First, we will employ the Riccati technique to establish the main oscillation criteria for (1.1) when $\gamma \geq 1$ and then consider the case when $0 < \gamma < 1$ and establish some criteria for oscillation which are essentially new. An example is considered to illustrate the main results.

2. MAIN RESULTS

In what follows and later, we assume

$$\int_{t_0}^{\infty} \delta^{\gamma}(s) q(s)[1 - p(\delta(s))]^{\gamma} \Delta s = \infty. \tag{2.1}$$

Before we state the main results we present the following Lemma which plays an important role in the proof of the main results and its proof is similar to the proof of Lemma 2.1 in [3] and hence is omitted. Also note that this lemma can be applied in the sublinear case as well as in the superlinear case.

Lemma 2.1. *Assume that* $(h_1) - (h_2)$ *hold and* (1.1) *has a positive solution* $y(t)$ *on* $[t_0, \infty)_{\mathbb{T}}$. *Define*

$$x(t) := y(t) + p(t) y(\tau(t)). \tag{2.2}$$

Then there exists a $T \in [t_0, \infty)_{\mathbb{T}}$, *sufficiently large, so that* $x(t)$ *satisfies the inequality*

$$\left(a(t) \left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + q(t)(1 - p(\delta(t)))^{\gamma} x^{\gamma}(\delta(t)) \leq 0, \ \textit{for } t \geq t_2, \tag{2.3}$$

and

(*i*) $x^{\Delta}(t) > 0$, $x(t) > t x^{\Delta}(t)$ *for* $t \in [\mathbb{T}, \infty)_{\mathbb{T}}$;

(*ii*) $x(t)/t$ *is strictly decreasing on* $[\mathbb{T}, \infty)_{\mathbb{T}}$.

In the following, we consider the case $\gamma \geq 1$ and establish new oscillation criteria for (1.1) of Hille and Nehari types. To simplify the calculations we introduce the following notation.

$$p_* := \liminf_{t \to \infty} \frac{t^{\gamma}}{a(t)} \int_{\sigma(t)}^{\infty} Q(s) \Delta s, \quad q_* := \liminf_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \frac{s^{\gamma+1}}{a(s)} Q(s) \Delta s, \tag{2.4}$$

where $Q(t) := q(t)(1 - p(\delta(t)))^{\gamma} \left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma}$ and assume that $l := \liminf_{t \to \infty} \frac{t}{\sigma(t)}$.

Theorem 2.1. *Assume that* $(h_1)-(h_2)$ *hold and* (1.1) *has a positive or a negative solution* $y(t)$ *on* $[t_0,\,\infty)_{\mathbb{T}}$ *such that* $y(t)$ *and* $y(\tau(t))>0$ *for* $t\geq t_1>t_0$. *Let* $x(t)$ *be as defined by* (2.2) *and let* $\omega(t)=a(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma}$,

$$r_*:=\liminf_{t\to\infty}\frac{t^{\gamma}\omega^{\sigma}(t)}{a(t)},\quad \textit{and}\quad R:=\limsup_{t\to\infty}\frac{t^{\gamma}\omega^{\sigma}(t)}{a(t)}. \tag{2.5}$$

Then

$$p_*\leq r-l^{\gamma}r^{1+\frac{1}{\gamma}}\quad \textit{and}\quad p_*+q_*\leq\frac{1}{l^{\gamma(\gamma+1)}}. \tag{2.6}$$

Proof. From Lemma 2.1 there is a $T\in[t_1,\infty)_{\mathbb{T}}$, sufficiently large, so that $x(t)$ satisfies the conclusions of Lemma 2.1. From the definition of $\omega(t)$ we see that $\omega(t)$ is a positive function and after using the quotient rule and inequality (2.3), we see that $\omega(t)$ satisfies

$$\omega^{\Delta}(t)\leq-\left(\frac{x(\tau(t))}{x^{\sigma}(t)}\right)^{\gamma}q(t)(1-p(\delta(t)))^{\gamma}-\frac{\left(a(t)x^{\Delta}(t)\right)^{\gamma}(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)\left(x^{\sigma}(t)\right)^{\gamma}}.$$

