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LYAPUNOV THEORY FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

Recently there has been a surge in the study of the theory of fractional differential equations. The existing results have been collected in the forthcoming monograph on that subject [1]. In this paper, we prove necessary comparison theorems utilizing Lyapunov-like functions, define stability concepts in terms of a norm and prove stability results parallel to Lyapunov's results relative to stability and asymptotic stability. Since there are many stability concepts and corresponding results in the literature for ODEs, it is useful to work with stability concepts in terms of two measures, which includes several important stability notions as special cases [2, 3]. A few choices of the two measures are given to demonstrate the versatility of this approach. We extend this versatile approach to provide necessary conditions for stability criteria in terms of two measures for fractional differential equations. We use Caputo's derivative of arbitrary order which is more suitable to discuss Lyapunov stability theory.

2. BASIC COMPARISON RESULTS

We shall consider the IVP of Caputo's fractional differential equation, namely

$$ {}^cD^q x = f(t,x), \quad x(t_0) = x_0, \quad t_0 \geq 0, \tag{2.1}$$

where $f \in C(R_+ \times \mathbb{R}^n, \mathbb{R}^n)$. For any Lyapunov-like function
$V \in C(R_+ \times \mathbb{R}^n, R_+)$, we define

$$ {}^cD^q_+ V(t,x) = \limsup_{h \to 0} \frac{1}{h^q}[V(t,x) - V(t-h, x-h^q f(t,x))] \tag{2.2}$$

for $(t,x) \in R_+ \times \mathbb{R}^n$. In this chapter, we shall employ Lyapunov-like functions to estimate the solutions $x(t) = x(t,t_0,x_0)$ of IVP (2.1) in terms of such functions, as

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the method offers a lot of flexibility in obtaining qualitative properties, including stability, of solutions of IVP (2.1). In fact, the Lyapunov-like function acts like a transformation of (2.1) into a relatively simple fractional differential equation, the properties of solutions of this simple system can be transferred back to the original more complicated system. This is known as the comparison principle, in general.

We can now formulate the basic comparison results via Lyapunov-like function.

Theorem 2.1. *Let $V \in C(R_+ \times \mathbb{R}^n, R_+)$ and $V(t,x)$ be locally Lipschitzian in x. Assume that*

$$^cD_+^q V(t,x) \leq g(t, V(t,x)), \quad (t,x) \in R_+ \times \mathbb{R}^n, \tag{2.3}$$

where $g \in C(R_+^2, \mathbb{R})$. Suppose that $r(t) = r(t, t_0, u_0)$ is the maximal solution of the scalar fractional differential equation

$$^cD^q u = g(t,u), \quad u(t_0) = u_0 \geq 0 \tag{2.4}$$

existing on $[t_0, \infty)$. Then $V(t_0, x_0) \leq u_0$ implies

$$V(t, x(t)) \leq r(t), \quad t \geq t_0, \tag{2.5}$$

where $x(t) = x(t, t_0, x_0)$ is any solution of the IVP (2.1) existing on $[t_0, \infty)$.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (2.1) existing on $[t_0, \infty)$ such that $V(t_0, x_0) \leq u_0$. Set $m(t) = V(t, x(t))$. Then for small $h > 0$, we have

$$\begin{aligned} V(t,x(t)) - V(t-h, S(x,h,r,q)) &= V(t,x(t)) - V(t-h, x - h^q f(t,x)) \\ &\quad + V(t-h, x-h^q f(t,x)) - V(t-h, S(x,h,r,q)), \end{aligned}$$

where $S(x,h,r,q) = \sum_{r=1}^{n} (-1)^{r+1} \binom{q}{r} x(t-rh)$.

Since $V(t,x)$ is locally Lipschitzian in x, we get, using (2.2), the fractional differential inequality

$$\begin{aligned} ^cD_+^q m(t) &= \limsup_{\substack{h\to 0 \\ nh = t - t_0}} \frac{1}{h^q} [V(t,x(t)) - V(t-h, x(t) - h^q f(t,x(t))) \\ &\qquad + V(t-h, x(t) - h^q f(t, x(t))) - V(t-h, S(x,h,r,q))] \\ &\leq g(t, V(t,x(t))) = g(t, m(t)), \end{aligned} \tag{2.6}$$

