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DUAL SOLUTIONS FOR AXISYMMETRIC STAGNATION POINT FLOW OVER A LUBRICATED SURFACE

YOSHITAKA ISHIHARA AND JOSEPH E. PAULLET

School of Science, Pennsylvania State University at Erie
Erie, PA 16563 USA
yvi5003@psu.edu and jep7@psu.edu

ABSTRACT. In this note we consider axisymmetric stagnation point flow of one fluid impinging on a disk covered with a second fluid. A similarity reduction is employed to reduce the governing PDEs to a nonlinear ODE boundary value problem. Previous numerical investigations of the problem in the literature indicate the existence of one solution. Here we prove the existence of at least two solutions to the BVP. We also obtain results concerning the possibility of further solutions and present numerical approximations to the solutions.

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1. INTRODUCTION

The flow of one fluid impinging on a impermeable surface covered with a second fluid has many industrial applications. Examples include the cooling of gas turbine blades [1] and cooling in grinding processes [2]. Others include surface cleaning [3] paper and filament manufacturing, and various coating and finishing processes [4].

Such applications can involve many different physical configurations. Santra *et. al.* [5] consider one such configuration, that of axi-symmetric stagnation point flow of a Newtonian fluid over a flat disk covered by a thin layer of a non-Newtonian fluid. After impinging orthogonally on the lubricated surface, the fluid then flows radially away from the stagnation point. The center of the disk is taken as the origin of a cylindrical coordinate system (r, θ, z).

In the fully developed flow, the velocities in the radial, $u(r, z)$, and axial, $w(r, z)$, directions satisfy

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{1.1}$$

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2}\right], \tag{1.2}$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right], \tag{1.3}$$

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where p is pressure, ρ is density and ν is the kinematic viscosity of the Newtonian fluid. Santra *et. al.* [5] consider a similarity solution of the form

$$\eta = z\sqrt{\frac{A}{\nu}}, \tag{1.4}$$

$$u = Arf(\eta), \quad w = \sqrt{A\nu}g(\eta), \quad p = A\mu p^*(\eta) - \rho A^2 r^2/2, \tag{1.5}$$

where A is a positive constant indicating the strength of the stagnation flow and μ is viscosity. See [5] for a full explanation and derivation of the model.

This results in the following ODE boundary value problem:

$$g' = -2f, \tag{1.6}$$

$$f'' = f^2 + gf' - 1, \tag{1.7}$$

$$p^{*\prime} = 2fg - 2f', \tag{1.8}$$

subject to

$$g(0) = 0, \quad f'(0) = \lambda[f(0)]^{2/3}, \quad p^*(0) = 0, \tag{1.9}$$

$$f(\infty) = 1. \tag{1.10}$$

As Santra *et. al.* [5] explain, the slip coefficient λ can be interpreted as the ratio of the viscous length scale to the lubrication length. If the lubrication length is small, λ becomes large, recovering the no-slip condition $f(0) = 0$ as $\lambda \to \infty$. Conversely, as the viscous length scale becomes infinitely large, the slip coefficient λ vanishes and the full slip condition, $f'(0) =$ is achieved. Thus, λ can be interpreted as an inverse measure of slip.

In the next section we prove, for each $\lambda > 0$, the existence of at least two solutions to this BVP. In section 3 we discuss the possibility of further solutions. In the last section we present numerical approximations to the solutions and discuss open questions.

2. EXISTENCE OF SOLUTIONS

Note that equations (1.6–1.7) are uncoupled from equation (1.8) and can thus be solved in isolation. Equation (1.8) can then be solved by quadrature. As noted in [5], by setting $g = -2F$ the boundary value problem can be recast as

$$F''' + 2FF'' + 1 - F'^2 = 0, \quad 0 < \eta < \infty, \tag{2.1}$$

subject to

$$F(0) = 0, \quad F''(0) = \lambda[F'(0)]^{2/3}, \quad F'(\infty) = 1. \tag{2.2}$$

To prove existence of solutions to this BVP we will investigate a related IVP, namely equation (2.1) subject to

$$F(0) = 0 \tag{2.3}$$

$$F'(0) = \alpha \tag{2.4}$$

$$F''(0) = \lambda[\alpha]^{2/3} \tag{2.5}$$

where α is a free parameter. Denote the solution of this IVP by $F(\eta; \alpha)$. By basic existence and uniqueness theory, this IVP will have a unique solution for all values of α, at least on some initial interval containing $\eta = 0$. We will show that for each $\lambda > 0$ there are at least two values of α for which the solution to this IVP will exist for all $\eta > 0$ and satisfy the condition $F'(\infty) = 1$, giving two solutions to the BVP.