Since

$$\frac{x(\tau(t))}{\tau(t)}\geq\frac{x(t)}{t}\geq\frac{x^{\sigma}(t)}{\sigma(t)}\quad\text{and}\quad x^{\Delta}(t)\geq\frac{a^{\sigma}}{a}x^{\Delta\sigma}(t),$$

we get the inequality

$$\omega^{\Delta}(t)\leq-\left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma}q(t)(1-p(\delta(t)))^{\gamma}-\frac{\left(a^{\sigma}x^{\Delta\sigma}(t)\right)^{\gamma}(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)\left(x^{\sigma}(t)\right)^{\gamma}}, \tag{2.7}$$

since $x^{\Delta\Delta}(t)<0$. By the Pötzsche chain rule, and the fact that $x^{\Delta}(t)>0$, we obtain when $\gamma>1$, that

$$\begin{aligned}(x^{\gamma}(t))^{\Delta} &= \gamma\int_0^1\left[x(t)+h\mu(t)x^{\Delta}(t)\right]^{\gamma-1}dh\,x^{\Delta}(t)\\ &\geq \gamma\int_0^1(x(t))^{\gamma-1}\,dh\,x^{\Delta}(t)=\gamma(x(t))^{\gamma-1}x^{\Delta}(t).\end{aligned} \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\omega^{\Delta}(t)\leq-Q(t)-\frac{\left(x^{\Delta\sigma}(t)\right)^{\gamma}\gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}}=-Q(t)-\frac{\gamma}{a^{\frac{1}{\gamma}}}\left(\omega^{\sigma}\right)^{\frac{\gamma+1}{\gamma}}.$$

Then $\omega(t)$ satisfies the dynamic Riccati inequality

$$\omega^{\Delta}(t)+Q(t)+\frac{\gamma}{a^{\frac{1}{\gamma}}}\left(\omega^{\sigma}\right)^{\frac{\gamma+1}{\gamma}}\leq0,\quad\text{for}\quad t\in[T,\infty)_{\mathbb{T}}. \tag{2.9}$$

Since $Q(t)>0$ and $\omega(t)>0$ for $t\geq T$, it follows from (2.9) that $\omega^{\Delta}(t)<0$ and hence $\omega(t)$ is strictly decreasing for $t\in[T,\infty)_{\mathbb{T}}$. Then $\omega(t)$ satisfies the dynamic inequality

$$\omega^{\Delta}(t)+Q(t)+\frac{\gamma}{a^{\frac{1}{\gamma}}(t)}\omega^{\sigma}\omega^{\frac{1}{\gamma}}(t)\leq0,\quad\text{for}\ \ t\geq T. \tag{2.10}$$

We have from (2.10) that

$$-\left(\omega^{\Delta}(t)/\gamma\omega^{\sigma}\omega^{\frac{1}{\gamma}}(t)\right)>1/a^{\frac{1}{\gamma}}(t),\quad\text{for}\ \ t\geq T. \tag{2.11}$$

Now, since $f^\sigma = f(t) + \mu(t) f^\Delta(t)$ when f is differentiable, we see (using $f = 1/\omega^{\frac{1}{\gamma}}(t)$) that

$$\left(1/\omega^{\frac{1}{\gamma}}(t)\right)^\Delta = \frac{1}{\sigma - t}\frac{\omega^{\frac{1}{\gamma}}(t) - (\omega^\sigma)^{\frac{1}{\gamma}}}{\omega^{\frac{1}{\gamma}}(t)(\omega^\sigma)^{\frac{1}{\gamma}}}.$$

Using the inequality (see p. 39 in [6])

$$x^\beta - y^\beta \geqslant \beta x^{\beta-1}(x-y) \text{ for all } x > y > 0 \text{ and } 0 < \beta \leq 1,$$

