Communications in Applied Analysis **12** (2008), no. 2, 209–220

OSCILLATION OF PERTURBED DELAY DYNAMIC EQUATIONS ON TIME SCALES

S. H. SAKER

Department of Mathematics, College of Science, King Saud University
Riyadh 11451, Saudi Arabia
Email: saker@mans.edu.eg

ABSTRACT. In this paper, we study the oscillation of second-order nonlinear perturbed delay dynamic equations on time scales. By using employing the Riccati substitution, we establish some sufficient conditions for oscillation. We also follow the technique that has been used in [20] and establish some different sufficient conditions for oscillation which can be considered as the extension of oscillation criteria established by Hille and Nehari for second-order differential equations and can be applied on the sublinear and superlinear cases. As a special case the results improve some oscillation criteria in the literature for perturbed dynamic equations without delay. Some examples are considered to illustrate the main results.

AMS (MOS) Subject Classification. 34K11, 34C10, 39A13

1. INTRODUCTION

The study of dynamic equations on time scales, which goes back to Stefan Hilger [11] is an area of mathematics that has recently received a lot of attention. It has been created to unify the study of differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A recent cover story article in New Scientist [21] discusses several possible applications. We suppose that the reader is familiar with the basic theory of the time scale calculus and for details we refer the reader to the books by Bohner and Peterson [4, 5] which summarize and organize much of time scale calculus.

Recently there has been much research activity concerning the qualitative theory of dynamic equations on time scales. One of the main subject of the qualitative analysis of the dynamic equations is studying oscillation and nonoscillation which seeks to harmonize the oscillation of the continuous and the discrete, to include

On leave from Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

Received June 10, 2008

them in one comprehensive mathematics and to eliminate obscurity from both. For convenience, we refer the reader to the results in [1, 2, 3, 6, 7, 8, 9, 10, 15, 16, 17, 18, 19, 20, 22] and the references cited therein.

For oscillation of perturbed dynamic equations on time scales, Bohner and Saker [6] considered the equation

$$(r(t)\left(x^{\Delta}\right)^{\gamma})^{\Delta} + F(t, x^{\sigma}) = G(t, x^{\sigma}, x^{\Delta}), \tag{1.1}$$

on a time scale $\mathbb{T}$, where $x^{\sigma} = x(\sigma(t))$, and $\sigma(t)$ is the forward jump operator defined on $\mathbb{T}$, γ is a quotient of odd positive integers and r is a positive real-valued rd-continuous function defined on the time scales interval $[a, b]$ (throughout a, $b \in \mathbb{T}$ with $a < b$), and assumed that there exist two positive rd-continuous functions p and q such that:

(H_1) $F : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ such that

$$uF(t, u) > 0 \ \text{ and } uG(t, u, v) > 0 \ \text{ for all } u \in \mathbb{R}\backslash\{0\},\ v \in \mathbb{R},\ t \in \mathbb{T}; \tag{1.2}$$

(H_2) $\frac{F(t,u)}{f(u)} \geq q(t)$, $\frac{G(t,u,v)}{f(u)} \leq p(t)$ for all u, $v \in \mathbb{R}\backslash\{0\}$, $t \in \mathbb{T}$;

(H_3) $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and nondecreasing, and $uf(u) > 0$ for all $u \in \mathbb{R}\backslash\{0\}$,

(H_4) $\int_{t_2}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} = \infty$

By using elementary calculus on time scales and the Riccati transformation techniques the authors in [6] established some sufficient conditions, in terms of the coefficients and the graininess function, which guarantee that every solution of (1.1) is oscillatory or converges to zero.

In 2005 Sun and Li [22] considered (1.1) when the conditions (H_1)–(H_3) hold and

$$\int_{t_0}^{\infty} [p(t) - q(t)]\Delta t = \infty, \tag{h_1}$$

and established some sufficient conditions which ensure that every solution of (1.1) is oscillatory or weakly oscillatory. One can easily note that the condition (h_1) that has been established Sun and Li [22] cannot be applied in the case when $p(t) - q(t) = \alpha/t^2$ for $\alpha > 0$.

In this paper, we are inspired to study the oscillation of second-order nonlinear delay perturbed dynamic equation

$$(r(t)\left(x^{\Delta}\right)^{\gamma})^{\Delta} + F(t, x(\tau(t))) = G(t, x(\tau(t)), x^{\Delta}(t)), \tag{1.3}$$

on a time scale $\mathbb{T}$, when (H_1)–(H_3) hold and the delay function satisfies the following condition:

(H_5) $\tau(t) : \mathbb{T} \to \mathbb{T}$ $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$.