since $\limsup_{h\to 0} L\frac{\epsilon(h^q)}{h^q} \to 0$, L being Lipschitzian constant. Moreover, we have $m(t_0) = V(t_0, x_0) \leq u_0$ and therefore it follows from Theorem 2.8.3 [1], the desired estimate

$$V(t, x(t)) \leq r(t), \qquad t \geq t_0.$$

Here we have employed the discussion of Section 2.8 [1]. □

Corollary 2.2. *If, in Theorem 2.1, we suppose that $g(t,u) \equiv 0$, then we arrive at*

$$V(t, x(t)) \leq V(t_0, x_0), \qquad t \geq t_0. \tag{2.7}$$

Corollary 2.3. *If, in Theorem* 2.1, *we suppose, instead of* (2.3)

$$^{c}D_{+}^{q}V(t,x) + d(|x|) \leq g(t, V(t,x)) \tag{2.8}$$

where $d \in C(R_+, R_+)$, $d(0) = 0$, $d(u)$ *is strictly increasing in* u *and* $g(t,u)$ *is nondecreasing in* u *for each* t, *then we obtain the estimate*

$$V(t,x(t)) + \frac{1}{\Gamma(q)} \int\limits_{t_0}^{t} (t-s)^{q-1} d(|x(s)|) ds \leq r(t), \quad t \geq t_0. \tag{2.9}$$

Proof. Consider $L(t,x(t)) = V(t,x(t)) + \frac{1}{\Gamma(q)} \int\limits_{t_0}^{t} (t-s)^{q-1} d(|x(s)|) ds$, then setting $m(t) = L(t,x(t))$ and using the fact $g(t,u)$ is nondecreasing in u coupled with (2.8), it is easy to obtain, as before,

$$^{c}D_{+}^{q}m(t) \leq g(t, m(t)), \quad t \geq t_0,$$

and the assertion (2.9) follows from Theorem 2.8.3 [1] and Theorem 2.1. □

The next result plays an important role when we utilize vector Lyapunov functions.

Theorem 2.4. *Let* $V \in C(R_+ \times \mathbb{R}^n, R_+^N)$ *be locally Lipschitzian in* x. *Assume that*

$$^{c}D_{+}^{q}V(t,x) \leq g(t, V(t,x)), \quad (t,x) \in R_+ \times \mathbb{R}^n, \tag{2.10}$$

where $g \in C(R_+ \times R_+^N, R^N)$ *and* $g(t,u)$ *is quasimonotone nondecreasing in* u *for each* t. *Let* $r(t) = r(t, t_0, u_0)$ *be the maximal solution of the fractional differential system*

$$^{c}D^{q}u = g(t,u), \quad u(t_0) = u_0, \tag{2.11}$$

existing on $[t_0, \infty)$ *and* $x(t) = x(t, t_0, x_0)$ *be any solution of* (2.1) *existing on* $[t_0, \infty)$. *Then* $V(t_0, x_0) \leq u_0$ *implies*

$$V(t, x(t)) \leq r(t), \qquad t \geq t_0.$$

We recall that inequalities between vectors are component-wise inequalities and quasimonotonicity of $g(t,u)$ means that $u \leq v, u_i = v_i$ for $1 \leq i \leq N$ implies $g_i(t,u) \leq g_i(t,v)$. The proof is similar to the proof of Theorem 2.1; working component-wise and using the Comparison Theorem 2.8.3 [1]. We leave it to the reader.

Corollary 2.5. *If, in Theorem* 2.4, *we specialize* $g(t,u) = Au$ *where* A *is an* $n \times n$ *matrix, we get the estimate*

$$V(t,x(t)) \leq V(t_0, x_0) E_q(A(t-t_0)^q), \quad t \geq t_0,$$

where E_q *is the corresponding Mittag-Leffler's function.*

3. STABILITY CRITERIA

In this section, we shall consider some simple stability results. We list a few definitions concerning the stability of the trivial solution of (2.1) which we assume to exist.

Definition 3.1. The trivial solution $x = 0$ of (2.1) is said to be

(S_1) equi-stable if, for each $\epsilon > 0$ and $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ such that $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| < \epsilon,\ t \geq t_0$;
(S_2) uniformly stable, if the δ in (S_1) is independent of t_0;
(S_3) quasi-equi-asymptotically stable, if for each $\epsilon > 0$ and $t_0 \in R_+$, there exist positive $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that $\|x_0\| < \delta_0$ implies $\|x(t, t_0, x_0)\| < \epsilon$ for $t \geq t_0 + T$;
(S_4) quasi-uniformly asymptotically stable, if δ_0 and T in (S_3) are independent of t_0;
(S_5) equi-asymptotically stable, if (S_1) and (S_3) hold simultaneously;
(S_6) uniformly asymptotically stable, if (S_2) and (S_4) hold simultaneously.