and the fact that $\omega(t)$ is nonincreasing, we see that

$$\begin{aligned}\left(1/\omega^{\frac{1}{\gamma}}(t)\right)^\Delta &\geq \frac{1}{\sigma - t}\frac{1}{\gamma\omega^{\frac{1}{\gamma}}(t)(\omega^\sigma)^{\frac{1}{\gamma}}}\omega^{\frac{1}{\gamma}-1}(t)(\omega(t) - \omega^\sigma)\\ &\geq \frac{1}{\sigma - t}\frac{1}{\gamma\omega^{\frac{1}{\gamma}}(t)(\omega^\sigma)^{\frac{1}{\gamma}}}(\omega^\sigma)^{\frac{1}{\gamma}-1}(\omega(t) - \omega^\sigma)\\ &= -\frac{\omega^\sigma - \omega(t)}{\sigma - t}\frac{1}{\gamma\omega^{\frac{1}{\gamma}}(t)(\omega^\sigma)^{\frac{1}{\gamma}}}(\omega^\sigma)^{\frac{1}{\gamma}-1}\\ &= \frac{-1}{\gamma}\frac{\omega^\Delta}{\omega^\sigma\omega^{\frac{1}{\gamma}}(t)}.\end{aligned}$$

This and (2.11) imply that

$$\left(1/\omega^{\frac{1}{\gamma}}(t)\right)^\Delta > \frac{1}{a^{\frac{1}{\gamma}}(t)}. \tag{2.12}$$

Integrating the last inequality from T to t, we obtain

$$\omega(t) < \frac{1}{\left(\int_T^t \frac{\Delta s}{a^{\frac{1}{\gamma}}(t)}\right)^\gamma}, \tag{2.13}$$

which implies by (h_2) that $\lim_{t\to\infty}\omega(t) = 0$, and that

$$0 \leq r \leq R < 1. \tag{2.14}$$

Now, we prove that the first inequality in (2.6) holds. Let $\epsilon > 0$, then from the definitions of p_* and r_* we can pick $t_2 \in [T,\infty)_{\mathbb{T}}$, sufficiently large, so that

$$\frac{t^\gamma}{a(t)}\int_{\sigma(t)}^\infty Q(s)\Delta s \geq p_* - \epsilon, \quad \text{and} \quad \frac{t^\gamma\omega^\sigma(t)}{a(t)} \geq r_* - \epsilon, \text{ for } t \in [t_2,\infty)_{\mathbb{T}}.$$

Integrating (2.10) from $\sigma(t)$ to ∞ and using $\lim_{t\to\infty}\omega(t) = 0$, we have

$$\omega^\sigma(t) \geq \int_{\sigma(t)}^\infty Q(s)\Delta s + \gamma\int_{\sigma(t)}^\infty \frac{\omega^{\frac{1}{\gamma}}(s)\omega^\sigma(s)}{a^{\frac{1}{\gamma}}(s)}\Delta s, \quad t \in [t_2,\infty)_{\mathbb{T}}. \tag{2.15}$$

It follows from (2.15) and the fact $a^\Delta(t) \geq 0$, that

$$\begin{aligned}\frac{t^\gamma}{a(t)}\omega^\sigma(t) &\geq \frac{t^\gamma}{a(t)}\int_{\sigma(t)}^\infty Q(s)\Delta s + \gamma\frac{t^\gamma}{a(t)}\int_{\sigma(t)}^\infty \frac{s(\omega^\sigma)^{\frac{1}{\gamma}}s^\gamma\omega^\sigma}{s^{\gamma+1}a^{\frac{1}{\gamma}}(s)}\Delta s\\ &\geq (p_* - \epsilon) + \frac{t^\gamma(r_* - \epsilon)^{1+\frac{1}{\gamma}}}{a(t)}\int_{\sigma(t)}^\infty \frac{\gamma r(s)}{s^{\gamma+1}}\Delta s\end{aligned}$$

$$\geq \quad (p_* - \epsilon) + (r_* - \epsilon)^{1+\frac{1}{\gamma}}\, t^\gamma \int_{\sigma(t)}^{\infty} \frac{\gamma}{s^{\gamma+1}} \Delta s,$$

so that

$$\frac{t^\gamma}{a(t)} \omega^\sigma(t) \geq (p_* - \epsilon) + (r_* - \epsilon)^{1+\frac{1}{\gamma}}\, t^\gamma \int_{\sigma(t)}^{\infty} \frac{\gamma}{s^{\gamma+1}} \Delta s. \tag{2.16}$$

Using the Pötzsche chain rule, we get

$$\left(\frac{-1}{s^\gamma}\right)^\Delta = \gamma \int_0^1 \frac{1}{[s + h\mu(s)]^{\gamma+1}} dh \leq \int_0^1 \left(\frac{\gamma}{s^{\gamma+1}}\right) dh = \frac{\gamma}{s^{\gamma+1}}. \tag{2.17}$$