One reason for this upsurge of interest in this types of equations is that the differential forms of these equations have important applications and are a powerful tool in the study of many problems in the natural sciences and in technology. In fact they are extensively employed in mechanics, astronomy, physics, and in many problems of chemistry and biology. The reason for it is the fact that objective laws governing certain phenomena (processes) can be written as differential equations, difference equations, so that the equations themselves are a quantitative expression of these laws. For instance, Newton's laws of mechanics make it possible to reduce the description of the motion of mass points or solid bodies to solving differential equations and to use computer simulations, these equations should be translated to difference equations. The computation of radiotechnical circuits or satellite trajectories, studies of the stability of a plane in flight, and explaining the course of chemical reactions are all carried out by studying and solving such equations. The most interesting and most important applications of these equations are in the theory of oscillations and in automatic control theory. Applied problems in turn produce new formulations of problems in the theory of differential and difference equations; the mathematical theory of optimal control in fact arose in this manner.

Since, we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. Our attention is restricted to those solutions of (1.3) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > t_1\} > 0$ for any $t_1 \geq t_x$. A solution $x(t)$ of (1.3) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

We note that (1.3) involves some different types of differential and difference equations depend on the type of the time scale. For example in the case when $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, $\mu(t) = 0$, $f^{\Delta}(t) = f'(t)$ and (1.3) becomes the second-order nonlinear perturbed delay differential equation

$$(r(t)(x'(t))^{\gamma})' + F(t, x(\tau(t))) = G(t, x(\tau(t)), x'(t)). \tag{1.4}$$

When $\mathbb{T} = \mathbb{N}$, we have $\sigma(t) = t + 1$, $\mu(t) = 1$, $y^{\Delta}(t) = \Delta y(t) = y(t+1) - y(t)$, and (1.3) becomes the second-order nonlinear perturbed delay difference equation

$$\Delta(r(t)(\Delta x(t))^{\gamma}) + F(t, x(\tau(t))) = G(t, x(\tau(t)), \Delta x(t)). \tag{1.5}$$

When $\mathbb{T} = h\mathbb{N}$, $h > 0$, we have $\sigma(t) = t + h$, $\mu(t) = h$, $x^{\Delta}(t) = \Delta_h x(t) = (x(t+h) - x(t))/h$, and (1.3) becomes the second-order nonlinear perturbed delay difference equation

$$\Delta_h(r(t)\,(\Delta_h x(t))^{\gamma}) + F(t, x(\tau(t))) = G(t, x(\tau(t)), \Delta_h x(t)). \tag{1.6}$$

When $\mathbb{T} = q^{\mathbb{N}} = \{t : t = q^k,\ k \in \mathbb{N},\ q > 1\}$, we have $\sigma(t) = qt$, $\mu(t) = (q-1)t$, $x^{\Delta}(t) = \Delta_q x(t) = (x(qt) - x(t))\,/(q-1)\,t$, and (1.3) becomes the second-order q-perturbed delay difference equation

$$\Delta_q(r(t)\,(\Delta_q x(t))^{\gamma}) + F(t, x(\tau(t))) = G(t, x(\tau(t)), \Delta_q x(t)). \tag{1.7}$$

which has applications in quantum physics, see [13].

The main oscillation results will be proved in the next section which is organized as follows: First, we use the simple consequence of Keller's chain rule ([4], Theorem 1.90] and give some information about the properties of the solutions. Second, by using the Riccati transformation technique, we establish some oscillation criteria for (1.3). Third, we consider $r(t) = 1$, and follows the technique that has been used in [20] establish some sharp oscillation criteria of Hille [12] and Nehari [14] types for (1.3) and prove that the oscillation condition is given by $\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}$ which as a special case when $\gamma = 1$ becomes the oscillation condition of second order differential equations. The results can be applied on the sublinear and superlinear cases and improve the results established in [22] since our results does not require the condition (h_1). Some examples are considered to illustrate the main results.

2. MAIN RESULTS

In what follows, it will be assumed that

$$\int_{t_0}^{\infty} \tau^{\gamma}(s)\,[p(s) - q(s)]\,\Delta s = \infty, \tag{2.1}$$

is fulfilled and $r^{\Delta}(t) \geq 0$. Before stating our main results, we begin with the following lemma which plays an important rule on the proof of our main oscillation results and gives us some information about the properties of the solutions.

Lemma 2.1. *Assume that* (H_1)–(H_5), (2.1) *hold and* (1.3) *has a positive solution* x *on* $[t_0, \infty)_{\mathbb{T}}$. *Then there exists a* $T \in [t_0, \infty)_{\mathbb{T}}$, *sufficiently large, so that*

(i) $x^{\Delta}(t) > 0$, $x^{\Delta\Delta}(t) < 0$, $x(t) > t x^{\Delta}(t)$ *for* $t \in [T, \infty)_{\mathbb{T}}$;

(ii) x *is strictly increasing and* $x(t)/t$ *is strictly decreasing on* $[T, \infty)_{\mathbb{T}}$.