To begin, consider the following subsets of the interval $(0, 1)$:

$$\mathcal{A} = \{\alpha \in (0,1) \mid F'(\eta;\alpha) = 1 \text{ strictly before } F''(\eta;\alpha) = 0\}$$

and

$$\mathcal{B} = \{\alpha \in (0,1) \mid F''(\eta;\alpha) = 0 \text{ strictly before } F'(\eta;\alpha) = 1\}.$$

Lemma 1. The set $\mathcal{A}$ is non-empty and open.

Proof. When $\alpha = 1$, $F(0) = 0$, $F'(0) = 1$, and $F''(0) = \lambda > 0$. Thus there exists an $\varepsilon_0 > 0$ such that $F' > 1$ and $F'' > 0$ on $(0, \varepsilon_0]$. By continuity of solutions of the IVP with respect to its initial conditions, for $\alpha \in (0, 1)$ sufficiently close to 1 we can arrange that $F'' > 0$ on $(0, \varepsilon_0]$ with $F'(\varepsilon_0) > 1$. But $F'(0) = \alpha < 1$. Thus there exists a first η_0 such that $F' = 1$ strictly before $F'' = 0$. Thus $\mathcal{A}$ is non-empty. Also note that whenever $F' = 1$, we must have $F'' \neq 0$ for if F'' were to equal zero then the ODE (2.1) implies that $F''' = 0$. But this would imply that $F'(\eta) \equiv 1$, which is a contradiction since $F'(0) = \alpha < 1$. Thus when $F' = 1$, $F'' \neq 0$ and $\mathcal{A}$ is open. □

Lemma 2. The set $\mathcal{B}$ is non-empty and open.

Proof. When $\alpha = 0$, $F(0) = 0$, $F'(0) = 0$, $F''(0) = 0$ and $F'''(0) = -1 < 0$. Thus there exists $\varepsilon_1 > 0$ such that $F' < 1$ and $F'' < 0$ on $(0, \varepsilon_1]$. By continuity of the solutions of the IVP in its initial conditions, for $\alpha \in (0, 1)$ sufficiently close to 0 we can arrange that $F' < 1$ on $(0, \varepsilon_1]$ with $F''(\varepsilon_1) < 0$. But $F''(0) = \lambda\alpha^{2/3} > 0$. Thus there exists a first η_1 such that $F''(\eta_1) = 0$ with $F' < 1$ on $(0, \eta_1]$. Thus for $\alpha > 0$ sufficiently small we have $F'' = 0$ strictly before $F' = 1$ and $\mathcal{B}$ is non-empty. As before we cannot have $F' = 1$ and $F'' = 0$ simultaneously and $\mathcal{B}$ is also open. □

The interval $(0, 1)$ is connected, thus $\mathcal{A} \cup \mathcal{B} \neq (0, 1)$. Thus there must exist at least one value α_1 such that $\alpha_1 \notin \mathcal{A}$ and $\alpha_1 \notin \mathcal{B}$. For such a value of α we cannot have $F' = 1$ before $F'' = 0$ nor can we have $F'' = 0$ before $F' = 1$. As we have already seen, we cannot have $F' = 1$ and $F'' = 0$ simultaneously. The only possibility left is $F'(\eta; \alpha_1) < 1$ and $F''(\eta; \alpha_1) > 0$ for all $\eta > 0$. From the ODE (2.1) we see that we must have $F'(\infty; \alpha_1) = 1$, giving a solution to the BVP.

Next consider the following subsets of the interval $(-1, 0)$:

$$\mathcal{C} = \{\alpha \in (-1,0) \mid F''(\eta;\alpha) = 0 \text{ strictly before } F'(\eta;\alpha) = 1\}$$

and

$$\mathcal{D} = \{\alpha \in (-1,0) \mid F'(\eta;\alpha) = 1 \text{ strictly before } F''(\eta;\alpha) = 0\}\,.$$

Lemma 3. The set $\mathcal{C}$ is non-empty and open.

Proof. The proof is essentially the same as that of **Lemma 2.** □

Lemma 4. The set $\mathcal{D}$ is non-empty and open.

Proof. Note that if $\alpha = -1$, the solution to the IVP is $F(\eta) = \lambda\eta^2/2 - \eta$. Thus $F' = 1$ when $\eta = 2/\lambda$ and $F''(\eta) \equiv \lambda > 0$ on the interval $[0, 2/\lambda]$. (i.e. $F' = 1$ strictly before $F'' = 0$.) Consider the interval $(0, \frac{2}{\lambda} + 1)$. By continuity of solutions of the IVP in its initial conditions, for all $\alpha \in (-1,0)$ sufficiently close to -1 there will exist an $\eta_2 \in (0, \frac{2}{\lambda} + 1)$ such that $F'(\eta_2;\alpha) = 1$ with $F''(\eta;\alpha) > 0$ for $\eta \in [0, \eta_2]$. Thus $\mathcal{D}$ is non-empty and can be shown to be open as in the other Lemmas. □

Reasoning as before we can conclude that there exists at least one value $\alpha_2 \in (-1,0)$ giving a solution to the boundary value problem (2.1-2.2). The results of this section establish the following theorem.

