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A NOTE ON POSITIVE NONOSCILLATORY SOLUTIONS OF THE DIFFERENCE EQUATION $x_{n+1} = \frac{p}{x_n} + \left(\frac{x_{n-2}}{x_n}\right)^\alpha$

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ABSTRACT. The aim of this note is to show that the following difference equation

$$x_{n+1} = \frac{p}{x_n} + \left(\frac{x_{n-2}}{x_n}\right)^\alpha$$

where $p, \alpha > 0$, has positive nonoscillatory solutions which converge to the positive equilibrium $\overline{x} = \dfrac{1+\sqrt{1+4p}}{2}$. In the proof of the result we use a method developed by L. Berg and S. Stević.

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1. INTRODUCTION AND PRELIMINARIES

Recently, there has been a lot of interest in studying the global attractivity, the boundedness character and the periodic nature of nonlinear difference equations. For some recent results see, for example, [1–27].

In [7] the authors have studied the behavior of all positive solutions of the difference equation

$$x_{n+1} = \frac{p + x_{n-2}}{x_n}, \quad n = 0, 1, 2, \dots$$

where p is a positive real parameter and the initial values x_{-2}, x_{-1}, x_0 are positive real numbers. For every value of (positive) parameter p, there exists a unique positive equilibrium $\overline{x}$ which satisfies the equation

$$\overline{x}^2 = \overline{x} + p.$$

In this note we investigate the behavior of positive solutions of the difference equation

$$x_{n+1} = \frac{p}{x_n} + \left(\frac{x_{n-2}}{x_n}\right)^\alpha, \quad n = 0, 1, 2, \dots \tag{1}$$

where $p, \alpha > 0$, and the initial values x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers. Note that the positive equilibrium of Eq (1) also satisfies the equation $\overline{x}^2 = \overline{x} + p$.

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We say that a solution (x_n) of equation (1) is bounded and persists if there exists positive constants P and Q such that

$$P \leq x_n \leq Q \quad \text{for} \quad n = -2, -1, 0, 1, \ldots$$

A positive semicycle of a solution (x_n) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to $\overline{x}$, with $l \geq -2$ and $m \leq +\infty$ and such that

$$\text{either} \;\; l = -2, \;\text{ or }\; l > -2 \text{ and } \; x_{l-1} < \overline{x},$$

and

$$\text{either} \;\; m = \infty, \;\text{ or }\; m < \infty \text{ and } \; x_{m+1} < \overline{x}.$$

A negative semicycle of a solution (x_n) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all less than to $\overline{x}$, with $l \geq -2$ and $m \leq \infty$ and such that

$$\text{either} \;\; l = -2, \;\text{ or }\; l > -2 \text{ and } \; x_{l-1} \geq \overline{x},$$

and

$$\text{either} \;\; m = \infty, \;\text{ or }\; m < \infty \text{ and } \; x_{m+1} \geq \overline{x}.$$

The first semicycle of a solution starts with the term x_{-2} and is positive if $x_{-2} \geq \overline{x}$. We now investigate oscillation and nonoscillation of positive solutions of the difference equation (1). We shall prove two following theorems, which are similar to the results from paper [7] (see also [19]).

Theorem 1.1. *Let* $\{x_n\}_{n=-2}^{+\infty}$ *be a positive solution of* Eq (1) *for which there exists* $N \geq -2$ *such that* $x_N < \overline{x}$ *and* $x_{N+1} \geq \overline{x}$, *or* $x_N \geq \overline{x}$ *and* $x_{N+1} < \overline{x}$. *Then the solution* $\{x_n\}_{n=-2}^{+\infty}$ *oscillates about the equilibrium* $\overline{x}$ *with every semicycle (except possibly the first) having at most two terms.*

Proof. Let $N \geq -2$ such that $x_N < \overline{x} \leq x_{N+1}$. The case where $x_{N+1} < \overline{x} \leq x_N$ is similar and will be omitted. Now suppose that the positive semicycle beginning with the term x_{N+1} has two terms. Then $x_N < \overline{x} \leq x_{N+2}$ and so

$$x_{N+3} = \frac{p}{x_{N+2}} + \left(\frac{x_N}{x_{N+2}}\right)^{\alpha} < \frac{p}{\overline{x}} + 1 = \overline{x}.$$

The proof is completed. □

Theorem 1.2. *All nonoscillatory solutions of* Eq (1) *converge to the positive equilibrium* $\overline{x}$.