Proof. Since $x(t)$ is a positive solution of (1.3) on $[t_0, \infty)_{\mathbb{T}}$, we can pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ so that $t_1 > t_0$ and so that $x(t) > 0$ and $x(\tau(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. (Note that in the case when $x(t)$ is negative the proof is similar, since the transformation $y(t) = -x(t)$ transforms the (1.3) into the same form). Since $x(\tau(t)) > 0$, we have from (1.3) and (H_3) that

$$(r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} \leq -(p(t) - q(t))x^{\gamma}(\tau(t)) < 0, \text{ for } t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.2}$$

Then $r(t)\left(x^{\Delta}(t)\right)^{\gamma}$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. We claim that $r(t)\left(x^{\Delta}(t)\right)^{\gamma}>0$ on $[t_1,\infty)_{\mathbb{T}}$. Assume not, then there is a $t_2\in[t_1,\infty)_{\mathbb{T}}$ such that $r(t_2)\left(x^{\Delta}(t_2)\right)^{\gamma}=:$ $c<0$. Then $r(t)\left(x^{\Delta}(t)\right)^{\gamma}\le r(t_2)\left(x^{\Delta}(t_2)\right)^{\gamma}=c$, for $t\in[t_2,\infty)_{\mathbb{T}}$, and therefore

$$x^{\Delta}(t)\le\frac{c}{r^{\frac{1}{\gamma}}(t)},\ \text{ for } t\in[t_2,\infty)_{\mathbb{T}}.$$

Integrating the last inequality form t_2 to t, we find by (H_4) that

$$x(t)=x(t_2)+\int_{t_2}^{t}x^{\Delta}(s)\Delta s\le x(t_2)+c\int_{t_2}^{t}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}\to-\infty\quad\text{ as } t\to\infty,\tag{2.3}$$

which implies that $x(t)$ is eventually negative. This is a contradiction. Hence $r(t)\left(x^{\Delta}(t)\right)^{\gamma}>0$ on $[t_1,\infty)_{\mathbb{T}}$ and so $x^{\Delta}(t)>0$ on $[t_1,\infty)_{\mathbb{T}}$. We now show that $x^{\Delta\Delta}(t)<0$. Since $(r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta}<0$ on $[t_1,\infty)_{\mathbb{T}}$, we have after differentiation that

$$r^{\Delta}(t)\left(x^{\Delta}(t)\right)^{\gamma}+r^{\sigma}\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}<0.\tag{2.4}$$

Using the Keller's chain rule

$$(x^{\gamma}(t))^{\Delta}=\gamma\int_{0}^{1}\left[hx^{\sigma}+(1-h)x\right]^{\gamma-1}dhx^{\Delta}(t),$$

we have

$$\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}=\frac{\gamma x^{\Delta\Delta}(t)}{\left(r^{\sigma}\right)^{\gamma-1}}\int_{0}^{1}\left[hr^{\sigma}(\left(x^{\Delta}(t)\right)^{\sigma}+r^{\sigma}(1-h)\left(x^{\Delta}(t)\right)\right]^{\gamma-1}dh.\tag{2.5}$$

Using the fact that $r(t)\left(x^{\Delta}(t)\right)^{\gamma}$ is nonincreasing and $r^{\sigma}\ge r(t)$ we have

$$x^{\Delta}(t)\ge\left(\frac{r^{\sigma}}{r(t)}\right)^{\frac{1}{\gamma}}\left(x^{\Delta}(t)\right)^{\sigma}>\left(x^{\Delta}(t)\right)^{\sigma}.$$

This and (2.5) imply that

$$\begin{aligned}\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}&\ge\frac{\gamma x^{\Delta\Delta}(t)}{\left(r^{\sigma}\right)^{\gamma-1}}\int_{0}^{1}\left[hr^{\sigma}(\left(x^{\Delta}(t)\right)^{\sigma}+r^{\sigma}(1-h)\left(x^{\Delta}(t)\right)^{\sigma}\right]^{\gamma-1}dh\\&=\gamma(\left(x^{\Delta}(t)\right)^{\sigma})^{\gamma-1}x^{\Delta\Delta}(t)\end{aligned}\tag{2.6}$$

since $r^{\Delta}(t)\ge0$. From the last inequality and (2.4), we have

$$r^{\Delta}(t)\left(x^{\Delta}(t)\right)^{\gamma}+\gamma(\left(x^{\Delta}(t)\right)^{\sigma})^{\gamma-1}x^{\Delta\Delta}(t)<0,\tag{2.7}$$

which implies that $x^{\Delta\Delta}(t)<-\frac{r^{\Delta}(t)\left(x^{\Delta}(t)\right)^{\gamma}}{\gamma((x^{\Delta}(t))^{\sigma})^{\gamma-1}}$, for $t\in[t_1,\infty)_{\mathbb{T}}$. Then $x^{\Delta\Delta}(t)<0$ since $r^{\Delta}(t)\ge0$ and $x^{\Delta}(t)>0$. Next, we show that $x(t)/t$ is strictly decreasing. To do this, let $U(t):=x(t)-tx^{\Delta}(t)$, so that $U^{\Delta}(t)=-\sigma(t)x^{\Delta\Delta}(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. This implies that $U(t)$ is strictly increasing on $[t_1,\infty)_{\mathbb{T}}$. We claim that there is a