Corresponding to the definitions (S_1) to (S_6), we can define the stability notions of the trivial solution $u = 0$ of (2.4). For example, the trivial solution $u = 0$ of (2.4) is equi-stable if, for each $\epsilon > 0$ and $t_0 \in R_+$, there exists a function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ, such that $u_0 < \delta$ implies $u(t, t_0, u_0) < \epsilon,\ t \geq t_0$.

Definition 3.2. A function $\phi(u)$ is said to belong to the class $\mathcal{K}$, if $\phi \in C([0, \rho), R_+)$, $\phi(0) = 0$ and $\phi(u)$ is strictly monotone increasing in u. A function $\phi(t, u)$ is said to belong to the class $C\mathcal{K}$, if for each $t, \phi(t, u) \in \mathcal{K}$ and $\phi \in C(R_+ \times [0, \rho), R_+)$.

Definition 3.3. A function $V(t, x)$ with $V(t, 0) \equiv 0$ is said to be positive definite if there exists a function $b \in \mathcal{K}$ such that

$$V(t, x) \geq b(|x|), \tag{3.1}$$

is satisfied for $(t, x) \in R_+ \times S(\rho)$, where $S(\rho) = [x \in \mathbb{R}^n : |x| < \rho]$, and it is said to be decrescent if a function $a \in \mathcal{K}$ exists such that

$$V(t, x) \leq a(|x|), \quad (t, x) \in R_+ \times S(\rho). \tag{3.2}$$

If, on the other hand, we have

$$V(t, x) \leq a(t, |x|), \quad (t, x) \in R_+ \times S(\rho), \tag{3.3}$$

where $a \in C\mathcal{K}$, then $V(t, x)$ is known as weakly decrescent.

We can now prove two results on stability theory of Lyapunov parallel to the analogous results in ODE in the present frame work.

Theorem 3.4. *Assume that there exists a Lyapunov function* $V \in C(R_+ \times S(\rho), R_+)$ *such that* $V(t,x)$ *is positive definite, weakly decrescent and satisfies the inequality*

$$^cD_+^q V(t,x) \leq 0, \quad (t,x) \in R_+ \times S(\rho). \tag{3.4}$$

Then the trivial solution $x(t) \equiv 0$ *of the IVP* (2.1) *is equistable. If, in addition,* $V(t,x)$ *is decrescent, then the trivial solution* $x(t) \equiv 0$ *of the IVP* (2.1) *is uniformly stable.*

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of the IVP (2.1). Suppose that $t_0 \in R_+$ and $0 < \epsilon < \rho$ is given. Then it is possible to find a $\delta = \delta(t_0, \epsilon)$ such that

$$a(t_0, \delta) < b(\epsilon). \tag{3.5}$$

Choose $|x_0| < \delta$. Then we claim $|x(t)| < \epsilon$, $t \geq t_0$. If this is not true, there would exist a solution $x(t) = x(t, t_0, x_0)$ with $|x_0| < \delta$ and $t_1 > t_0$ such that

$$|x(t_1)| = \epsilon, \quad |x(t)| \leq \epsilon, \quad t \in [t_0, t_1], \tag{3.6}$$

so that we have, because of (3.1),

$$V(t_1, x(t_1)) \geq b(\epsilon). \tag{3.7}$$

This means $|x(t)| < \rho$ for $t \in [t_0, t_1]$ and hence condition (3.4) yields by Corollary 2.2, the inequality

$$V(t_1, x(t_1)) \leq V(t_0, x_0).$$

This gives using (3.3), (3.5), (3.6), (3.7),

$$b(\epsilon) \leq V(t_1, x(t_1)) \leq V(t_0, x_0) \leq a(t_0, |x_0|) \leq a(t_0, \delta) < b(\epsilon),$$

a contradiction, proving our claim. □

If $V(t,x)$ is only decrescent, we get by (3.2)

$$V(t_0, x_0) \leq a(|x_0|)$$

and choose $\delta = \delta(\epsilon) > 0$ such that $a(\delta) < b(\epsilon)$. Since δ is now independent of t_0, we have uniform stability of the trivial solution of IVP (2.1) by the foregoing considerations. The proof is therefore complete.