Proof. We will show the proof of the theorem in the case of a single positive semicycle. The case of a single negative semicycle is similar and will be omitted. Assume that $x_n \geq \overline{x}$ for all $n \geq -2$. We first claim that for this solution

$$x_{n-2} \geq x_n \;\; \text{for all } n = 0, 1, 2, \ldots$$

For the sake of contradiction assume that there exists $N \geq 0$ such that $x_{N-2} < x_N$. Using Eq (1) we have

$$x_{N+1} = \frac{p}{x_N} + \left(\frac{x_{N-2}}{x_N}\right)^{\alpha} < \frac{p}{x_N} + 1 \leq \frac{p}{\overline{x}} + 1 = \overline{x}$$

which is a contradiction and so

$$\overline{x} \leq x_n \leq x_{n-2} \quad \text{for } n = 0, 1, 2, \ldots$$

In addition for $i = 0, 1$ there exists α_0, α_1 such that

$$\lim_{n\to\infty} x_{2n+i} = \alpha_i; \quad i = 0, 1.$$

It follows that $\{\alpha_0, \alpha_1, \alpha_0, \alpha_1, \ldots\}$ is a periodic solution of not necessarily prime period two. On the other hand Eq (1) has no prime period two solutions, and so $\alpha_0 = \alpha_1 = \overline{x}$. The proof is completed. □

2. ON POSITIVE NONOSCILLATORY SOLUTIONS OF THE DIFFERENCE EQUATION (1)

Our aim in this note is to solve the following problem. Do there exists nonoscillatory solutions of Eq (1)? We will solve this problem by a method due to L. Berg and S. Stević, see, for example, [1]–[5], [20, 22, 23, 24, 26, 27].

Note that the linearized equation for Eq (1) about the positive equilibrium $\overline{x}$ is

$$y_{n+1} = -\frac{p + \alpha\overline{x}}{\overline{x}^2} y_n + \frac{\alpha}{\overline{x}} y_{n-2}$$

$$y_{n+1} + \frac{p + \alpha\overline{x}}{\overline{x}^2} y_n - \frac{\alpha}{\overline{x}} y_{n-2} = 0 \tag{2}$$

From Eq (1) we have $\overline{x} = \frac{p}{\overline{x}} + 1$, $\overline{x}^2 = \overline{x} + p$, and

$$y_{n+1} + \frac{\overline{x}^2 - \overline{x} + \alpha\overline{x}}{\overline{x}^2} y_n - \frac{\alpha}{\overline{x}} y_{n-2} = 0$$

$$\overline{x} y_{n+1} + (\overline{x} + \alpha - 1) y_n - \alpha y_{n-2} = 0 \tag{3}$$

The characteristic polynomial associated with Eq (3) is

$$p(t) = \overline{x} t^3 + (\overline{x} + \alpha - 1) t^2 - \alpha = 0 \tag{4}$$