$t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $U(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Assume not, then $U(t) < 0$ on $[t_1,\infty)_{\mathbb{T}}$. Therefore,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} > 0, \quad t \in [t_2,\infty)_{\mathbb{T}}, \tag{2.8}$$

which implies that $x(t)/t$ is strictly increasing on $[t_2,\infty)_{\mathbb{T}}$. Pick $t_3 \in [t_2,\infty)_{\mathbb{T}}$ so that $\tau(t) \geq \tau(t_2)$, for $t \geq t_3$. Then $x(\tau(t))/\tau(t) \geq x(\tau(t_2))/\tau(t_2) =: d > 0$, so that $x(\tau(t)) \geq d\tau(t)$ for $t \geq t_3$. Now by integrating both sides of (2.2) from t_3 to t, we have

$$r(t)\left(x^{\Delta}(t)\right)^{\gamma} - r(t_3)\left(x^{\Delta}(t_3)\right)^{\gamma} + \int_{t_3}^{t} [p(s) - q(s)]\, x^{\gamma}(\tau(s))\Delta s \leq 0.$$

This implies that

$$r(t_3)\left(x^{\Delta}(t_3)\right)^{\gamma} \geq \int_{t_3}^{t} [p(s) - q(s)]\, x^{\gamma}(\tau(s))\Delta s \geq d^{\gamma}\int_{t_3}^{t} [p(s) - q(s)]\, \tau^{\gamma}(s)\Delta s, \tag{2.9}$$

which contradicts (2.1). Hence there is a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $U(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Consequently,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} < 0, \quad t \in [t_2,\infty)_{\mathbb{T}},$$

and we have that $\frac{x(t)}{t}$ is strictly decreasing on $[t_2,\infty)_{\mathbb{T}}$. The proof is complete.

In following, we obtain new oscillation criteria which can be considered as the extension of Kamenev-type oscillation criteria. First, we define $\Re$ by $H \in \Re$ provided $H : [t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}$ satisfies

$$H(t,t) \geq 0, \quad t \geq t_0, \quad H(t,s) > 0, \quad t > s \geq t_0,$$

$H^{\Delta_s}(t,s) \leq 0$, for $t \geq s \geq t_0$, and for each fixed t, $H(t,s)$ is right-dense continuous with respect to s. As a simple and important example, note that if $\mathbb{T} = R$, then $H(t,s) := (t-s)^n$ is in $\Re$.

Theorem 2.1. *Assume that* (H_1)–(H_5) *and* (2.1) *hold. Furthermore assume that there exists a positive* rd*-continuous* Δ*-differentiable function* $\alpha(t)$ *and let* $H : D \to \mathbb{R}$ *be* rd*-continuous function belonging to the class* $\Re$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} K(t,s)\,\Delta s = \infty, \tag{2.10}$$

where

$$K(t,s) = H(t,s)\alpha(s)P(s) - \frac{(\alpha^{\sigma})^{\gamma+1} r(s)\left[H(t,s)\frac{\alpha^{\Delta}(s)}{\alpha^{\sigma}} - H^{\Delta_s}(t,s)\right]^{\gamma+1}}{\alpha^{\gamma}(s)(\gamma+1)^{\gamma+1}H^{\gamma}(t,s)}. \tag{2.11}$$

Then every solution of (1.3) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.3), and let $t_1 \geq t_0$ be such that $x(t) \neq 0$ for all $t \geq t_1$. Without loss of generality, we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_1$. Therefore from Lemma 2.1, we have

$$x(t) > 0,\ x^{\Delta}(t) \geq 0,\ (r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} < 0,\ t \geq t_1. \tag{2.12}$$

Define $w(t)$ by

$$w(t) := \alpha(t)\frac{r(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(t)},\ \text{for } t \geq t_1. \tag{2.13}$$

Then $w(t) > 0$, and satisfies

$$w^{\Delta}(t) = \frac{x^{\gamma}(\tau(t))\alpha(t)}{x^{\gamma}(t)}\frac{\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}}{x^{\gamma}(\tau(t))} + \left(r\left(x^{\Delta}\right)^{\gamma}\right)^{\sigma}\left[\frac{x^{\gamma}(t)\alpha^{\Delta}(t) - \alpha(t)(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\right]. \tag{2.14}$$

