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SINGULAR THIRD-ORDER m-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. This paper is concerned with the following third-order m-point boundary value problem

$$\begin{cases} u'''(t) = f(t, u(t), u'(t), u''(t)) + e(t), \quad 0 < t < 1, \\ u(0) = \sum_{i=1}^{m-2} k_i u(\xi_i), u'(0) = u'(1) = 0, \end{cases}$$

where $f : (0,1) \times R^3 \to R$ is a function satisfying Carathéodory's conditions, $e : (0,1) \to R$ and $t(1-t)e(t) \in L^1[0,1]$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $k_i \in R$ $(i = 1, 2, \ldots, m-2)$ and $\sum_{i=1}^{m-2} k_i \neq 1$. Some existence criteria of at least one solution are established by using the well-known Leray-Schauder Continuation Principle.

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1. INTRODUCTION

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [10]. Recently, third-order two-point or three-point boundary value problems (BVPs for short) have received much attention [1, 2, 3, 6, 7, 11, 12, 13, 14, 17, 18]. Although there are many excellent works on third-order two-point or three-point BVPs, a little work has been done for more general third-order m-point BVPs or high-order multi-point BVPs [4, 5, 8, 9, 16] (either singular or non-singular).

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As we know, the study on singular multi-point BVPs proceeded very slowly. For the singular second-order m-point BVP

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t), & 0 < t < 1, \\ x'(0) = 0, x(1) = \sum\limits_{i=1}^{m-2} a_i x(\xi_i), \end{cases} \tag{1.1}$$

Ma [15] studied the existence of at least one solution by using Leray-Schauder Continuation Principle.

Motivated greatly by the above-mentioned excellent works, in this paper we will investigate the third-order m-point BVP

$$\begin{cases} u'''(t) = f(t, u(t), u'(t), u''(t)) + e(t), & 0 < t < 1, \\ u(0) = \sum\limits_{i=1}^{m-2} k_i u(\xi_i), u'(0) = u'(1) = 0. \end{cases} \tag{1.2}$$

Throughout this paper, we always assume that $f : (0,1) \times R^3 \to R$ is a function satisfying Carathéodory's conditions, $e : (0,1) \to R$ and $t(1-t)e(t) \in L^1[0,1]$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $k_i \in R$ $(i = 1, 2, \ldots, m-2)$ and $\sum\limits_{i=1}^{m-2} k_i \neq 1$. It is interesting that our f and e may be singular at $t = 0$ and $t = 1$. Some existence results of at least one solution for the BVP (1.2) are established by applying the well-known Leray-Schauder Continuation Principle [19], which we state here for convenience of the reader.

Theorem 1.1. *Let X be a Banach space and $T : X \to X$ be a compact map. Suppose that there exists an $R > 0$ such that if $u = \lambda T u$ for some $\lambda \in (0,1)$, then $\|u\| \leq R$. Then T has a fixed point.*

In the remainder of this section, we introduce some useful spaces. We will use the classical Banach spaces $C[0,1]$, $C^k[0,1]$, $L^1[0,1]$ and denote the space of absolutely continuous functions on the interval $[0,1]$ by $AC[0,1]$. We also denote

$$AC_{loc}(0,1) = \left\{ y \mid \; y|_{[a,b]} \in AC[a,b] \text{ for every compact interval } [a,b] \subseteq (0,1) \right\}.$$

Let E be the Banach space

$$E = \left\{ y \in L^1_{loc}(0,1) \middle| \; t(1-t)y(t) \in L^1[0,1] \right\}$$

equipped with the norm

$$\|y\|_E = \int_0^1 t(1-t)|y(t)|\,dt,$$

where

$$L^1_{loc}(0,1) = \left\{ y \mid \; y|_{[a,b]} \in L^1[a,b] \text{ for every compact interval } [a,b] \subseteq (0,1) \right\}.$$

Moreover, we also use the Banach space

$$X = \left\{ u \in C^2(0,1) \middle| \; u \in C[0,1], u' \in C[0,1] \text{ and } t(1-t)u'' \in C[0,1] \right\}$$

equipped with the norm

$$\|u\|_X = \max\left\{\|u\|_\infty, \|u'\|_\infty, \|t(1-t)u''\|_\infty\right\},$$

where $\|\cdot\|_\infty$ denotes the sup norm.