Since $p(0) = -\alpha < 0$, $p(1) = 2\overline{x} - 1 > 0$ with $\overline{x} = \frac{1 + \sqrt{1 + 4p}}{2} > 1$ for every $p > 0, \alpha > 0$ and

$$p'(t) = 3\overline{x} t^2 + 2(\overline{x} + \alpha - 1) t > 0$$

when $t \in (0, 1]$, it follows that for each $p > 0$, $\alpha > 0$, there is unique positive root t_0 of the polynomial belonging to the interval $(0, 1)$. As suggested by Stević in [24], this fact motivates us to believe that there are solutions of Eq (1) which have the following asymptotics

$$x_n = \overline{x} + at_0^n + o(t_0^n) \tag{5}$$

where $a \in \mathbb{R}$ and t_0 is the above mentioned root of the polynomial (4). Asymptotics for solutions of difference equations have been investigated by L. Berg and S. Stević, see, for example, [1–6], [9–27] and the reference therein. The problem is solved by constructing two appropriate sequences y_n and z_n with

$$y_n \le x_n \le z_n \tag{6}$$

for sufficiently large n. In [1], [2] some methods can be found for the construction of these bounds, see, also [3, 4].

From (5) and results in Berg's paper [3, 4] we expect that for $k \ge 2$ such solutions have the first three members in their asymptotics in the following form

$$\varphi_n = \overline{x} + at^n + bt^{2n}. \tag{7}$$

Since Eq (1) is an autonomous one, the parameter a remains arbitrarily, whereas b must have the structure $b = a^2 A$ with A independent from a (see the proof of Theorem 3.1). We need the following result in the proof of the main theorem. The proof of the result can be found in [23] and [24].

Theorem 2.1. *Let $f : I^{k+2} \to I$ be a continuous and nondecreasing function in each argument on the interval $I \subset \mathbb{R}$, and let (y_n) and (z_n) be sequences in I with $y_n < z_n$ for $n \ge n_0$ and such that*

$$y_{n-k} \le f(n, y_{n-k+1}, \dots, y_{n+1}), \tag{8}$$
$$f(n, z_{n-k+1}, \dots, z_{n+1}) \le z_{n-k}, \ \textit{for}\ n > n_0 + k - 1.$$

Then there is a solution of the following difference equation

$$x_{n-k} = f(n, x_{n-k+1}, \dots, x_{n+1}) \tag{9}$$

with property (6) *for $n \ge n_0$.*

3. THE MAIN RESULT

In this section, we prove the main result in this note.

Theorem 3.1. *For each $p, \alpha > 0$ there is a nonoscillatory solution of* Eq (1) *converging to the positive equilibrium*

$$\overline{x} = \frac{1 + \sqrt{1+4p}}{2},$$

with the asymptotic behavior (7).

Proof. First note that Eq (1) can be written in the following equivalent form

$$x_{n-2} = \left(x_{n+1} - \frac{p}{x_n}\right)^{\frac{1}{\alpha}} x_n.$$

Since

$$x_{n+1}x_n = p + x_{n-2}^{\alpha} x_n^{1-\alpha},$$

we have

$$x_{n+1}x_n > p,$$

$$x_{n-2} = (x_{n+1}x_n - p)^{\frac{1}{\alpha}} x_n^{1-\frac{1}{\alpha}}$$

and

$$F(x_{n-2}, x_{n-1}, x_n, x_{n+1}) = (x_{n+1}x_n - p)^{\frac{1}{\alpha}} x_n^{\frac{\alpha-1}{\alpha}} - x_{n-2} = 0 \tag{10}$$

Let

$$f(x, y) = (x^{\alpha} y - p x^{\alpha-1})^{\frac{1}{\alpha}}.$$

Then it is obvious that f increases in y on the interval $(0, \infty)$ for each fixed $x \in (0, \infty)$. On the other hand

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y) &= \frac{1}{\alpha}(x^{\alpha} y - p x^{\alpha-1})^{\frac{1}{\alpha}-1}(\alpha x^{\alpha-1} y - p(\alpha - 1)x^{\alpha-2}) \\ &= \frac{1}{\alpha}(x^{\alpha-1}(xy - p))^{\frac{1}{\alpha}-1} x^{\alpha-2}(\alpha xy + p(1 - \alpha)),\end{aligned}$$

which is obviously positive on the set $A = \{(x, y) \in \mathbb{R}_+^2 : xy > p\}$, if $\alpha \in (0, 1]$.