In view of Lemma 2.1, since $x(t)/t$ is strictly decreasing, we see that $\frac{x^{\gamma}(\tau(t))}{x^{\gamma}(t)} \geq \frac{\tau^{\gamma}(t))}{t^{\gamma}}$. This, (2.2) and (2.14) imply that

$$w^{\Delta}(t) \leq -\alpha(t)P(t) + \frac{\alpha^{\Delta}(t)}{\alpha^{\sigma}}w^{\sigma} - \frac{\alpha(t)\left(r\left(x^{\Delta}\right)^{\gamma}\right)^{\sigma}(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}. \tag{2.15}$$

where $P(t) = \left(\frac{\tau(s)}{s}\right)^{\gamma}[p(s) - q(s)]$. Using the Keller's chain rule and the fact that $x^{\sigma} \geq x(t)$, we obtain

$$(x^{\gamma}(t))^{\Delta} \geq \gamma\int_0^1 [hx + (1-h)x]^{\gamma-1}\,dhx^{\Delta}(t) = \gamma(x(t))^{\gamma-1}x^{\Delta}(t), \tag{2.16}$$

which implies that

$$(x^{\gamma}(t))^{\Delta} \geq \gamma x^{\gamma-1}(t)\left(x^{\Delta}(t)\right). \tag{2.17}$$

From (2.12) since $\left(r(t)\left(x^{\gamma}(t)\right)^{\Delta}\right)^{\Delta} < 0$, we have for $t \geq t_1$ that

$$r(t)\left(x^{\Delta}(t)\right)^{\gamma} \geq \left(r\left(x^{\Delta}\right)^{\gamma}\right)^{\sigma}. \tag{2.18}$$

It follows from (2.15), (2.17) and (2.18) that

$$w^{\Delta}(t) \leq -\alpha(t)P(t) + \frac{\alpha^{\Delta}(t)}{\alpha^{\sigma}}w^{\sigma} - \frac{\gamma\alpha(t)\left(r\left(x^{\Delta}\right)^{\gamma+1}\right)^{\sigma}}{r^{\frac{1}{\gamma}}(t)x^{\gamma+1}(\sigma(t))}, \tag{2.19}$$

since $x(t)$ is nondecreasing. From (2.13) and (2.19), we obtain

$$w^{\Delta}(t) \leq -\alpha(t)P(t) + \frac{\alpha^{\Delta}(t)}{\alpha^{\sigma}}w^{\sigma} - \frac{\gamma\alpha(t)}{r^{\frac{1}{\gamma}}(t)(\alpha^{\sigma})^{\frac{\gamma+1}{\gamma}}}\left(w^{\sigma}\right)^{\frac{\gamma+1}{\gamma}}. \tag{2.20}$$

The remainder of the proof is similar to that of the proof of Theorem 2.2 in [2] by using the inequality (2.20) and hence is omitted.

As an immediate consequence of Theorem 2.1 we can establish some different sufficient conditions for oscillation by choosing different form of the function $H(t,s)$. For example by using $H(t,s) := 1$, we have the following oscillation criteria.

Corollary 2.1. *Assume that* (H_1)–(H_5) *and* (2.1) *hold. Let* $\alpha(t)$ *be as defined in Theorem* 2.1 *and*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\alpha(s)\left(\frac{\tau(s)}{s}\right)^{\gamma}[p(s)-q(s)]-\frac{r(s)(\alpha^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\alpha^{\gamma}(s)}\right]\Delta s=\infty, \tag{2.21}$$

then every solution of (1.3) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

From Corollary 2.1, we can obtain different conditions for oscillation of all solutions of (1.3) by different choices of $\alpha(t)$. For instance, if $\alpha(t)=t$ for $t\geq t_0$, we have the following result.

Corollary 2.2. *Assume that* (H_1)–(H_5) *and* (2.1) *hold. Furthermore, assume that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[s\left(\frac{\tau(s)}{s}\right)^{\gamma}[p(s)-q(s)]-\frac{r(s)}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\Delta s=\infty, \tag{2.22}$$

then, every solution of (1.3) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

To illustrate the above results we consider the following example.

