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RIEMANN-LIOUVILLE FRACTIONAL OPIAL INEQUALITIES FOR SEVERAL FUNCTIONS WITH APPLICATIONS

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ABSTRACT. A large variety of very general $L_p (1 \leq p \leq \infty)$ form Opial type inequalities ([15]) is presented involving Riemann-Liouville fractional derivatives ([5], [12], [13], [14]) of several functions in different orders and powers.

From the established results derive several other particular results of special interest. Applications of some of these special inequalities are given in proving uniqueness of solution and in giving upper bounds to solutions of initial value fractional problems involving a very general system of several fractional differential equations. Upper bounds to various Riemann-Liouville fractional derivatives of the solutions that are involved in the above systems are given too.

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0. INTRODUCTION

Opial inequalities appeared for the first time in [15] and then many authors dealt with them in different directions and for various cases. For a complete recent account on the activity of this field see [3], and still it remains a very active area of research. One of their main attractions to these inequalities is their applications, especially to proving uniqueness and upper bounds of solution of initial value problems in differential equations. The author was the first to present Opial inequalities involving fractional derivatives of functions in [4], [5] with applications to fractional differential equations. See also [8], [9].

Fractional derivatives come up naturally in a number of fields, especially in Physics, see the recent book [11]. To name a few topics such as, fractional Kinetics of Hamiltonian Chaotic systems, Polymer Physics and Rheology, Regular variation in Thermodynamics, Biophysics, fractional time evolution, fractal time series, etc. One there deals also with stochastic fractional-difference equations and fractional diffusion equations. Great applications of these can be found in the study of DNA sequences. Other fractional differential equations arise in the study of suspensions, coming from

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the fluid dynamical modeling of certain blood flow phenomena. An excellent account in the study of fractional differential equations is in the recent book [16].

The study of fractional calculus started from 1695 by L'Hospital and Leibniz, also continued later by J. Fourier in 1822 and Abel in 1823, and continues to our days in an increased fashion due to its many applications and necessity to deal with fractional phenomena and structures.

In this paper the author is greatly motivated and inspired by the very important papers [1], [2]. Of course there the authors are dealing with other kinds of derivative. Here the author continues his study of Riemann-Liouville fractional Opial inequalities now involving several different functions and produces a wide variety of corresponding results with important applications to systems of several fractional differential equations. This article is a generalization of the author's earlier article [6].

We start in Section 1 with Background, we continue in Section 2 with the main results and we finish in Section 3 with applications.

To give an idea to the reader of the kind of inequalities we are dealing with, briefly we mention a simple one

$$\int_0^x \left(\sum_{j=1}^M |(D^\gamma f_j)(w)|\,|(D^\nu f_j)(w)|\right) dw$$
$$\leq \left(\frac{x^{\nu-\gamma}}{2\Gamma(\nu-\gamma)\sqrt{\nu-\gamma}\sqrt{2\nu-2\gamma-1}}\right)\left\{\int_0^x \left(\sum_{j=1}^M ((D^\nu f_j)(w))^2\right) dw\right\}, \qquad (*)$$

$x \geq 0$, for functions $f_j \in L_1(0,x)$, $j = 1,\ldots,M \in \mathbb{N}$; $\nu > \gamma \geq 0$ etc. Here $D^\beta f$ stands for the Riemann-Liouville fractional derivative of f of order $\beta \geq 0$. Furthermore one system of fractional differential equations we are dealing with briefly is of the form

$$(D^\nu f_j)(t) = F_j(t, \{(D^{\gamma_i} f_1)(t)\}_{i=1}^r, \quad \{(D^{\gamma_i} f_2)(t)\}_{i=1}^r, \ldots,$$
$$\{(D^{\gamma_i} f_M)(t)\}_{i=1}^r), \qquad \text{all } t \in [a,b], \qquad (**)$$

$j = 1,\ldots,M$; $D^{\nu-k} f_j(0) = \alpha_{kj} \in \mathbb{R}$, $k = 1,\ldots,[\nu]+1$. Here $[\nu]$ is the integral part of ν.

1. BACKGROUND

We need

Definition 1 (see [10], [13], [14]). Let $\alpha \in \mathbb{R}_+ - \{0\}$. For any $f \in L_1(0,x)$; $x \in \mathbb{R}_+ - \{0\}$, the *Riemann-Liouville fractional integral* of f of order α is defined by

$$(J_\alpha f)(s) := \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} f(t)dt, \quad \forall s \in [0,x], \qquad (1)$$

and the *Riemann-Liouville fractional derivative of* f of order α by

$$D^{\alpha} f(s) := \frac{1}{\Gamma(m-a)} \left(\frac{d}{ds}\right)^{m} \int_{0}^{s} (s-t)^{m-\alpha-1} f(t) dt, \tag{2}$$

where $m := [\alpha]+1$, $[\cdot]$ is the integral part. In addition, we set $-D^{0} f := f := J_{0} f$, $J_{-\alpha} f = D^{\alpha} f$ if $\alpha > 0$, $D^{-\alpha} f := J_{\alpha} f$, if $0 < \alpha \leq 1$. If $\alpha \in \mathbb{N}$, then $D^{\alpha} f = f^{(\alpha)}$ the ordinary derivative.