Theorem. For any $\lambda > 0$, there exists $\alpha_1 \in (0,1)$ such that $F'(\eta;\alpha_1)$ is a solution to the BVP (2.1-2.2). This solution is monotonically increasing and satisfies $\alpha_1 < F'(\eta;\alpha_1) < 1$ for all $\eta > 0$. There also exists an $\alpha_2 \in (-1,0)$ such that $F'(\eta;\alpha_2)$ is a solution to the BVP (2.1-2.2). This solution is monotonically increasing and satisfies $\alpha_2 < F'(\eta;\alpha_2) < 1$ for all $\eta > 0$.

3. FURTHER SOLUTIONS

In this section we show that there is precisely one value $\alpha_1 \in (0,1)$ which gives a solution to the BVP. Further, no values of α outside of the intervals (0,1) and $(-1,0)$ give a solution to the BVP.

Differentiating the ODE (2.1) results in

$$F^{(iv)} + 2FF''' = 0, \tag{3.1}$$

which can be integrated to give

$$F'''(\eta) = (\alpha^2 - 1)\exp\left(-2\int_0^\eta F(t)\,dt\right) \tag{3.2}$$

Thus F''' is of one sign. If $\alpha > 1$, then $F'(0) > 1$, $F''(0) > 0$ and for all $\eta > 0$, $F'''(\eta) > 0$. Thus the boundary condition $F'(\infty) = 1$ cannot be satisfied. If $\alpha < -1$, then $F'(0) < -1$ and again $F'''(\eta) > 0$ for all $\eta > 0$. Thus F' starts out below 1 and is always concave up and therefore cannot satisfy $F'(\infty) = 1$.

As we have already seen, if $\alpha = -1$, $F(\eta) = \lambda\eta^2/2 - \eta$ which does not satisfy $F'(\infty) = 1$. If $\alpha = 1$, then the solution to the IVP is $F(\eta) = \lambda\eta^2/2 + \eta$, which does not satisfy $F'(\infty) = 1$. Finally, if $\alpha = 0$, then $F'(0) = 0$, $F''(0) = 0$ and $F'''(\eta) < 0$ for all $\eta > 0$ and again the boundary condition at infinity cannot be satisfied.

Next suppose that there are two values $\alpha_1, \alpha_3 \in (0,1)$ such that $F_1(\eta;\alpha_1)$ and $F_3(\eta;\alpha_3)$ are both solutions to the BVP. Without loss of generality assume $\alpha_3 > \alpha_1$. Let $u = F_3 - F_1$. Then using (3.1) and adding and subtracting in the term $2F_3F_1'''$, u satisfies

$$u^{(iv)} + 2F_3u''' + 2F_1'''u = 0, \tag{3.3}$$

subject to

$$u(0) = 0,$$
$$u'(0) = \alpha_3 - \alpha_1 > 0,$$
$$u''(0) = \lambda(\alpha_3^{2/3} - \alpha_1^{2/3}) > 0,$$
$$u'''(0) = \alpha_3^2 - \alpha_1^2 > 0,$$

and at infinity

$$u'(\infty) = 0.$$

Since u' initially starts off positive, increasing and concave up, in order to satisfy the condition at infinity there must be a first change in concavity where $u''' = 0$ and $u^{(iv)} \leq 0$. Note that u will be positive at this first change in concavity and recall that $F_1'''(\eta) < 0$ from (3.2). Using this information in (3.3) implies that at the first change in concavity

$$u^{(iv)} = -2F_1'''u > 0,$$

contradicting the fact that $u^{(iv)} \leq 0$ at such a point. Thus the value $\alpha_1 \in (0,1)$ which gives a solution to the BVP is unique (in the interval $(0,1)$).

4. NUMERICAL RESULTS AND OPEN QUESTIONS

In this section we numerically integrate the IVP (2.1, 2.3-2.5) to approximate the solution to the BVP. In all calculations a fourth order Runge-Kutta scheme with a step size of .005 is employed on a interval of η of length 20 until an accuracy of 10^{-6} is achieved for the right boundary condition. Figure 1 shows the dual solutions for $\lambda = 2$. Figure 2 shows a plot of the values of α_1 (top curve) and α_2 (bottom curve) as a function of λ. For initial conditions on the bottom curve, there is an region of flow reversal where $F' < 0$ making such solutions less likely physically.