From (2.16) and (2.17), we have

$$\frac{t^\gamma \omega^\sigma(t)}{a(t)} \geq (p_* - \epsilon) + (r - \epsilon)^{1+\frac{1}{\gamma}} \left(\frac{t}{\sigma(t)}\right)^\gamma.$$

Taking the $\liminf$ of both sides as $t \to \infty$, we get

$$r_* \geq p_* - \epsilon + (r_* - \epsilon)^{1+\frac{1}{\gamma}}\, l^\gamma.$$

Since $\epsilon > 0$ is arbitrary, we get the desired result

$$p_* \leq r_* - (r_*)^{1+\frac{1}{\gamma}}\, l^\gamma.$$

To complete the proof it remains to prove the second inequality in (2.6). To do this we will use the inequality (2.9). Multiplying (2.9) by $\frac{t^{\gamma+1}}{a(t)}$, and integrating from t_2 to t $(t \geq t_2)$, we get

$$\int_{t_2}^{t} \frac{s^{\gamma+1}}{a(s)} \omega^\Delta(s) \Delta s \leq -\int_{t_2}^{t} \frac{s^{\gamma+1}}{a(s)} Q(s) \Delta s - \gamma \int_{t_2}^{t} \left(\frac{s^\gamma \omega^\sigma(s)}{a(s)}\right)^{\frac{\gamma+1}{\gamma}} \Delta s. \tag{2.18}$$

Using integration by parts, we obtain

$$\begin{aligned} \frac{t^{\gamma+1}}{a(t)} \omega(t) \quad &\leq \quad \frac{t_2^{\gamma+1} \omega(t_2)}{a(t_2)} - \int_{t_2}^{t} \frac{s^{\gamma+1}}{a(s)} Q(s) \Delta s - \gamma \int_{t_2}^{t} \left(\frac{s^\gamma \omega^\sigma(s)}{a(s)}\right)^{\frac{\gamma+1}{\gamma}} \Delta s \\ &\quad + \int_{t_2}^{t} \left(\frac{s^{\gamma+1}}{a(s)}\right)^\Delta \omega^\sigma(s) \Delta s. \end{aligned}$$

But, by the Pötzsche chain rule and $a^\Delta(t) \geq 0$, we have

$$\begin{aligned} \left(\frac{s^{\gamma+1}}{a(s)}\right)^\Delta \quad &= \quad \frac{1}{\sigma(s) - s} \left[\frac{\sigma^{\gamma+1}}{a^\sigma(s)} - \frac{s^{\gamma+1}}{a(s)}\right] \\ &\leq \quad \frac{1}{a(s)(\sigma(s) - s)} \left[\sigma^{\gamma+1} - s^{\gamma+1}\right] \leq \frac{(\gamma+1)\sigma^\gamma(s)}{a(s)}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{t^{\gamma+1}}{a(t)} \omega(t) \quad &\leq \quad \frac{t_2^{\gamma+1} \omega(t_2)}{a(t_2)} - \int_{t_2}^{t} \frac{s^{\gamma+1}}{r(s)} Q(s) \Delta s + \int_{t_2}^{t} (\gamma+1) \frac{(\sigma(s))^\gamma \omega^\sigma(s)}{a(s)} \Delta s \\ &\quad - \quad \gamma \int_{t_2}^{t} \left(\frac{s^\gamma \omega^\sigma(s)}{a(s)}\right)^{\frac{\gamma+1}{\gamma}} \Delta s. \end{aligned}$$

Let $\epsilon > 0$ be given, then using the definition of l, we can assume, without loss of generality, that t_2 is sufficiently large so that $\frac{s}{\sigma(s)} > l - \epsilon$, $s \geq t_2$. It follows that $\sigma(s) \leq Ks$, $s \geq t_2$ and $K := \frac{1}{l-\epsilon}$. We then get that

$$\begin{aligned} \frac{t^{\gamma+1}}{a(t)}\omega(t) &\leq \frac{t_2^{\gamma+1}\omega(t_2)}{a(t_2)} - \int_{t_2}^{t} \frac{s^{\gamma+1}}{r(s)} Q(s)\Delta s \\ &\quad + \int_{t_2}^{t} [(\gamma+1)K^{\gamma}\frac{s^{\gamma}\omega^{\sigma}(s)}{a(s)} - \gamma \left(\frac{s^{\gamma}\omega^{\sigma}(s)}{a(s)}\right)^{\frac{\gamma+1}{\gamma}}]\Delta s. \end{aligned}$$