The next result offers the uniform asymptotic stability.

Theorem 3.5. *Let the assumptions Theorem* 3.4 *hold except that the condition* (3.4) *is replaced by*

$$^cD_+^q V(t,x) \leq -d(|x|), \quad (t,x) \in R_+ \times S(\rho), \quad d \in \mathcal{K}, \tag{3.8}$$

and $V(t,x)$ *is decrescent. Then the trivial solution* $x(t) \equiv 0$ *of the IVP* (2.1) *is uniformly asymptotically stable.*

Proof. Since (3.8) implies (3.4) and $V(t,x)$ is assumed to be decrescent, we get from Theorem 3.4 that the trivial solution of (2.1) is uniformly stable. Let $0<\epsilon<\rho$ and $\delta=\delta(\epsilon)>0$ correspond to uniform stability. Choose an $\epsilon_0\leq\rho$ and designate by $\delta_0=\delta(\epsilon_0)>0$ where ϵ_0 is fixed. Let us now choose $|x_0|<\delta_0$ and $T(\epsilon)=[\frac{a(\delta_0)}{d(\delta(\epsilon))}\Gamma(1+q)]^{\frac{1}{q}}$, where $\delta(\epsilon)$ corresponds to uniform stability. Suppose that $|x_0|<\delta_0$ and that we would have $|x(t)|\geq\delta(\epsilon)$ for $t_0\leq t\leq t_0+T(\epsilon)$. We then get using (3.8) and Corollary 2.3 with $g(t,u)\equiv 0$,

$$\begin{aligned} V(t,x(t)\leq V(t_0,x_0)) &- \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}d(|x(s)|)ds \\ &\leq a(|x_0|)-\frac{d(\delta)}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}ds\leq a(\delta_0)-\frac{d(\delta)}{\Gamma(1+q)}(t-t_0)^q, \end{aligned}$$

which for $t=t_0+T(\epsilon)$, reduces to

$$0<b(\delta(\epsilon))\leq V(t_0+T,x(t_0+T))\leq a(\delta_0)-\frac{d(\delta)}{\Gamma(1+q)}T^q\leq 0.$$

This contradiction proves that there exists a $t_1\in[t_0,t_0+T(\epsilon)]$ such that $|x(t_1)|<\delta(\epsilon)$. Thus, in any case, we have $|x(t)|<\epsilon$, $t\geq t_0+T(\epsilon)$, whenever $|x_0|<\delta_0$, proving the uniform stability of the trivial solution of the IVP (2.1) and the proof is complete. $\square$

4. STABILITY CONCEPTS IN TERMS OF TWO MEASURES

Let us begin by defining the following classes of functions for future use:

$$\begin{aligned} \mathcal{K} &= \{a\in C[R_+,R_+]: a(u) \text{ is strictly increasing in } u \text{ and } a(0)=0\}, \\ \mathcal{L} &= \{\sigma\in C[R_+,R_+]: \sigma(u) \text{ is strictly decreasing in } u \text{ and } \lim_{u\to\infty}\sigma(u)=0\}, \\ \mathcal{KL} &= \{a\in C[R_+^2,R_+]: a(t,s)\in\mathcal{K} \text{ for each } s \text{ and } a(t,s)\in\mathcal{L} \text{ for each } t\}, \\ \mathcal{CK} &= \{a\in C[R_+^2,R_+]: a(t,s)\in\mathcal{K} \text{ for each } t\}, \\ \Gamma &= \{h\in C[R_+\times R^n,R_+]: \inf h(t,x)=0\}, \\ \Gamma_0 &= \{h\in\Gamma: \inf_x h(t,x)=0 \text{ for each } t\in R_+\}. \end{aligned}$$

We shall now define the stability concepts for the system (2.1) in terms of two measures $h_0,h\in\Gamma$.