On the other hand,

$$\frac{\partial f}{\partial x}(x, y) = \frac{1}{\alpha}(x^{\alpha-1}(xy - p))^{\frac{1}{\alpha}-1} x^{\alpha-2}(\alpha(xy - p) + p) > 0,$$

on the set A, also for $\alpha > 1$.

Let $I = [\overline{x}, \infty)$. Since for $x, y \in [\overline{x}, \infty)$, $xy \geq \overline{x}^2 = \overline{x} + p > p$, we have that $[\bar{x}, \infty)^2 \subset A$, so that

$$\min_{(x,y)\in I^2} f(x, y) = f(\overline{x}, \overline{x}) = \overline{x},$$

that is,

$$f : [\overline{x}, \infty)^2 \to [\overline{x}, \infty),$$

and f is increasing in both variables on $[\overline{x}, \infty)$.

We expect that solutions of Eq (1) have the asymptotics approximation (7) with $a>0$. Thus, we can calculate $F(\varphi_{n-2},\varphi_{n-1},\varphi_n,\varphi_{n+1})$. We have

$$F=[(\overline{x}+at^n+bt^{2n})(\overline{x}+at^{n+1}+bt^{2n+2})-p]^{\frac{1}{\alpha}}(\overline{x}+at^n+bt^{2n})^{\frac{\alpha-1}{\alpha}}- \\ -(\overline{x}+at^{n-2}+bt^{2n-4}).$$

$$F=\Big(\overline{x}^2+\overline{x}at^{n+1}+\overline{x}bt^{2n+2}+\overline{x}at^n+a^2t^{2n+1}+abt^{3n+2}+ \\ +\overline{x}bt^{2n}+abt^{3n+1}+b^2t^{4n+2}-p\Big)^{\frac{1}{\alpha}}(\overline{x})^{\frac{\alpha-1}{\alpha}}\left(1+\frac{at^n+bt^{2n}}{\overline{x}}\right)^{\frac{\alpha-1}{\alpha}}- \\ -(\overline{x}+at^{n-2}+bt^{2n-4}).$$

From $\overline{x}^2=p+\overline{x}$, we have

$$F=(\overline{x}+\overline{x}at^{n+1}+\overline{x}bt^{2n+2}+\overline{x}at^n+a^2t^{2n+1}+abt^{3n+2}+ \\ +\overline{x}bt^{2n}+abt^{3n+1}+b^2t^{4n+2})^{\frac{1}{\alpha}}\overline{x}^{\frac{\alpha-1}{\alpha}}\left(1+\frac{at^n+bt^{2n}}{\overline{x}}\right)^{\frac{\alpha-1}{\alpha}}-(\overline{x}+at^{n-2}+bt^{2n-4}).$$

$$F=\overline{x}\Bigg[1+\frac{1}{\alpha\overline{x}}(\overline{x}at^{n+1}+\overline{x}bt^{2n+2}+\overline{x}at^n+a^2t^{2n+1}+abt^{3n+2}+\overline{x}bt^{2n}+abt^{3n+1}+ \\ +b^2t^{4n+2})+\frac{(1-\alpha)}{2\alpha^2\overline{x}^2}(\overline{x}at^n+a^2t^{2n+1}+abt^{3n+2}+\overline{x}at^{n+1}+\overline{x}bt^{2n+2}+\overline{x}bt^{2n}+ \\ +abt^{3n+1}+b^2t^{4n+2})^2+\cdots\Bigg]\times\left[1+\frac{\alpha-1}{\alpha\overline{x}}(at^n+bt^{2n})+\frac{(1-\alpha)}{2\alpha^2\overline{x}^2}(at^n+bt^{2n})^2+\cdots\right]- \\ -(\overline{x}+at^{n-2}+bt^{2n-4}).$$