Let $u(s) := \frac{s^{\gamma}\omega^{\sigma}(s)}{a(s)}$, then $(u(s))^{\frac{\gamma+1}{\gamma}} = \left(\frac{s^{\gamma}\omega^{\sigma}(s)}{a(s)}\right)^{\frac{\gamma+1}{\gamma}}$. It follows that

$$\frac{t^{\gamma+1}}{a(t)}\omega(t) \leq \frac{t_2^{\gamma+1}\omega(t_2)}{a(t_2)} - \int_{t_2}^{t} \frac{s^{\gamma+1}Q(s)}{a(s)}\Delta s + \int_{t_2}^{t} [(\gamma+1)K^{\gamma}u(s) - \gamma[u(s)]^{\frac{\gamma+1}{\gamma}}]\Delta s.$$

Using the inequality

$$Bu - Au^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^{\gamma}},$$

where A, B are constants, we get

$$\begin{aligned} \frac{t^{\gamma}}{a(t)}\omega(t) &\leq \frac{t_2^{\gamma+1}\omega(t_2)}{a(t_2)} - \int_{t_2}^{t} \frac{s^{\gamma+1}}{a(s)} Q(s)\Delta s + \int_{t_2}^{t} \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{[(\gamma+1)K^{\gamma}]^{\gamma+1}}{\gamma}\Delta s \\ &= \frac{t_2^{\gamma+1}\omega(t_2)}{t_2 a(t_2)} - \frac{1}{t_2}\int_{t_2}^{t} s^{\gamma+1}Q(s)\Delta s + \frac{K^{\gamma(\gamma+1)}(t_2 - t_2)}{t_2}. \end{aligned}$$

It follows from this that

$$\frac{t^{\gamma}}{a(t)}\omega(t) \leq \frac{t_2^{\gamma+1}\omega(t_2)}{a(t_2)t} - \frac{1}{t}\int_{t_2}^{t} \frac{s^{\gamma+1}}{a(s)} Q(s)\Delta s + K^{\gamma(\gamma+1)}\frac{(t-t_2)}{t}.$$

Since $\omega^{\sigma}(t) \leq \omega(t)$, we get

$$\frac{t^{\gamma}}{a(t)}\omega(t) \leq \frac{t_2^{\gamma+1}\omega(t_2)}{a(t_2)t} - \frac{1}{t}\int_{t_2}^{t} \frac{s^{\gamma+1}}{a(s)} Q(s)\Delta s + K^{\gamma(\gamma+1)}\frac{(t-t_2)}{t}.$$

Taking the lim sup of both sides as $t \to \infty$, we obtain

$$R \leq -q_* + K^{\gamma(\gamma+1)} = -q_* + \frac{1}{(l-\epsilon)^{\gamma(\gamma+1)}}.$$

Since $\epsilon > 0$ is an arbitrary, we get that

$$R \leq -q_* + \frac{1}{l^{\gamma(\gamma+1)}}.$$

Using this and the first inequality in (2.14), we get

$$p_* \leq r_* - l^{\gamma} r^{1+\frac{1}{\gamma}} \leq r_* \leq R \leq -q_* + \frac{1}{l^{\gamma(\gamma+1)}}$$

which gives us the desired second inequality in (2.6). The proof is complete.

As a consequence of Theorem 2.1 we have the following oscillation results.