Definition 4.1. The differential system (2.1) is said to be

(S_1) (h_0,h)-equistable, if for each $\epsilon>0$ and $t_0\in R_+$, there exists a function $\delta=\delta(t_0,\epsilon)>0$ which is continuous in t_0 for each ϵ such that

$$h_0(t_0,x_0)<\delta$$

implies

$$h(t, x(t)) < \epsilon, \quad t \geq t_0,$$

where $x(t) = x(t, t_0, x_0)$ is any solution of (2.1);

(S_2) (h_0, h)-uniformly stable, if (S_1) holds with δ being independent of t_0;

(S_3) (h_0, h)-quasi-equi-asymptotically stable, if for each $\epsilon > 0$ and $t_0 \in R_+$, there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that

$$h_0(t_0, x_0) < \delta_0$$

implies

$$h(t, x(t)) < \epsilon, \quad t \geq t_0 + T;$$

(S_4) (h_0, h)-quasi-uniform asymptotically stable if (S_3) holds with δ_0 and T being independent of t_0;

(S_3^*) (h_0, h)-quasi-equi-asymptotically stable, if for any $\epsilon > 0$ and $\alpha > 0, t_0 \in R_+$, there exists a positive number $T = T(t_0, \epsilon, \alpha)$ such that

$$h_0(t_0, x_0) < \alpha$$

implies

$$h(t, x(t)) < \epsilon, \quad t \geq t_0 + T;$$

(S_4^*) (h_0, h)-quasi uniform asymptotically stable if (S_3^*) holds with T being independent of t_0;

(S_5) (h_0, h)-asymptotically stable if (S_1) holds and given $t_0 \in R_+$, there exists a $\delta_0 = \delta_0(t_0) > 0$ such that

$$h_0(t_0, x_0) < \delta_0$$

implies $\lim\limits_{t \to \infty} h(t, x(t)) = 0$;

(S_6) (h_0, h)-equi-asymptotically stable, if (S_1) and (S_3) hold together;

(S_7) (h_0, h)-uniformly asymptotically stable if (S_2) and (S_4) hold simultaneously;

(S_8) (h_0, h)-unstable if (S_1) fails to hold.

Sometimes the notion of quasi asymptotic stability may be relaxed somewhat as in (S_3^*) and (S_4^*). We shall use these notions to define Lagrange stability later.

A few choices of the two measures (h_0, h) given below will demonstrate the generality of the Definition 4.1. Furthermore, (h_0, h)-stability concepts enable us to unify a variety of stability notions found in the literature. It is easy to see that Definition 4.1 reduces to

(1) the well known stability of the trivial solution $x(t) \equiv 0$ of (2.1) or equivalently, of the invariant set $\{0\}$, if $h(t, x) = h_0(t, x) = \|x\|$;

(2) the stability of the prescribed motion $x_0(t)$ of (2.1) if

$$h(t, x) = h_0(t, x) = \|x - x_0(t)\|;$$

(3) the partial stability of the trivial solution of (2.1) if

$$h(t,x) = \|x\|s, \ 1 \leq s \leq n$$

and

$$h_0(t,x) = \|x\|$$

(4) the stability of asymptotically invariant set $\{0\}$, if

$$h(t,x) = h_0(t,x) = \|x\| + \sigma(t)$$

where $\sigma \in \mathcal{L}$:

(5) the stability of the invariant set $A \subset \mathbb{R}^n$ if $h(t,x) = h_0(t,x) = d(x,A)$, where $d(x,A)$ is the distance of x from the set A;

(6) the stability of conditionally invariant set B with respect to A, where $A \subset B \subset \mathbb{R}^n$, if $h(t,x) = d(x,B)$ and $h_0(t,x) = d(x,A)$.

We recall that the set $\{0\}$ is said to be asymptotically invariant if given $\epsilon > 0$, there exists a $\tau(\epsilon) > 0$ such that $x_0 = 0$ implies $\|x(t,t_0,0)\| < \epsilon$ for $t \geq t_0 \geq \tau(\epsilon)$. Several other combinations of choices are possible for h_0, h in addition to those given in (1) through (6).

Definition 4.2. Let $h_0, h \in \Gamma$. Then, we say that

(i) h_0 is finer than h if there exists a $\rho > 0$ and a function $\phi \in \mathcal{CK}$ such that $h_0(t,x) < \rho$ implies $h(t,x) \leq \phi(t, h_0(t,x))$;

(ii) h_0 is uniformly finer than h if in (i) ϕ is independent of t;

(iii) h_0 is asymptotically finer than h if there exists a $\rho > 0$ and a function $\phi \in \mathcal{KL}$ such that $h_0(t,x) < \rho$ implies $h(t,x) \leq \phi(h_0(t,x),t)$.