$$F=\overline{x}\Bigg[1+\frac{1}{\alpha\overline{x}}(\overline{x}at^{n+1}+\overline{x}bt^{2n+2}+\overline{x}at^n+a^2t^{2n+1}+abt^{3n+2}+\overline{x}bt^{2n}+abt^{3n+1}+ \\ +b^2t^{4n+2})+\frac{1-\alpha}{2\alpha^2\overline{x}^2}(\overline{x}^2a^2t^{2n}+\overline{x}^2a^2t^{2n+2}+2\overline{x}^2a^2t^{2n+1})+\cdots\Bigg]\times \\ \times\left[1+\frac{\alpha-1}{\alpha\overline{x}}at^n+\frac{(\alpha-1)b}{\alpha\overline{x}}t^{2n}+\frac{1-\alpha}{2\alpha^2\overline{x}^2}a^2t^{2n}+\cdots\right]-(\overline{x}+at^{n-2}+bt^{2n-4}).$$

$$F=a\left(\frac{\overline{x}}{\alpha}+\frac{\alpha-1}{\alpha}+\frac{\overline{x}t}{\alpha}-\frac{1}{t^2}\right)t^n+\Bigg\{b\left[\frac{\overline{x}}{\alpha}+\frac{\overline{x}}{\alpha}t^2+\frac{\alpha-1}{\alpha}-\frac{1}{t^4}\right]+ \\ +a^2\left[\frac{t}{\alpha}+\frac{(1-\alpha)\overline{x}}{2\alpha^2}+\frac{(1-\alpha)\overline{x}t}{\alpha^2}+\frac{(1-\alpha)\overline{x}}{2\alpha^2}t^2+\frac{\alpha-1}{\alpha^2}+\frac{\alpha-1}{\alpha^2}t\right]\Bigg\}t^{2n}+o(t^{2n}).$$

$$F=a\left[\frac{(\overline{x}+\alpha-1+\overline{x}t)t^2-\alpha}{\alpha t^2}\right]t^n+\Bigg[b\frac{(\overline{x}+\alpha-1+\overline{x}t^2)t^4-\alpha}{\alpha t^4}+ \\ +a^2\frac{(1-\alpha)\overline{x}+2\alpha-2+(4\alpha-2+2\overline{x}-2\alpha\overline{x})t+(1-\alpha)\overline{x}t^2}{2\alpha^2}\Bigg]t^{2n}+o(t^{2n}).$$

We have

$$\frac{(\overline{x}+\alpha-1+\overline{x}t)t^2-\alpha}{\alpha t^2}=\frac{\overline{x}t^3+(\overline{x}+\alpha-1)t^2-\alpha}{\alpha t^2}=\frac{p(t)}{\alpha t^2},$$

where $p(t)$ is the characteristic polynomial (4). We know that there exists the unique root $t_0 \in (0,1)$ such that $p(t_0) = 0$. Let

$$\frac{p(t_0^2)}{\alpha t_0^4} = \frac{(\overline{x}+\alpha-1+\overline{x}t_0^2)t_0^4-\alpha}{\alpha t_0^4} = \frac{\overline{x}t_0^6+(\overline{x}+\alpha-1)t_0^4-\alpha}{\alpha t_0^4}.$$

From this, with $t = t_0$, we have

$$F = \left\{ b\frac{p(t_0^2)}{\alpha t_0^4} + a^2\left[\frac{(1-\alpha)\overline{x}+2\alpha-2+(4\alpha-2+2\overline{x}-2\alpha\overline{x})t_0+(1-\alpha)\overline{x}t_0^2}{2\alpha^2}\right]\right\} t_0^{2n} +$$
$$+ o(t_0^{2n}),\ 0 < t_0^2 < t_0 < 1,\ p(t_0^2) < p(t_0) = 0.$$

Thus, the coefficient of b is negative: $\dfrac{p(t_0^2)}{\alpha t_0^4} < 0$.