In the remainder of this paper, we suppose that the following condition is satisfied:

(H) There exist $\alpha_0, \alpha_1, \delta \in E$ and $\alpha_2 \in L^1[0,1]$ such that

$$|f(t, x_0, x_1, x_2)| \le \sum_{i=0}^{2} \alpha_i(t)|x_i| + \delta(t), \ \text{a.e. } t \in (0,1), (x_0, x_1, x_2) \in R^3.$$

2. PRELIMINARY LEMMAS

In this section, we present several important preliminary lemmas.

Lemma 2.1. *Let $y \in E$. Then the BVP*

$$\begin{cases} u'''(t) + y(t) = 0, & t \in (0,1), \\ u(0) = \sum\limits_{i=1}^{m-2} k_i u(\xi_i), & u'(0) = u'(1) = 0 \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G_0(t,s)y(s)\,ds + \frac{1}{1 - \sum\limits_{i=1}^{m-2} k_i} \sum_{i=1}^{m-2} k_i \int_0^1 G_0(\xi_i, s)y(s)\,ds,$$

which satisfies

$$u'(t) = \int_0^1 G_1(t,s)y(s)\,ds \ \text{ and } u''(t) = \int_0^1 G_2(t,s)y(s)\,ds,$$

where

$$G_0(t,s) = \begin{cases} \frac{2st - s^2 - st^2}{2}, & 0 \le s \le t \le 1, \\ \frac{(1-s)t^2}{2}, & 0 \le t \le s \le 1, \end{cases} \tag{2.2}$$

$$G_1(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1 \end{cases} \tag{2.3}$$

and

$$G_2(t,s) = \begin{cases} -s, & 0 \le s \le t \le 1, \\ 1-s, & 0 \le t \le s \le 1. \end{cases} \tag{2.4}$$

Proof. In fact, if u is a solution of the BVP (2.1), then we may suppose that

$$u(t) = -\int_0^t \frac{(t-s)^2}{2} y(s)\,ds + At^2 + Bt + C.$$

By the boundary conditions, we get $A = \int_0^1 \frac{1-s}{2} y(s)\,ds$, $B = 0$ and

$$C = \frac{1}{1 - \sum\limits_{i=1}^{m-2} k_i} \sum_{i=1}^{m-2} k_i \int_0^1 \frac{1-s}{2}\xi_i^2 y(s)\,ds - \frac{1}{1 - \sum\limits_{i=1}^{m-2} k_i} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i - s)^2}{2} y(s)\,ds.$$

Therefore, the BVP (2.1) has a unique solution

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^2}{2} y(s)\,ds + \int_0^1 \frac{1-s}{2} t^2 y(s)\,ds \\ &\quad + \frac{1}{1-\sum_{i=1}^{m-2} k_i} \sum_{i=1}^{m-2} k_i \int_0^1 \frac{1-s}{2} \xi_i^2 y(s)\,ds - \frac{1}{1-\sum_{i=1}^{m-2} k_i} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i - s)^2}{2} y(s)\,ds \\ &= \int_0^t \frac{2st - s^2 - st^2}{2} y(s)\,ds + \int_t^1 \frac{(1-s)t^2}{2} y(s)\,ds \\ &\quad + \frac{1}{1-\sum_{i=1}^{m-2} k_i} \sum_{i=1}^{m-2} k_i \left(\int_0^{\xi_i} \frac{2s\xi_i - s^2 - s\xi_i^2}{2} y(s)\,ds + \int_{\xi_i}^1 \frac{(1-s)\xi_i^2}{2} y(s)\,ds \right). \end{aligned}$$