Definition 4.3. Let $V \in C[R_+ \times \mathbb{R}^n, R_+]$. Then V is said to be

(i) h-positive definite if there exists a $\rho > 0$ and a function $b \in \mathcal{K}$ such that $b(h(t,x)) \leq V(t,x)$ whenever $h(t,x) < \rho$;

(ii) h-decrescent if there exists a $\rho > 0$ and a function $a \in \mathcal{K}$ such that $V(t,x) \leq a(h(t,x))$ whenever $h(t,x) < \rho$;

(iii) h-weakly decrescent if there exists a $\rho > 0$ and a function $a \in \mathcal{CK}$ such that $V(t,x) \leq a(t, h(t,x))$ whenever $h(t,x) < \rho$;

(iv) h-asymptotically decrescent if there exists a $\rho > 0$ and a function $a \in \mathcal{KL}$ such that $V(t,x) \leq a(h(t,x),t)$ whenever $h(t,x) < \rho$.

Corresponding to Definition 4.1, we need the stability definition for the trivial solution of the comparison equation

$$^cD^q u = g(t,u), \quad u(t_0) = u_0 \geq 0, \tag{4.1}$$

where $g \in C[R_+^2, \mathbb{R}]$ and $g(t,0) \equiv 0$. We merely state one of the concepts.

Definition 4.4. The trivial solution $u(t) \equiv 0$ of (4.1) is said to be equistable if for any $\epsilon > 0$ and $t_0 \in R_+$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ which is continuous in t_0 such that $u_0 < \delta$ implies $u(t, t_0, u_0) < \epsilon$, $t \geq t_0, u(t, t_0, u_0)$ being any solution of (4.1).

5. STABILITY CRITERIA IN TERMS OF TWO MEASURES

Let us now establish some sufficient conditions for the (h_0, h) stability properties of the differential system (2.1).

Theorem 5.1. *Assume that*

(A_0) $h, h_0 \in \Gamma$ *and* h_0 *is uniformly finer than* h*;*
(A_1) $V \in C[R_+ \times \mathbb{R}^n, R_+], V(t,x)$ *is locally Lipschitzian in* x, V *is* h*-positive definite and* h_0*-decrescent;*
(A_2) $g \in C[R_+^2, \mathbb{R}]$ *and* $g(t, 0) \equiv 0$*;*
(A_3) ${}^cD_+^q V(t,x) \leq g(t, V(t,x))$ *for* $(t,x) \in S(h, \rho)$ *for some* $\rho > 0$*, where* $S(h,\rho) = \{(t,x) \in R_+ \times \mathbb{R}^n : h(t,x) < \rho\}$.

Then, the stability properties of the trivial solution of (4.1) *imply the corresponding* (h_0, h)*-stability properties of* (2.1)*.*

Proof. We shall only prove (h_0, h)-equi-asymptotic stability of (2.1). For this purpose, let us first prove (h_0, h) equistability.

Since V is h-positive definite, there exists a $\lambda \in (0, \rho]$ and $b \in \mathcal{K}$ such that

$$b(h(t,x)) \leq V(t,x), \quad (t,x) \in S(h, \lambda). \tag{5.1}$$

Let $0 < \epsilon < \lambda$ and $t_0 \in R_+$ be given and suppose that the trivial solution of (4.1) is equistable. Then, given $b(\epsilon) > 0$ and $t_0 \in R_+$, there exists a function $\delta_1 = \delta_1(t_0, \epsilon)$ that is continuous in t_0 such that $u_0 < \delta_1$ implies

$$u(t, t_0, u_0) < b(\epsilon), \quad t \geq t_0, \tag{5.2}$$

where $u(t, t_0, u_0)$ is any solution of (4.1). We choose $u_0 = V(t_0, x_0)$. Since V is h_0-decrescent and h_0 is uniformly finer than h, there exists a $\lambda_0 > 0$ and a function $a \in \mathcal{K}$ such that for $(t_0, x_0) \in S(h_0, \lambda_0)$,

$$h(t_0, x_0) < \lambda \quad \text{and} \quad V(t_0, x_0) \leq a(h_0(t_0, x_0)). \tag{5.3}$$

It then follows from (5.1) that

$$b(h(t_0, x_0)) \leq V(t_0, x_0) \leq a(h_0(t_0, x_0)), \quad (t_0, x_0) \in S(h_0, \lambda_0). \tag{5.4}$$