We set

$$A = \frac{(1-\alpha)\overline{x}+2\alpha-2+(4\alpha-2+2\overline{x}-2\alpha\overline{x})t_0+(1-\alpha)\overline{x}t_0^2}{2\alpha^2}.$$

Then

$$F = \left[b\frac{p(t_0^2)}{\alpha t_0^4} + a^2A\right]t_0^{2n} + o(t_0^{2n}).$$

Set

$$q = -\frac{a^2A\alpha t_0^4}{p(t_0^2)} \quad \text{and} \quad H_{t_0}(q) = q\frac{p(t_0^2)}{\alpha t_0^4} + a^2A.$$

Note that

$$H'_{t_0}(q) = \frac{p(t_0^2)}{\alpha t_0^4} < 0.$$

If

$$\hat{\varphi}_n = \overline{x} + at_0^n + bt_0^{2n} = \frac{1+\sqrt{4p+1}}{2} + at_0^n + bt_0^{2n},$$

we obtain

$$F(\hat{\varphi}_{n-2}, \hat{\varphi}_{n-1}, \hat{\varphi}_n, \hat{\varphi}_{n+1}) \sim \left[b\frac{p(t_0^2)}{\alpha t_0^4} + a^2A\right]t_0^{2n} = H_{t_0}(b)t_0^{2n}.$$

Since $H'_{t_0}(q) = \dfrac{p(t_0^2)}{\alpha t_0^4} < 0$, we obtain that there are $q_1 < b$ and $q_2 > b$ such that $H_{t_0}(q_1) > 0$ and $H_{t_0}(q_2) < 0$.

With the notations

$$y_n = \overline{x} + at_0^n + q_1t_0^{2n},\ z_n = \overline{x} + at_0^n + q_2t_0^{2n}.$$

We get

$$F(y_{n-2}, y_{n-1}, y_n, y_{n+1}) \sim \left[q_1\frac{p(t_0^2)}{\alpha t_0^4} + a^2A\right]t_0^{2n} > 0$$
$$F(z_{n-2}, z_{n-1}, z_n, z_{n+1}) \sim \left[q_2\frac{p(t_0^2)}{\alpha t_0^4} + a^2A\right]t_0^{2n} < 0.$$

These relations show that inequalities (8) are satisfied for sufficiently large n, where $f = F + x_{n-2}$ and F is given by (10). Since for all n, $y_n > 0$, we can apply Theorem 2.1

with $I = [\overline{x}, \infty)$ and see that there is an $n_0 > 0$ and a solution of Eq (1) with the asymptotics $x_n = \hat{\varphi}_n + o(t_0^{2n})$, for $n \geq n_0$, where $\hat{\varphi}_n$ is defined by (7) and $b = q$. In particular, the solution converges monotonically to the positive equilibrium

$$\overline{x} = \frac{1 + \sqrt{1 + 4p}}{2}, \text{ for } n \geq n_0.$$

Hence, the solution x_{n+n_0+2} is also such a solution when $n \geq -2$. □

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REFERENCES

[1] L. Berg, *Asymptoticsche Darstellungen und Entwicklungen*, Dt. Verlag Wiss, Berlin, 1968.

[2] L. Berg, On the asymptotics of nonlinear difference equations, *Z. Anal. Anwendungen* **21** (2002), 1061–1074.

[3] L. Berg, Inclusion theorems for nonlinear difference equations with applications, *J. Differ. Equations. Appl.* **10** (2004), 399–408.

[4] L. Berg, Oscillating solutions of rational difference equations, *Rostock. Math. Kolloq.* **58** (2004), 31–35.

[5] L. Berg, Corrections to "Inclusion theorems for nonlinear difference equations with applications", *J. Differ. Equations Appl.* **11** (2005), 181–182.

[6] L. Berg and L. v. Wolfersdorf, On a class of generalized autoconvolution equations of the third kind, *Z. Anal. Anwendungen* **24** (2005), 217–250.