Moreover,

$$u'(t) = \int_0^t s(1-t)\,y(s)\,ds + \int_t^1 t(1-s)\,y(s)\,ds$$

and

$$u''(t) = \int_0^t (-s)\,y(s)\,ds + \int_t^1 (1-s)\,y(s)\,ds.$$

□

Lemma 2.2. *For all* $(t,s) \in [0,1] \times [0,1]$, *we have*

$$0 \le G_0(t,s) \le \frac{1}{2} s(1-s) \tag{2.5}$$

and

$$0 \le G_1(t,s) \le s(1-s). \tag{2.6}$$

Proof. Since it is obvious that (2.6) holds, we only prove (2.5). If $s \le t$, then

$$G_0(t,s) = \frac{2st - s^2 - st^2}{2} = \frac{s\left(1 - s - (1-t)^2\right)}{2} \le \frac{1}{2} s(1-s).$$

If $t \le s$, then

$$G_0(t,s) = \frac{(1-s)t^2}{2} \le \frac{(1-s)s^2}{2} \le \frac{1}{2} s(1-s).$$

□

Lemma 2.3. *Let* $y \in E$. *Then the unique solution of the BVP* (2.1) *satisfies*

$$\left\| u^{(i)} \right\|_\infty \le A_i \|y\|_E, \ i = 0, 1 \tag{2.7}$$

and

$$\|t(1-t)u''(t)\|_\infty \le A_2 \|y\|_E, \tag{2.8}$$

where $A_0 = \frac{1}{2}\left(1 + \frac{\sum_{i=1}^{m-2} |k_i|}{\left|1 - \sum_{i=1}^{m-2} k_i\right|}\right)$ *and* $A_1 = A_2 = 1$.

Proof. In view of Lemma 2.2, for all $t \in [0,1]$, we have

$$\begin{aligned}
|u(t)| &= \left| \int_0^1 G_0(t,s)\, y(s)\, ds + \frac{1}{1-\sum\limits_{i=1}^{m-2} k_i} \sum_{i=1}^{m-2} k_i \int_0^1 G_0(\xi_i,s)\, y(s)\, ds \right| \\
&\leq \int_0^1 G_0(t,s)\, |y(s)|\, ds + \frac{1}{\left|1-\sum\limits_{i=1}^{m-2} k_i\right|} \sum_{i=1}^{m-2} |k_i| \int_0^1 G_0(\xi_i,s)\, |y(s)|\, ds \\
&\leq \frac{1}{2} \int_0^1 s(1-s)\, |y(s)|\, ds + \frac{1}{2\left|1-\sum\limits_{i=1}^{m-2} k_i\right|} \sum_{i=1}^{m-2} |k_i| \int_0^1 s(1-s)\, |y(s)|\, ds \\
&= \frac{1}{2} \left(1 + \frac{\sum\limits_{i=1}^{m-2} |k_i|}{\left|1-\sum\limits_{i=1}^{m-2} k_i\right|} \right) \|y\|_E ,
\end{aligned}$$

and thus

$$\|u\|_\infty \leq \frac{1}{2} \left(1 + \frac{\sum\limits_{i=1}^{m-2} |k_i|}{\left|1-\sum\limits_{i=1}^{m-2} k_i\right|} \right) \|y\|_E .$$

Similarly, for all $t \in [0,1]$, we get

$$|u'(t)| = \left| \int_0^1 G_1(t,s)\, y(s)\, ds \right| \leq \int_0^1 s(1-s)\, |y(s)|\, ds = \|y\|_E ,$$

and so

$$\|u'\|_\infty \leq \|y\|_E .$$

Finally, for all $t \in [0,1]$,

$$\begin{aligned}
|t(1-t)\, u''(t)| &= \left| \int_0^t t(1-t)(-s)\, y(s)\, ds + \int_t^1 t(1-t)(1-s)\, y(s)\, ds \right| \\
&\leq \int_0^t (1-t)\, s\, |y(s)|\, ds + \int_t^1 t(1-s)\, |y(s)|\, ds \\
&\leq \int_0^t (1-s)\, s\, |y(s)|\, ds + \int_t^1 s(1-s)\, |y(s)|\, ds \\
&= \int_0^1 s(1-s)\, |y(s)|\, ds = \|y\|_E ,
\end{aligned}$$

which implies that

$$\|t(1-t)\, u''(t)\|_\infty \leq \|y\|_E .$$

□

Now, we define an integral mapping $T : E \to X$ by

$$(Ty)(t) = \int_0^1 G_0(t,s)\,y(s)\,ds + \frac{1}{1-\sum\limits_{i=1}^{m-2} k_i} \sum_{i=1}^{m-2} k_i \int_0^1 G_0(\xi_i, s)\,y(s)\,ds,\ \ t \in [0,1].$$