Choose $\delta = \delta(t_0, \epsilon)$ such that $\delta \in (0, \lambda_0]$, $a(\delta) < \delta_1$ and let $h_0(t_0, x_0) < \delta$. Then (5.4) shows that $h(t_0, x_0) < \epsilon$ since $\delta_1 < b(\epsilon)$. We claim that

$$h(t, x(t)) < \epsilon, \quad t \geq t_0$$

whenever $h_0(t_0, x_0) < \delta$, where $x(t) = x(t, t_0, x_0)$ is any solution of (2.1) with $h_0(t_0, x_0) < \delta$. If this is not true, then there exists a $t_1 > t_0$ and a solution $x(t)$ of (2.1) such that

$$h(t_1, x(t_1)) = \epsilon \quad \text{and} \quad h(t, x(t)) < \epsilon, \quad t_0 \le t < t_1, \tag{5.5}$$

in view of the fact that $h(t_0, x_0) < \epsilon$ whenever $h_0(t_0, x_0) < \delta$. This means that $x(t) \in S(h, \lambda)$ for $[t_0, t_1]$ and hence by Theorem 2.1 we have

$$V(t, x(t)) \le r(t, t_0, u_0), \quad t_0 \le t \le t_1, \tag{5.6}$$

where $r(t, t_0, u_0)$ is the maximal solution of (4.1). Now the relations (5.1), (5.2), (5.5) and (5.6) yield

$$b(\epsilon) \le V(t_1, x(t_1)) \le r(t_1, t_0, u_0) < b(\epsilon),$$

a contradiction proving (h_0, h)-equistability of (2.1).

Suppose next that the trivial solution of (4.1) is quasi-equi-asymptotically stable. From the (h_0, h)-equistability, we set $\epsilon = \lambda$ so that $\hat{\delta_0} = \delta(t_0, \lambda)$. Now let $0 < \eta < \lambda$. Then, by quasi-equi-asymptotic stability of (4.1), we have that, given $b(\eta) > 0$ and $t_0 \in R_+$, there exist positive numbers $\delta_1^* = \delta_1^*(t_0)$ and $T = T(t_0, \eta) > 0$ such that

$$u_0 < \delta_1^* \quad \text{implies} \quad u(t, t_0, u_0) < b(\eta), \quad t \ge t_0 + T. \tag{5.7}$$

Choosing $u_0 = V(t_0, x_0)$ as before, we find a $\delta_0^* = \delta_0^*(t_0) > 0$ such that $\delta_0^* \in (0, \lambda_0]$ and $a(\delta_0^*) < \delta_1^*$. Let $\delta_0 = \min(\delta_0^*, \hat{\delta_0})$ and $h_0(t_0, x_0) < \delta_0$. This implies that $h(t, x(t)) < \lambda$, $t \ge t_0$ and hence the estimate (5.6) is valid for all $t \ge t_0$. Suppose now that there exists a sequence $\{t_k\}, t_k \ge t_0 + T$, $t_k \to \infty$ as $k \to \infty$ such that $\eta \le h(t_k, x(t_k))$ where $x(t)$ is any solution of (2.1) such that $h_0(t_0, x_0) < \delta_0$. This leads to a contradiction

$$b(\eta) \le V(t_k, x(t_k)) \le r(t_k, t_0, u_0) < b(\eta)$$

because of (5.6) and (5.7). Hence the system (2.1) is (h_0, h)-equi-asymptotically stable and the proof is complete. $\square$

We have assumed in Theorem 5.1 stronger requirements on V, h, h_0 only to unify all the stability criteria in one theorem. This obviously puts burden on the comparison equation (4.1). However, to obtain only non-uniform stability criteria, we could weaken certain assumptions of Theorem 5.1 as in the next result. The details of proof are omitted.

Theorem 5.2. *Assume that conditions* (A_0)–(A_3) *hold with the following changes:*

(i) h_0, $h \in \Gamma_0$ *and* h_0 *is finer than* h *and*

(ii) V *is* h_0*-weakly decrescent.*

Then, the equi or uniform stability properties of the trivial solution of (4.1) *imply the corresponding equi* (h_0, h)*-stability properties of* (2.1).

We shall next consider a result on (h_0, h)-asymptotic stability which generalizes classical results.