[7] E. Camouzis, R. Devault, and W. Kosmala, On the period five trichotomy of all positive solutions of $x_{n+1} = \frac{p + x_{n-2}}{x_n}$, *J. Math. Anal. Appl.* **291** (2004), 40–49.

[8] R. Devault, C. Kent and W. Kosmala, On the recursive sequence $x_{n+1} = p + \frac{x_{n-k}}{x_n}$, *J. Differ. Equations Appl.* **9** (2003), 721–730.

[9] S. Stević, Asymptotic behaviour of a sequence defined by iteration, *Math. Vesnik* **48** (1996), 99–105.

[10] S. Stević, Behaviour of the positive solutions of the generalized Beddington-Holt equation, *Panamer. Math. J.* **4** (2000), 77–85.

[11] S. Stević, Asymptotic behaviour of a sequence defined by a recurrence formula, *Austral. Math. Soc. Gaz.* **28** (2001), 243–245.

[12] S. Stević, On the recursive sequence $x_{n+1} = -\frac{1}{x_n} + \frac{A}{x_{n-1}}$, *Inter. J. Math. Sci.* **27** (2001).

[13] S. Stević, A global convergence results with applications to periodic solutions, *Indian J. Pure Appl. Math.* **33** (2002), 45–53.

[14] S. Stević, A note on the difference equations $x_{n+1} = \sum_{i=0}^{k} \frac{\alpha_i}{x_{n-i}^{p_i}}$, *J. Differ. Equations Appl.* **8** (2002), 641–647.

[15] S. Stević, Asymptotic behaviour of a sequence defined by a recurrence formula II, *Austral. Math. Soc. Gaz.* **29** (2002), 209–215.

[16] S. Stević, Asymptotic behaviour of a nonlinear difference equation, *Indian J. Pure Appl. Math.* **34** (2003), 1681–1689.

[17] S. Stević, On the recursive sequence $x_{n+1} = x_n + \dfrac{x_n^\alpha}{n^\beta}$, *Bull. Calcuta Math. Soc.* **95** (2003), 39–46.

[18] S. Stević, More on the difference equation $x_{n+1} = \dfrac{x_{n-1}}{1 + x_{n-1}x_n}$, *Appl. Math. E-notes* **4** (2004), 80–85.

[19] S. Stević, On the recursive sequence $x_{n+1} = \alpha + \dfrac{x_{n-1}^p}{x_n^p}$, *J. Appl. Math and Computing* **18** (2005), 229–234.

[20] S. Stević, Asymptotic behaviour of a class of nonlinear difference equations, *Discrete Dyn. Nat. Soc.* **Vol. 2006**, Article ID 47156, pp. 10.

[21] S. Stević, Global stability and asymptotics of some classes of rational difference equations, *J. Math. Anal. Appl.* **316** (2006), 60–68.

[22] S. Stević, On monotone solutions of some classes of difference equations, *Discrete Dyn. Nat. Soc.* **Vol. 2006**, Article ID 53890, (2006), pp. 9.

[23] S. Stević, On positive solutions of a $(k+1)$-th order difference equation, *Appl. Math. Lett.* **19** (2006), 427–431.

[24] S. Stević, Asymptotics of some classes of higher order difference equations, *Discrete Dyn. Nat. Soc.* **Vol. 2007**, Article ID 56813, (2007), 20 pages.

[25] S. Stević, Existence of nontrivial solutions of a rational difference equation, *Appl. Math. Lett.* **20** (2007), 28–31.

[26] S. Stević and K. Berenhaut, A note on positive nonoscillatory solutions of the difference equation $x_{n+1} = \alpha + \dfrac{x_{n-k}^p}{x_n^p}$, *J. Differ. Equations Appl.* **12** (2006), 495–499.

[27] S. Stević and K. Berenhaut, The difference equation $x_{n+1} = \alpha + \dfrac{x_{n-k}}{\sum_{i=0}^{k-1} c_i x_{n-i}}$ has solutions converging to zero, *J. Math. Anal. Appl.* **326** (2007), 1466–1471.