Theorem 2.2. *Assume that* $(h_1)-(h_2)$ *hold. If*

$$p_* = \liminf_{t\to\infty} t^\gamma \int_{\sigma(t)}^{\infty} Q(s)\Delta s > \frac{\gamma^\gamma}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}, \tag{2.19}$$

then (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume (1.1) is nonoscillatory on $[t_0,\infty)_{\mathbb{T}}$, then the hypotheses of Theorem 2.1 hold. From the first inequality in (2.6), we have that

$$p_* \le r_* - l^\gamma r_*^{\frac{\gamma+1}{\gamma}}.$$

Using the inequality

$$Bu - Au^{\frac{\gamma+1}{\gamma}} \le \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}$$

with $B=1$ and $A=l^\gamma$ we get that

$$p_* \le \frac{\gamma^\gamma}{l^{\gamma^2}(\gamma+1)^{\gamma+1}},$$

which contradicts (2.19). The proof is complete.

Also as a consequence of Theorem 2.1 we have the following oscillation results and since the proof is similar to that of Theorem 2.2 by using the results in Theorem 2.1 we omitted it.

Theorem 2.3. *Assume that* $(h_1)-(h_2)$ *hold. If*

$$p_* + q_* > \frac{1}{l^{\gamma(\gamma+1)}},$$

then (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

As a consequence of Theorem 2.3, we have the following result.

Corollary 2.1. *Assume that* $(h_1)-(h_2)$ *hold. If*

$$p_* > \frac{1}{l^{\gamma(\gamma+1)}}, \quad \text{or } q_* > \frac{1}{l^{\gamma(\gamma+1)}},$$

then (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

The following theorem gives sufficient condition for oscillation of (1.1) in the superlinear and sublinear cases.

Theorem 2.4. *Assume that* $(h_1)-(h_2)$ *hold. If*

$$\limsup_{t\to\infty} \frac{t^\gamma}{r(t)} \int_t^\infty q(s)(1-p(\delta(s)))^\gamma \left(\frac{\tau(s)}{s}\right)^\gamma \Delta s > 1. \tag{2.20}$$

Then every solution of (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume y is an eventually positive solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$ and let $x(t)$ be as defined by (2.2). Then by Lemma 2.1 there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)$ is a solution of the inequality (2.3) and

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad x^{\Delta}(t) > 0, \quad x^{\Delta\Delta}(t) < 0, \quad \frac{x(t)}{t} > x^{\Delta}(t),$$

on $[t_1, \infty)_{\mathbb{T}}$ and $\frac{x(t)}{t}$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. Integrating (2.3) from t to T, $T \geq t \geq t_1$ we obtain

$$\int_t^T q(s)(1 - p(\delta(s)))^{\gamma} x^{\gamma}(\tau(s))\Delta s \leq r(t)(x^{\Delta}(t))^{\gamma} - r(T)(x^{\Delta}(T))^{\gamma}.$$

Since $x^{\Delta}(t) > 0$, we get that

$$\frac{1}{r(t)} \int_t^T q(s)(1 - p(\delta(s)))^{\gamma} x^{\gamma}(\tau(s))\Delta s \leq (x^{\Delta}(t))^{\gamma}.$$

Since $\frac{x(t)}{t}$ is strictly decreasing and using $x^{\Delta}(t) < \frac{x(t)}{t}$ we obtain

$$\frac{1}{r(t)} \int_t^T q(s)(1 - p(\delta(s)))^{\gamma} \left(\frac{\tau(s)}{s}\right)^{\gamma} x^{\gamma}(s)\Delta s \leq \frac{x^{\gamma}(t)}{t^{\gamma}}.$$

Since $x(t)$ is increasing we get

$$\frac{t^{\gamma}}{r(t)} \int_t^T q(s)(1 - p(\delta(s)))^{\gamma} \left(\frac{\tau(s)}{s}\right)^{\gamma} \Delta s \leq 1,$$

which implies that

$$\frac{t^{\gamma}}{r(t)} \int_t^{\infty} q(s)(1 - p(\delta(s)))^{\gamma} \left(\frac{\tau(s)}{s}\right)^{\gamma} \Delta s \leq 1,$$

which gives us the contradiction

$$\limsup_{t\to\infty} \frac{t^{\gamma}}{r(t)} \int_t^{\infty} q(s)(1 - p(\delta(s)))^{\gamma} \left(\frac{\tau(s)}{s}\right)^{\gamma} \Delta s \leq 1.$$

The proof is complete.

In the following, we consider the case $0 < \gamma < 1$ and establish new oscillation criteria for (1.1) of Hille and Nehari types.