Theorem 5.3. *Assume that*

(i) h_0, $h \in \Gamma_0$ *and* h_0 *is finer than* h*;*
(ii) $V \in C[R_+ \times \mathbb{R}^n, R_+]$, $V(t,x)$ *is locally Lipschitzian in* x, V *is* h*-positive definite and* h_0*-weakly decrescent;*
(iii) $W \in C[R_+ \times \mathbb{R}^n,\ R_+], W(t,x)$ *is locally Lipschitzian in* x, W *is* h*-positive definite,* $D_-^+ W(t,x)$ *is bounded from above or from below on* $S(h,\rho)$ *and for* $(t,x) \in S(h,\rho)$*,*

$$^cD_+^q V(t,x) \le -C(W(t,x)), \quad C \in \mathcal{K}.$$

Then, the system (2.1) *is* (h_0,h)*-asymptotically stable.*

Proof. By Theorem 5.2 with $g \equiv 0$, if follows that the system (2.1) is (h_0,h)-equistable. Hence it is enough to prove that given $t_0 \in R_+$, there exists a $\delta_0 = \delta_0(t_0) > 0$ such that $h_0(t_0,x_0) < \delta_0$ implies $h(t,x(t)) \to 0$ as $t \to \infty$.

For $\epsilon = \lambda$, let $\delta_0 = \delta(t_0,\lambda)$ be associated with (h_0,h)-equistability. We suppose that $h_0(t_0,x_0) < \delta_0$. Since $W(t,x)$ is h-positive definite, it is enough to prove that $\lim\limits_{t\to\infty} W(t,x(t)) = 0$ for any solution $x(t)$ of (2.1) with

$$h_0(t_0,x_0) < \delta_0.$$

We first note that $\liminf\limits_{t\to\infty} W(t,x(t)) = 0$. For otherwise, in view of (iii), we get $V(t,x(t)) \to -\infty$ as $t \to \infty$.

Suppose that $\lim\limits_{t\to\infty} W(t,x(t)) \neq 0$. Then, for any $\epsilon > 0$, there exist divergent sequences $\{t_n\}, \{t_n^*\}$ such that $t_i < t_i^* < t_{i+1}$, $i = 1,2,\ldots$, and

$$\begin{cases} W(t_i,x(t_i)) = \frac{\epsilon}{2}, \\ W(t_i^*,x(t_i^*)) = \epsilon, \end{cases}$$

and

$$\frac{\epsilon}{2} < W(t,x(t)) < \epsilon, \quad t \in (t_i,t_i^*). \tag{5.8}$$

Of course, we could also have, instead of (5.8),

$$W(t_i,x(t_i)) = \epsilon, \quad W(t_i^*,x(t_i^*)) = \frac{\epsilon}{2}, \quad W(t,x(t)) \in (\frac{\epsilon}{2},\epsilon), \quad t \in (t_i,t_i^*) \tag{5.9}$$

Suppose that $D^+W(t,x) \leq M$. Then, it is easy to obtain, using (5.8), the relation $t_i^* - t_i > \frac{\epsilon}{2M}$. In view of (iii), we have for large n,

$$0 \leq V(t_n^*, x(t_n^*)) \leq V(t_0, x_0) - \frac{1}{\Gamma(q)} \sum_{1\leq i\leq n} \int_{t_i}^{t_i^*} (t_i^* - s)^{q-1} C(W(s, x(s)))ds$$
$$\leq V(t_0, x_0) - nC(\frac{\epsilon}{2}) \frac{\epsilon}{2M} \frac{1}{\Gamma(q+1)} < 0,$$

which is a contradiction. Thus, $W(t, x(t)) \to 0$ as $t \to \infty$ and hence $h(t, x(t)) \to 0$ as $t \to \infty$. The argument is similar when D^+W is bounded from below and we use (5.9). The proof is therefore complete. □

Corollary 5.4 (Marachkov's theorem)**.** *Suppose that f is bounded on $R_+ \times S(\rho)$. Then the trivial solution of* (2.1) *is asymptotically stable if there exist $C \in \mathcal{K}$ and $V \in C[R_+ \times S(\rho), R_+]$ such that*

(i) *V is positive definite, $V(t,0) \equiv 0$ and $V(t,x)$ is locally Lipschitzian in x;*
(ii) *${}^cD_+^q V(t,x) \leq -C(\|x\|)$, $(t,x) \in R_+ \times S(\rho)$, $C \in \mathcal{K}$.*

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