Similar to the proof of part one of Lemma 2.3, we have

$$\begin{aligned} & \left| \int_0^1 G_0(t,s)\,y(s)\,ds + \frac{1}{1-\sum\limits_{i=1}^{m-2} k_i} \sum_{i=1}^{m-2} k_i \int_0^1 G_0(\xi_i, s)\,y(s)\,ds \right| \\ \le\ & \frac{1}{2}\left(1 + \frac{\sum\limits_{i=1}^{m-2} |k_i|}{\left|1-\sum\limits_{i=1}^{m-2} k_i\right|}\right) \int_0^1 s(1-s)\,|y(s)|\,ds < \infty, \end{aligned}$$

which shows that T is well-defined.

Lemma 2.4. *Let $y \in E$. Then $Ty \in X$ and*

$$\begin{cases} (Ty)'''(t) + y(t) = 0,\ \ a.e.\ t \in (0,1), \\ (Ty)(0) = \sum\limits_{i=1}^{m-2} k_i (Ty)(\xi_i),\ (Ty)'(0) = (Ty)'(1) = 0. \end{cases} \tag{2.9}$$

Proof. For $y \in E$, we know that $t(1-t)\,y(t) \in L^1[0,1]$. By Lemma 2.1, we get

$$(Ty)'(t) = \int_0^1 G_1(t,s)\,y(s)\,ds \tag{2.10}$$

and

$$(Ty)''(t) = \int_0^1 G_2(t,s)\,y(s)\,ds. \tag{2.11}$$

Now, since

$$\int_0^1 \left|(Ty)'(t)\right| dt = \int_0^1 \left| \int_0^1 G_1(t,s)\,y(s)\,ds \right| dt \le \int_0^1 s(1-s)\,|y(s)|\,ds < \infty,$$

we have $Ty \in AC[0,1]$. A simple computation (by interchanging the order of integration) yields

$$\begin{aligned} \int_0^1 \left|(Ty)''(t)\right| dt &= \int_0^1 \left| \int_0^1 G_2(t,s)\,y(s)\,ds \right| dt \\ &\le \int_0^1 \int_0^t s\,|y(s)|\,ds dt + \int_0^1 \int_t^1 (1-s)\,|y(s)|\,ds dt \\ &= \int_0^1 \int_s^1 s\,|y(s)|\,dt ds + \int_0^1 \int_0^s (1-s)\,|y(s)|\,dt ds \\ &= 2\int_0^1 s(1-s)\,|y(s)|\,ds < \infty, \end{aligned}$$

which shows that $(Ty)' \in AC[0,1]$. Now (2.11) together with the fact $y \in L^1[a,b]$, for any $a, b \in (0,1)$, imply that $(Ty)'' \in AC[a,b]$. So

$$(Ty)'''(t) + y(t) = 0, \ a.e.\ t \in (0,1).$$

Set

$$\phi(t) := \left[t(1-t)(Ty)''(t)\right]', \ t \in [0,1].$$

We first show $\phi \in L^1[0,1]$. If this is true, then $t(1-t)(Ty)'' \in AC[0,1]$, and accordingly, $t(1-t)(Ty)'' \in C[0,1]$. In fact, a simple computation yields

$$\begin{aligned}\int_0^1 |\phi(t)|\,dt &= \int_0^1 \left|(1-2t)(Ty)''(t) + t(1-t)(Ty)'''(t)\right| dt \\ &\le \int_0^1 \left|(Ty)''(t)\right| dt + \int_0^1 t(1-t)\left|(Ty)'''(t)\right| dt \\ &\le 2\int_0^1 s(1-s)|y(s)|\,ds + \int_0^1 t(1-t)|y(t)|\,dt < \infty.\end{aligned}$$