Theorem 2.5. *Assume that* $(h_1) - (h_2)$ *hold and* (1.1) *has a positive or a negative solution* $y(t)$ *on* $[t_0, \infty)_{\mathbb{T}}$ *such that* $y(t)$ *and* $y(\tau(t)) > 0$ *for* $t \geq t_1 > t_0$. *Let* $x(t)$ *be as defined by* (2.2) *and let* $\omega(t)$, r_* *and* p_* *be as defined in Theorem* 2.1. *Then* (2.6) *holds.*

Proof. From Lemma 2.1 there is a $T \in [t_1, \infty)_{\mathbb{T}}$, sufficiently large, so that $x(t)$ satisfies the conclusions of Lemma 2.1. This implies that $\omega(t)$ is positive. Proceeding as in the proof of we get the inequality

$$\omega^{\Delta}(t) \leq -\left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma} Q(t) - \frac{\left(a^{\sigma} x^{\Delta\sigma}(t)\right)^{\gamma} (x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t) \left(x^{\sigma}(t)\right)^{\gamma}}. \tag{2.21}$$

By the Pötzsche chain rule, and the fact that $x^{\Delta}(t) > 0$, we obtain

$$\begin{aligned}(x^{\gamma}(t))^{\Delta} &= \gamma \int_0^1 \left[x(t) + h\mu(t)x^{\Delta}(t)\right]^{\gamma-1} dh\, x^{\Delta}(t) \\ &= \gamma \int_0^1 [(1-h)x(t) + hx^{\sigma}(t)]^{\gamma-1} dh\, x^{\Delta}(t) \\ &\geq \gamma \int_0^1 (x^{\sigma}(t))^{\gamma-1} dh\, x^{\Delta}(t) = \gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t),\end{aligned}$$

so that

$$(x^{\gamma}(t))^{\Delta} \geq \gamma \int_0^1 (x^{\sigma}(t))^{\gamma-1} dh\, x^{\Delta}(t) = \gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t). \tag{2.22}$$

It follows from (2.21) and (2.22) that

$$\omega^{\Delta}(t) \leq -Q(t) - \frac{\left(x^{\Delta\sigma}(t)\right)^{\gamma} \gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}} = -Q(t) - \frac{\gamma}{a^{\frac{1}{\gamma}}} (\omega^{\sigma})^{\frac{\gamma+1}{\gamma}}.$$

Then $\omega(t)$ satisfies the dynamic Riccati inequality

$$\omega^{\Delta}(t) + Q(t) + \frac{\gamma}{a^{\frac{1}{\gamma}}} (\omega^{\sigma})^{\frac{\gamma+1}{\gamma}} \leq 0, \quad \text{for} \quad t \in [T, \infty)_{\mathbb{T}}. \tag{2.23}$$

We have from (2.23) that

$$-\left(\omega^{\Delta}(t)/\gamma\omega^{\sigma}\omega^{\frac{1}{\gamma}}(t)\right) > 1/a^{\frac{1}{\gamma}}(t), \qquad \text{for} \;\; t \geq T. \tag{2.24}$$

As in the proof of Theorem 2.1, we see that

$$\left(1/\omega^{\frac{1}{\gamma}}(t)\right)^{\Delta} = \frac{1}{\sigma - t} \frac{\omega^{\frac{1}{\gamma}}(t) - (\omega^{\sigma})^{\frac{1}{\gamma}}}{\omega^{\frac{1}{\gamma}}(t)(\omega^{\sigma})^{\frac{1}{\gamma}}}.$$

Using the inequality (see p. 39 in [6])

$$x^{\beta} - y^{\beta} \geqslant \beta y^{\beta-1}(x-y) \text{ for all } x > y > 0 \text{ and } \beta \geq 1,$$

and the fact that $\omega(t)$ is nonincreasing, we see that

$$\begin{aligned}\left(1/\omega^{\frac{1}{\gamma}}(t)\right)^{\Delta} &\geq \frac{1}{\sigma - t} \frac{1}{\gamma\omega^{\frac{1}{\gamma}}(t)(\omega^{\sigma})^{\frac{1}{\gamma}}} (\omega^{\sigma})^{\frac{1}{\gamma}-1}(\omega(t) - \omega^{\sigma}) \\ &= -\frac{\omega^{\sigma} - \omega(t)}{\sigma - t} \frac{1}{\gamma\omega^{\frac{1}{\gamma}}(t)(\omega^{\sigma})^{\frac{1}{\gamma}}} (\omega^{\sigma})^{\frac{1}{\gamma}-1} \\ &= \frac{-1}{\gamma} \frac{\omega^{\Delta}}{\omega^{\sigma}\omega^{\frac{1}{\gamma}}(t)}.\end{aligned}$$