Example 2.1. Consider the following second-order perturbed delay dynamic equation

$$(t^{\gamma-1}\left(x^{\Delta}(t)\right)^{\gamma})+\left(\frac{\lambda t^{\gamma-2}}{\tau^{\gamma}(t)}+t^2x^2(\tau(t))\right)x^{\gamma}(\tau(t))=\frac{\beta t^{\gamma-2}x^{\gamma+2}(\tau(t))}{\tau^{\gamma}(t)(1+x^2(\tau(t)))(1+(x^{\Delta}(t))^2)}, \tag{2.23}$$

for $t\in[t_0,\infty)_{\mathbb{T}}=[1,\infty)\cap\mathbb{T}$ where $\mathbb{T}$ is a time scale, $\gamma>1$, α and β are positive constants, $r(t)=t^{\gamma-1}$ and $\tau(t)\leq t$ and $\lim_{t\to\infty}\tau(t)=\infty$. We consider

$$p(t):=\frac{\lambda t^{\gamma-2}}{\tau^{\gamma}(t)},\ \text{and}\ q(t):=\frac{\beta t^{\gamma-2}}{\tau^{\gamma}(t)}. \tag{2.24}$$

It is clear that the conditions (H_1)–(H_5) and (2.1) hold. To apply Corollary 2.2, it remains to prove that the condition (2.22) holds. For (2.23), the condition (2.22) reads

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^{t}\left[s[p(s)-q(s)s]\left(\frac{\tau(s)}{s}\right)^{\gamma}-\frac{r(s)}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\Delta s\\&=\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{(\lambda-\beta)(\gamma+1)^{\gamma+1}-1}{s}\right]\Delta s=\infty,\end{aligned} \tag{2.25}$$

if $(\lambda-\beta)>\frac{1}{(\gamma+1)^{\gamma+1}}$. Then by Corollary 2.2, every solution of (2.23) is oscillatory when

$$(\lambda-\beta)>\frac{1}{(\gamma+1)^{\gamma+1}}. \tag{2.26}$$

Remark 1. Note that the results that has been established in [3, 6, 22] cannot be applied on (2.23).

In the following, we consider (1.3), when $r(t) = 1$, i.e., we consider the perturbed delay equation

$$((x^{\Delta})^{\gamma})^{\Delta} + F(t, x(\tau(t))) = G(t, x(\tau(t)), x^{\Delta}(t)), \tag{2.27}$$

and establish some oscillation criteria of Hille [12] and Nehari [14] types. Note that when $r(t) = 1$, the condition (H_4) already satisfied, and then the conclusions of Lemma 2.1 still true for (2.27). We introduce the following notations

$$p_* := \liminf_{t\to\infty} t^{\gamma} \int_{\sigma(t)}^{\infty} P(s)\Delta s \quad \text{and} \quad q_* := \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} s^{\gamma+1}P(s)\Delta s, \tag{2.28}$$

where $P(s) = [p(s) - q(s)]\left(\frac{\tau(s)}{s}\right)^{\gamma}$.

Theorem 2.2. *Assume that (H_1)–(H_3), (H_5) and (2.1) hold. Let $x(t)$ be a nonoscillatory solution of (2.27) such that $x(t)$ and $x(\tau(t)) > 0$ for $t \geq t_1 > t_0$. Let $u(t) = \left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma}$ and*

$$r := \liminf_{t\to\infty} t^{\gamma}u^{\sigma}, \quad \textit{and} \quad R := \limsup_{t\to\infty} t^{\gamma}u^{\sigma}. \tag{2.29}$$

Then

$$p_* \leq r - l^{\gamma}r^{1+\frac{1}{\gamma}} \quad \textit{and} \quad p_* + q_* \leq \frac{1}{l^{\gamma(\gamma+1)}}. \tag{2.30}$$

where $l := \liminf_{t\to\infty} \frac{t}{\sigma(t)} \leq 1$.

Proof. From the definition of $u(t)$ and using Lemma 2.1, we see that $u(t)$ is positive. Follows the proof of Theorem 2.1, we get

$$u^{\Delta}(t) \leq -\left(\frac{\tau(t)}{t}\right)^{\gamma}[p(t) - q(t)] - \frac{((x^{\Delta})^{\gamma})^{\sigma}(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)\,(x^{\gamma})^{\sigma}}. \tag{2.31}$$

Using the chain rule, and the fact that $x^{\Delta}(t) > 0$, we see that $(x^{\gamma}(t))^{\Delta} \geq \gamma(x(t))^{\gamma-1}x^{\Delta}(t)$. Using this in the inequality (2.34), we get

$$\begin{aligned} u^{\Delta}(t) &\leq -\left(\frac{\tau(t)}{t}\right)^{\gamma}[p(t) - q(t)] - \frac{((x^{\Delta})^{\gamma})^{\sigma}\,\gamma(x(t))^{\gamma-1}x^{\Delta}(t)}{x^{\gamma}(t)(x^{\gamma})^{\sigma}} \\ &= -\left(\frac{\tau(t)}{t}\right)^{\gamma}p(t) - \gamma u^{\sigma}u^{\frac{1}{\gamma}}(t). \end{aligned} \tag{2.32}$$

Thus $u(t)$ satisfies the dynamic inequality

$$u^{\Delta}(t) + P(t) + \gamma u^{\sigma}u^{\frac{1}{\gamma}}(t) \leq 0, \quad \text{for} \ \ t \geq t_2. \tag{2.33}$$