Next,

$$(Ty)(0) = \frac{1}{1-\sum\limits_{i=1}^{m-2} k_i}\sum_{i=1}^{m-2} k_i \int_0^1 G_0(\xi_i, s)\,y(s)\,ds$$

and

$$(Ty)(\xi_i) = \int_0^1 G_0(\xi_i, s)\,y(s)\,ds + \frac{1}{1-\sum\limits_{i=1}^{m-2} k_i}\sum_{i=1}^{m-2} k_i \int_0^1 G_0(\xi_i, s)\,y(s)\,ds$$

imply that

$$(Ty)(0) = \sum_{i=1}^{m-2} k_i (Ty)(\xi_i).$$

Similarly, we can obtain that

$$(Ty)'(0) = (Ty)'(1) = 0.$$

□

For $u \in X$, we define a nonlinear operator $N : X \to E$ by

$$(Nu)(t) = -f(t, u(t), u'(t), u''(t)) - e(t), \ t \in (0,1).$$

From (H), we can conclude that N is well-defined. In fact,

$$\begin{aligned}\|Nu\|_E &= \int_0^1 t(1-t)\left|f(t,u(t),u'(t),u''(t))+e(t)\right|dt\\ &\le \int_0^1 t(1-t)|\alpha_0(t)||u(t)|\,dt+\int_0^1 t(1-t)|\alpha_1(t)||u'(t)|\,dt\\ &\quad+\int_0^1 |\alpha_2(t)|t(1-t)|u''(t)|\,dt\\ &\quad+\int_0^1 t(1-t)|\delta(t)|\,dt+\int_0^1 t(1-t)|e(t)|\,dt\\ &\le \|\alpha_0\|_E\|u\|_\infty+\|\alpha_1\|_E\|u'\|_\infty+\|\alpha_2\|_1\|t(1-t)u''\|_\infty+\|\delta\|_E+\|e\|_E<\infty.\end{aligned}$$

Lemma 2.5. *$TN : X \to X$ is compact.*

Proof. Let $D \subset X$ be a bounded set. We will prove that $TN(D)$ is relative compact in X. Suppose that $\{w_k\}_{k=1}^{\infty} \subset TN(D)$ is an arbitrary sequence. Then there is $\{u_k\}_{k=1}^{\infty} \subset D$ such that $TN(u_k) = w_k$. Set

$$M = \sup\{\|u\|_X : u \in D\}.$$

Then it is easy to see that

$$\begin{aligned}|(Nu_k)(t)| \le |\alpha_0(t)|M+|\alpha_1(t)|M+\frac{|\alpha_2(t)|}{t(1-t)}M\\ +|\delta(t)|+|e(t)|:=\chi(t),\quad t\in(0,1).\end{aligned}$$

Obviously, $\chi \in E$, i.e., $\int_0^1 t(1-t)\chi(t)\,dt < \infty$. Thus, by Lemma 2.2, we have

$$\begin{aligned}|w_k(t)| &= |((TN)u_k)(t)|\\ &= \left|\int_0^1 G_0(t,s)(Nu_k)(s)\,ds+\frac{1}{1-\sum\limits_{i=1}^{m-2}k_i}\sum_{i=1}^{m-2}k_i\int_0^1 G_0(\xi_i,s)(Nu_k)(s)\,ds\right|\\ &\le \frac{1}{2}\left(1+\frac{\sum\limits_{i=1}^{m-2}|k_i|}{\left|1-\sum\limits_{i=1}^{m-2}k_i\right|}\right)\int_0^1 s(1-s)|(Nu_k)(s)|\,ds\\ &\le \frac{1}{2}\left(1+\frac{\sum\limits_{i=1}^{m-2}|k_i|}{\left|1-\sum\limits_{i=1}^{m-2}k_i\right|}\right)\int_0^1 s(1-s)\chi(s)\,ds,\ t\in[0,1],\end{aligned}$$

which implies that $\{w_k\}_{k=1}^{\infty}$ is uniformly bounded. Similarly, we get

$$\begin{aligned}\left|w_k'(t)\right| &= \left|\left((TN)u_k\right)'(t)\right| = \left|\int_0^1 G_1(t,s)(Nu_k)(s)\,ds\right| \\ &\leq \int_0^1 s(1-s)\chi(s)\,ds,\ t\in[0,1],\end{aligned}$$

which shows that $\{w_k'\}_{k=1}^{\infty}$ is also uniformly bounded. Therefore, $\{w_k\}_{k=1}^{\infty}$ is equicontinuous. By the Arzela-Ascoli theorem, $\{w_k\}_{k=1}^{\infty}$ has a convergent subsequence in $C[0,1]$. Without loss of generality, we may assume that $\{w_k\}_{k=1}^{\infty}$ converges in $C[0,1]$.