This and (2.24) imply again that

$$\left(1/\omega^{\frac{1}{\gamma}}(t)\right)^{\Delta} > \frac{1}{a^{\frac{1}{\gamma}}(t)}.$$

The remainder of the proof is similar to the proof of Theorem 2.1 and hence is omitted.

Remark 1. It is clear that the inequality (2.23) is similar to the inequality (2.10) in Theorem 2.1. So as a consequence of Theorem 2.5 and as proved in Theorem 2.2 and Theorem 2.3, we can prove the similar results for equation (1.1) in the case when $0<\gamma<1$. So the results in Theorem 2.2 and Theorem 2.3 can be applied in the case when $0<\gamma<1$ as well as in the case when $\gamma>1$.

In the following, we give an example to illustrate the main results.

Example 2.1. Consider the following second-order neutral delay dynamic equation

$$\left(\left[\left(y(t)+\frac{\delta^{-1}(t)-1}{\delta^{-1}(t)}y(\tau(t))\right)^{\Delta}\right]^{\gamma}\right)^{\Delta}+\frac{\beta\sigma^{\gamma}(s)}{t\tau^{\gamma}(t)}y^{\gamma}(\delta(t))=0, \tag{2.25}$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, $t_0>0$. Here $r(t)=1$, $q(t)=\frac{\beta}{t\tau^{\gamma}(t)}$, and $\tau(t)$ and $\delta(t)$ are delay functions satisfy $\tau(t)=\mathbb{T}\to\mathbb{T}$, $\delta(t):\mathbb{T}\to\mathbb{T}$ for all $t\in\mathbb{T}$, and $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\delta(t)=\infty$, $\tau(t)\le t$, $\delta(t)\le t$. It is clear that (h_1), (h_2) and (2.1) hold. To apply Theorem 2.2 it remains to prove that the condition (2.19) holds. In this case (2.19) reads

$$\begin{aligned}\liminf_{t\to\infty}t^{\gamma}\int_{\sigma(t)}^{\infty}Q(s)\Delta s &= \liminf_{t\to\infty}t^{\gamma}\int_{\sigma(t)}^{\infty}\frac{\beta\sigma^{\gamma}(s)}{s\tau^{\gamma}(s)}(\frac{1}{s})^{\gamma}\left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma}\Delta s\\ &= \liminf_{t\to\infty}t^{\gamma}\int_{\sigma(t)}^{\infty}\frac{\beta}{s^{\gamma+1}}\Delta s\ge\liminf_{t\to\infty}t^{\gamma}\int_{\sigma(t)}^{\infty}\frac{\beta}{s^{\gamma}\sigma(s)}\Delta s\\ &\ge \beta\liminf_{t\to\infty}t^{\gamma}\int_{\sigma(t)}^{\infty}(\frac{-1}{s^{\gamma}})^{\Delta}\Delta s=\beta\liminf_{t\to\infty}\frac{t^{\gamma}}{\sigma^{\gamma}(t)}=\beta l^{\gamma}\end{aligned}$$

where $Q(t):=q(t)(1-p(\delta(t)))^{\gamma}\left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma}$. Then by Theorem 2.2 the equation (2.25) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$ if $\beta l^{\gamma}>\frac{\gamma^{\gamma}}{l\gamma^{2}(\gamma+1)^{\gamma+1}}$. In particular, if $\mathbb{T}=\mathbb{R}$, $t_0>0$, and the delay is the constant delay $\delta(t)=t-\alpha$, where $\alpha>0$, then if $\beta>\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}$, equation (2.25) is oscillatory; if $\mathbb{T}=\mathbb{Z}$, $t_0=1$, and the delay is the constant delay $\delta(t)=t-n$, n a positive integer, then if $\beta>\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}$, equation (2.25) is oscillatory.

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