Since $P(t) > 0$ and $u(t) > 0$ for $t \geq t_2$, we have from (2.35) that $u^{\Delta}(t) < 0$ and

$$-\left(u^{\Delta}(t)/\gamma u^{\sigma}u^{\frac{1}{\gamma}}(t)\right) > 1, \quad \text{for} \ \ t \geq t_2. \tag{2.34}$$

Since $\left(1/u^{\frac{1}{\gamma}}(t)\right)^{\Delta} = \frac{-1}{\gamma}\left(u^{\frac{1}{\gamma}-1}(\eta)u^{\Delta}(t)/u^{\frac{1}{\gamma}}(t)(u^{\sigma})^{\frac{1}{\gamma}}\right)$, where $\eta \in [t, \sigma(t)]$, we have

$$\left(1/u^{\frac{1}{\gamma}}(t)\right)^{\Delta} \geq \frac{-1}{\gamma}\left((u^{\sigma})^{\frac{1}{\gamma}-1}u^{\Delta}(t)/u^{\frac{1}{\gamma}}(t)(u^{\sigma})^{\frac{1}{\gamma}}\right).$$

This and (2.34) imply (since $u(t)$ is nonincreasing), that

$$\left(1/u^{\frac{1}{\gamma}}(t)\right)^{\Delta} > 1. \tag{2.35}$$

Integrating (2.35) from t_2 to t, we obtain

$$u(t) < \frac{1}{(t-t_2)^{\gamma}}, \qquad t \geq t_2, \tag{2.36}$$

which implies that $\lim_{t\to\infty} u(t) = 0$. By (2.29) and (2.36), we have

$$0 < r \leq 1, \quad \text{and} \quad 0 < R \leq 1.$$

First, we prove that (2.30) holds. Integrating (2.33) from $\sigma(t)$ to ∞ ($\sigma(t) \geq t_2$) and using $\lim_{t\to\infty} u(t) = 0$, we have

$$u^{\sigma} \geq \int_{\sigma(t)}^{\infty} P(s)\Delta s + \gamma \int_{\sigma(t)}^{\infty} u^{\frac{1}{\gamma}}(s) u^{\sigma} \Delta s \quad \text{for } t \geq t_2.$$

The remainder of the proof is similar to the proof of Theorem 2.1 in [20] and hence is omitted.

Theorem 2.3. *Assume that* (H_1)–(H_3), (H_5) *and* (2.1) *hold. If*

$$p_* = \liminf_{t\to\infty} t^{\gamma} \int_{\sigma(t)}^{\infty} [p(s) - q(s)] \left(\frac{\tau(s)}{s}\right)^{\gamma} \Delta s > \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{1}{l^{\gamma^2}}, \tag{2.37}$$

then every solution of (2.27) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. Let $x(t)$ be a nonoscillatory solution of (2.37) such that $x(t)$ and $x(\tau(t)) > 0$ for $t \geq t_1 > t_0$. Let $u(t)$ be as defined in Theorem 2.2 and let $r = \liminf_{t\to\infty} t^{\gamma} u^{\sigma}$. Then from Theorem 2.2, we see that p_* and r satisfy the inequality $p_* \leq r - l^{\gamma} r^{1+\frac{1}{\gamma}}$. Using the fact that $Bu - Au^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^{\gamma}}$, for $A > 0$, we see that

$$p_* \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{1}{l^{\gamma^2}},$$

which contradicts (2.37). Thus every solution of (2.27) oscillates. The proof is complete.

Theorem 2.4. *Assume that* (H_1)–(H_4) *and* (2.1) *hold. If*

$$p_* + q_* > \frac{1}{l^{\gamma(\gamma+1)}},$$

then (2.7) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

As a consequence of Theorem 2.3, we have the following result.

Corollary 2.3. *Assume that* (H_1)–(H_3), (H_5) *and* (2.1) *hold. If*

$$p_* > \frac{1}{l^{\gamma(\gamma+1)}}, \ \text{or } q_* > \frac{1}{l^{\gamma(\gamma+1)}}, \tag{2.38}$$

then (2.27) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Remark 2. It is clear that the inequality (2.35) is similar to the inequality (2.10) in Theorem 2.1 in [20]. So as a consequence of Theorem 2.2 and as proved in Theorem 2.2 and Theorem 2.3 in [20], we can prove the similar results for equation (2.27) in the case when $0 < \gamma < 1$. So the results in Theorem 2.3 and Theorem 2.4 can be applied in the case when $0 < \gamma < 1$ as well as in the case when $\gamma > 1$.