Next,

$$\begin{aligned}\int_0^1\int_0^1 |G_2(t,s)|\chi(s)\,dsdt &= \int_0^1\int_0^t s\chi(s)\,dsdt + \int_0^1\int_t^1 (1-s)\chi(s)\,dsdt \\ &= \int_0^1\int_s^1 s\chi(s)\,dtds + \int_0^1\int_0^s (1-s)\chi(s)\,dtds \\ &= 2\int_0^1 s(1-s)\chi(s)\,ds,\end{aligned}$$

that is to say, $\int_0^1 |G_2(t,s)|\chi(s)\,ds \in L^1[0,1]$, which together with

$$\begin{aligned}\left|w_k'(t_1)-w_k'(t_2)\right| &= \left|\int_{t_2}^{t_1} w_k''(t)\,dt\right| \leq \int_{t_2}^{t_1}\left|w_k''(t)\right|dt = \int_{t_2}^{t_1}\left|\left((TN)u_k\right)''(t)\right|dt \\ &= \int_{t_2}^{t_1}\left|\int_0^1 G_2(t,s)(Nu_k)(s)\,ds\right|dt \\ &\leq \int_{t_2}^{t_1}\int_0^1 |G_2(t,s)|\,|(Nu_k)(s)|\,dsdt \\ &\leq \int_{t_2}^{t_1}\int_0^1 |G_2(t,s)|\chi(s)\,dsdt\end{aligned}$$

for every $t_1,t_2\in[0,1]$ with $t_2<t_1$, imply that $\{w_k'\}_{k=1}^{\infty}$ is equicontinuous. As a result, without loss of generality, we may put that $\{w_k'\}_{k=1}^{\infty}$ is also convergent in $C[0,1]$.

Finally,

$$\begin{aligned}\left|t(1-t)w_k''(t)\right| &= \left|t(1-t)\left((TN)u_k\right)''(t)\right| \\ &= \left|\int_0^1 t(1-t)G_2(t,s)(Nu_k)(s)\,ds\right| \\ &\leq \int_0^t t(1-t)s\,|(Nu_k)(s)|\,ds + \int_t^1 t(1-t)(1-s)\,|(Nu_k)(s)|\,ds \\ &\leq \int_0^t s(1-s)\,|(Nu_k)(s)|\,ds + \int_t^1 s(1-s)\,|(Nu_k)(s)|\,ds \\ &\leq \int_0^1 s(1-s)\chi(s)\,ds,\ t\in[0,1],\end{aligned}$$

which shows that $\{t(1-t)w_k''\}_{k=1}^{\infty}$ is uniformly bounded. If we let
$\varphi(t)=\int_0^1 |G_2(t,s)|\chi(s)\,ds + t(1-t)\chi(t),\ t\in[0,1]$, then it is easy to know that

$\varphi \in L^1[0,1]$ and

$$\begin{aligned}\left|\left(t(1-t)w_k''(t)\right)'\right| &= \left|(1-2t)\left((TN)u_k\right)''(t)+t(1-t)\left((TN)u_k\right)'''(t)\right| \\ &\le \int_0^1 |G_2(t,s)|\,|(Nu_k)(s)|\,ds + t(1-t)\,|(Nu_k)(t)| \\ &\le \varphi(t),\ t\in[0,1].\end{aligned}$$