The argument used to show that there cannot be two values of α in the interval (0,1) that give solutions to the BVP cannot be extended to the interval $(-1,0)$. Thus it remains an open question whether there exist more than one value of α in the interval $(-1,0)$ which give solutions to the BVP. Our numerical investigations indicate that there are no further values in the interval $(-1,0)$.

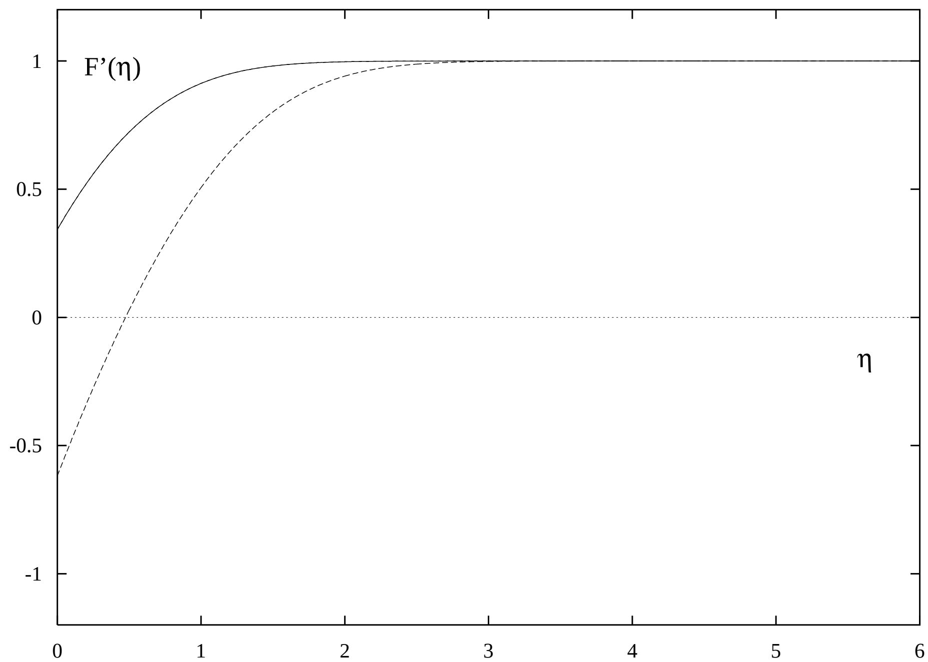

FIGURE 1. Dual solutions to the BVP (2.1-2.2) for $\lambda = 2$. $F'(0) = \alpha_1 = 0.34210655$ (top curve), $F'(0) = \alpha_2 = -0.61896015$ (bottom curve).

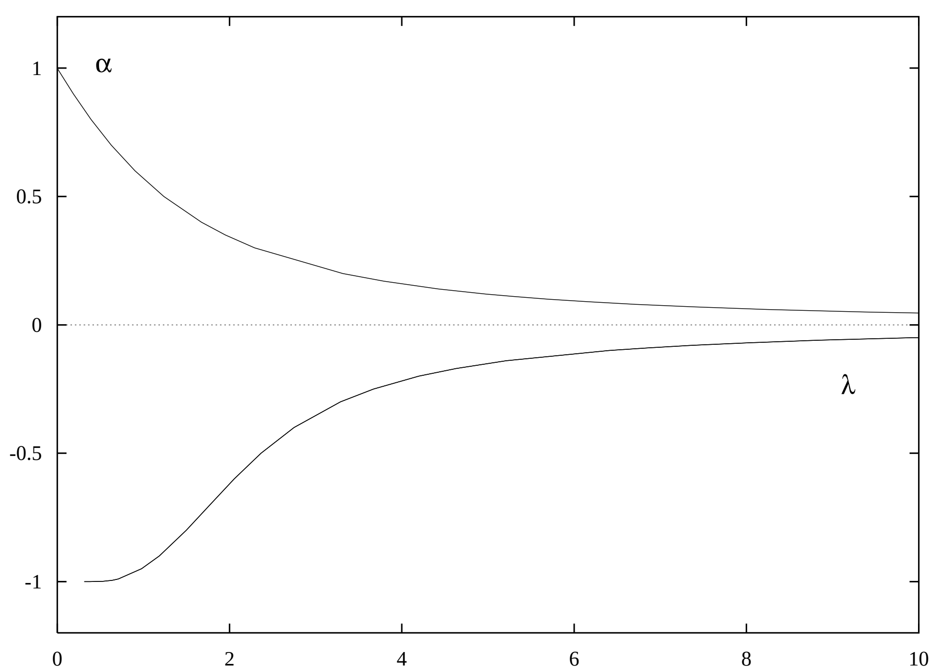

FIGURE 2. Graph of $F'(0) = \alpha$ versus λ for the two solution branches.

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