Example 2.2. Consider the delay perturbed dynamic equation

$$\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \left(\frac{\alpha}{t\tau^{\gamma}(t)} + t^2 x^2(\tau(t))\right) x^{\gamma}(\tau(t)) = \frac{\beta x^{\gamma+2}(\tau(t))}{t\tau^{\gamma}(t)(1+x^2(\tau(t)))(1+(x^{\Delta}(t))^2)}, \tag{2.39}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$. We consider $p(t) = \frac{\alpha}{t\tau^{\gamma}(t)}$ and $q(t) = \frac{\beta}{t\tau^{\gamma}(t)}$ such that $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$, $\gamma \geq 1$ is a ratio of odd positive integers and α and β are positive constants such that $(\alpha - \beta) > 0$. It is clear that the conditions (H_1)–(H_3), (H_5) and (2.1) hold. To apply Corollary 2.4, it remains to prove that (2.38) holds. For (2.39), the condition (2.38) reads

$$\begin{aligned} q_* &= \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} s^{\gamma+1} P(s)\Delta s = \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} s^{\gamma+1} \left(\frac{\tau(s)}{s}\right)^{\gamma} [p(s) - q(s)]\Delta s \\ &= (\alpha - \beta) \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \Delta s = (\alpha - \beta)\,. \end{aligned}$$

Then by Corollary 2.4, every solution of (2.39) is oscillatory if $(\alpha - \beta) > \frac{1}{l^{\gamma(\gamma+1)}}$.

Remark 3. Note the results that has been established in [3, 6, 22] cannot be applied on (2.39).

Acknowledgement. The author is very grateful to the anonymous referee for valuable remarks and comments which significantly contributed to the quality of the paper.

REFERENCES

[1] R. Agarwal, M. Bohner and S. H. Saker, Oscillation criteria for second order delay dynamic equation, Canadian Applied Mathematics Quarterly 13 (2005), 1–17.

[2] R. P. Agarwal, D. O'Regan and S. H. Saker, Oscillation results for second-order nonlinear neutral delay dynamic equations on time scales, Appl. Analysis 86 (2007), 1–17.

[3] R. P. Agarwal, D. O'Regan and S. H. Saker, Oscillation criteria for nonlinear perturbed dynamic equations of second-order on time scales, J Appl. Math. Computing. 20 (2006), 133–147.

[4] M. Bohner and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.

[5] M. Bohner and A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.

[6] M. Bohner and S. H. Saker, Oscillation criteria for perturbed nonlinear dynamic equations, Mathl. Comp. Modeling 40 (2004), 3-4, 249–260.

[7] L. Erbe, Oscillation criteria for second order linear equations on a time scale, Canadian Appl. Math. Quart. 9 (2001), 1–31.

[8] L. Erbe, A. Peterson and S. H. Saker, Oscillation criteria for second-order nonlinear dynamic equations on time scales, J. London Math. Soc. 76 (2003), 701–714.

[9] L. Erbe, A. Peterson and S. H. Saker, Kamenev-type oscillation criteria for second-order linear delay dynamic equations, Dynamic Syst. & Appl. 15 (2006), 65–78.

[10] L. Erbe, A. Peterson and S. H. Saker, Asymptotic behavior of solutions of a third-order nonlinear dynamic equation on time scales, J. Comp. Appl. Math. 181, No. 1 (2005), 92–102.

[11] S. Hilger, Analysis on measure chains — a unified approach to continuous and discrete calculus, Results Math. 18 (1990), 18–56.

[12] E. Hille, Non-oscillation theorems, Trans. Amer. Math. Soc. 64 (1948), 234–252.

[13] V. Kac and P. Cheung, Quantum Calculus, Springer, New York, 2001.

[14] Z. Nehari, Oscillation criteria for second-order linear differential equations, Trans. Amer. Math. Soc. 85 (1957), 428–445.

[15] S. H. Saker, Boundedness of solutions of second-order forced nonlinear dynamic equations, Rocky Mountain. J. Math. 36 (2006), 2027–2039.

[16] S. H. Saker, Oscillation criteria of second-order half-linear dynamic equations on time scales, J. Comp. Appl. Math. 177(2005), 375–387.

[17] S. H. Saker, Oscillation criteria for a certain class of second-order neutral delay dynamic equations, Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms (accepted).

[18] S. H. Saker, Oscillation of second-order forced nonlinear dynamic equations on time scales, EJQTDE. 23 (2005), 1–17.

[19] S. H. Saker, New oscillation criteria for second-order nonlinear dynamic equations on time scales, Nonlinear Fun. Anal. Appl. 11 (2006), 351–370.

[20] S. H. Saker, Oscillation of superlinear and sublinear neutral delay dynamic equations, Comm. Appl. Anal. (accepted).

[21] V. Spedding, Taming Nature's Numbers, New Scientist, July 19, 2003, 28–31.

[22] H. R. Sun and W. Tong Li, Oscillation of perturbed nonlinear dynamic equations on time scales, ZAMM, Z. Angew. Math. Mech. 85 (2005), 755–760.