And then for every $t_1, t_2 \in [0,1]$ with $t_2 < t_1$, we have

$$\begin{aligned}\left|t_1(1-t_1)w_k''(t_1)-t_2(1-t_2)w_k''(t_2)\right| &= \left|\int_{t_2}^{t_1}\left(t(1-t)w_k''(t)\right)'dt\right| \\ &\le \int_{t_2}^{t_1}\left|\left(t(1-t)w_k''(t)\right)'\right|dt \\ &\le \int_{t_2}^{t_1}\varphi(t)\,dt,\end{aligned}$$

which shows that $\{t(1-t)w_k''\}_{k=1}^{\infty}$ is equicontinuous. Again, by the Arzela-Ascoli theorem, we know that $\{t(1-t)w_k''\}_{k=1}^{\infty}$ has a convergent subsequence in $C[0,1]$. Therefore, $\{w_k\}_{k=1}^{\infty}$ has a convergent subsequence in X. □

3. MAIN RESULTS

Now, we apply the Leray-Schauder Continuation Principle to establish the existence of at least one solution for the BVP (1.2).

Theorem 3.1. *Assume that* (H) *holds. Then the BVP* (1.2) *has at least one solution in* X *provided*

$$\frac{1}{2}\|\alpha_0\|_E\left(1+\frac{\sum_{i=1}^{m-2}|k_i|}{\left|1-\sum_{i=1}^{m-2}k_i\right|}\right)+\|\alpha_1\|_E+\|\alpha_2\|_1<1. \tag{3.1}$$

Proof. To complete the proof, it suffices to verify that the set of all possible solutions of the BVP

$$\begin{cases} u'''(t)=\lambda f\left(t,u(t),u'(t),u''(t)\right)+\lambda e(t), \quad 0<t<1, \\ u(0)=\sum_{i=1}^{m-2}u(\xi_i),\, u'(0)=u'(1)=0 \end{cases} \tag{3.2}$$

is, a priori, bounded in X by a constant independent of $\lambda \in (0,1)$.

Suppose that $u \in X$ is a solution of the BVP (3.2) for some $\lambda \in (0,1)$. Then it follows from (H) and Lemma 2.3 that

$$\begin{aligned}
\|u'''\|_E &= \int_0^1 t(1-t)|u'''(t)|\,dt = \int_0^1 \lambda t(1-t)|f(t,u(t),u'(t),u''(t)) + e(t)|\,dt \\
&\le \int_0^1 t(1-t)(|\alpha_0(t)||u(t)| + |\alpha_1(t)||u'(t)| + |\alpha_2(t)||u''(t)| + |\delta(t)| + |e(t)|)\,dt \\
&\le \|\alpha_0\|_E \|u\|_\infty + \|\alpha_1\|_E \|u'\|_\infty + \|\alpha_2\|_1 \|t(1-t)u''\|_\infty + \|\delta\|_E + \|e\|_E \\
&\le \left(\frac{1}{2}\|\alpha_0\|_E \left(1 + \frac{\sum_{i=1}^{m-2}|k_i|}{\left|1-\sum_{i=1}^{m-2}k_i\right|} \right) + \|\alpha_1\|_E + \|\alpha_2\|_1 \right) \|u'''\|_E + \|\delta\|_E + \|e\|_E .
\end{aligned}$$

In view of (3.1), there exists a constant $c = \dfrac{\|\delta\|_E+\|e\|_E}{1-\left(\frac{1}{2}\|\alpha_0\|_E \left(1+\frac{\sum_{i=1}^{m-2}|k_i|}{\left|1-\sum_{i=1}^{m-2}k_i\right|} \right) + \|\alpha_1\|_E + \|\alpha_2\|_1 \right)}$, independent of $\lambda \in (0,1)$, such that

$$\|u'''\|_E \le c.$$

By Lemma 2.3, we obtain

$$\|u\|_\infty \le \frac{1}{2}\left(1 + \frac{\sum_{i=1}^{m-2}|k_i|}{\left|1-\sum_{i=1}^{m-2}k_i\right|} \right) c$$

and

$$\|u'\|_\infty \le c, \quad \|t(1-t)u''\|_\infty \le c.$$

Then,

$$\|u\|_X \le \max\left\{ 1,\ \frac{1}{2}\left(1 + \frac{\sum_{i=1}^{m-2}|k_i|}{\left|1-\sum_{i=1}^{m-2}k_i\right|} \right) \right\} c.$$

It is now immediate from Theorem 1.1 that TN has at least one fixed point, which is a desired solution of the BVP (1.2). □

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