Definition 2 ([10])**.** We say that $f \in L_1(0,x)$ has an L_∞ *fractional derivative* $D^{\alpha} f$ in $[0,x]$, $x \in \mathbb{R}_{+} - \{0\}$, iff $D^{\alpha-k} f \in C([0,x])$, $k = 1, \ldots, m := [\alpha]+1$; $\alpha \in \mathbb{R}_{+} - \{0\}$, and $D^{\alpha-1} f \in AC([0,x])$ (absolutely continuous functions) and $D^{\alpha} f \in L_\infty(0,x)$.

We need

Lemma 3 ([10])**.** *Let $\alpha \in \mathbb{R}_{+}$, $\beta > \alpha$, let $f \in L_1(0,x)$, $x \in \mathbb{R}_{+} - \{0\}$, have an L_∞ fractional derivative $D^{\beta} f$ in $[0,x]$, and let $D^{\beta-k} f(0) = 0$ for $k = 1, \ldots, [\beta]+1$. Then*

$$D^{\alpha} f(s) = \frac{1}{\Gamma(\beta-\alpha)} \int_{0}^{s} (s-t)^{\beta-\alpha-1} D^{\beta} f(t) dt, \quad \forall\, s \in [0,x]. \tag{3}$$

Clearly here $D^{\alpha} f \in AC([0,x])$ for $\beta-\alpha \geq 1$ and in $C([0,x])$ for $\beta - \alpha \in (0,1)$, hence $D^{\alpha} f \in L_\infty(0,x)$ and $D^{\alpha} f \in L_1(0,x)$.

2. MAIN RESULTS

Here we use a lot the following basic inequalities. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$, $n \in \mathbb{N}$, then

$$a_1^r + \cdots + a_n^r \leq (a_1 + \cdots + a_n)^r, \quad r \geq 1, \tag{4}$$

and

$$a_1^r + \cdots + a_n^r \leq n^{1-r} (a_1 + \cdots + a_n)^r, \quad 0 \leq r \leq 1. \tag{5}$$

Our first result follows next

Theorem 4. *Let $\alpha_1, \alpha_2 \in \mathbb{R}_{+}$, $\beta > \alpha_1, \alpha_2$ and let $f_j \in L_1(0,x)$, $j = 1, \ldots, M \in \mathbb{N}$, $x \in \mathbb{R}_{+} - \{0\}$ have, respectively, L_∞ fractional derivatives $D^{\beta} f_j$ in $[0,x]$, and let $D^{\beta-k} f_j(0) = 0$, for $k = 1, \ldots, [\beta]+1$; $j = 1, \ldots, M$. Consider also $p(t) > 0$ and $q(t) \geq 0$, with all $p(t)$, $\frac{1}{p(t)}$, $q(t) \in L_\infty(0,x)$. Let $\lambda_\beta > 0$ and $\lambda_{\alpha_1}, \lambda_{\alpha_2} \geq 0$, such that $\lambda_\beta < p$, where $p > 1$. Set*

$$P_i(s) := \int_{0}^{s} (s-t)^{\frac{p(\beta-\alpha_i-1)}{p-1}} (p(t))^{-1/(p-1)}\, dt, \quad i = 1,2; \quad 0 \leq s \leq x, \tag{6}$$

$$A(s) := \frac{q(s) (P_1(s))^{\lambda_{\alpha_1}(\frac{p-1}{p})} (P_2(s))^{\lambda_{\alpha_2}(\frac{p-1}{p})} (p(s))^{-\lambda_\beta/p}}{(\Gamma(\beta-\alpha_1))^{\lambda_{\alpha_1}} (\Gamma(\beta-\alpha_2))^{\lambda_{\alpha_2}}}, \tag{7}$$

$$A_0(x) := \left(\int_{0}^{x} (A(s))^{p/(p-\lambda_\beta)} ds\right)^{(p-\lambda_\beta)/p}, \tag{8}$$

and

$$\delta_1^* := \begin{cases} M^{1-((\lambda_{\alpha_1}+\lambda_\beta)/p)}, & \text{if } \lambda_{\alpha_1}+\lambda_\beta \le p, \\ 2^{\left(\frac{\lambda_{\alpha_1}+\lambda_\beta}{p}\right)-1}, & \text{if } \lambda_{\alpha_1}+\lambda_\beta \ge p. \end{cases} \tag{9}$$

Call

$$\varphi_1(x) := (A_0(x)\,|_{\lambda_{\alpha_2}=0}) \left(\frac{\lambda_\beta}{\lambda_{\alpha_1}+\lambda_\beta}\right)^{\lambda_\beta/p} \tag{10}$$

If $\lambda_{\alpha_2}=0$, *we obtain that,*

$$\int_0^x q(s)\left(\sum_{j=1}^M |D^{\alpha_1} f_j(s)|^{\lambda_{\alpha_1}} |D^\beta f_j(s)|^{\lambda_\beta}\right) ds$$
$$\le \delta_1^* \varphi_1(x)\left[\int_0^x p(s)\left(\sum_{j=1}^M |D^\beta f_j(s)|^p\right) ds\right]^{\left(\frac{\lambda_{\alpha_1}+\lambda_\beta}{p}\right)}. \tag{11}$$

Proof. By Theorem 4 of [7] we obtain

$$\int_0^x q(s)\left[\left|D^{\alpha_1} f_j(s)\right|^{\lambda_{\alpha_1}}\left|D^\beta f_j(s)\right|^{\lambda_\beta} + \left|D^{\alpha_1} f_{j+1}(s)\right|^{\lambda_{\alpha_1}}\left|D^\beta f_{j+1}(s)\right|^{\lambda_\beta}\right] ds$$
$$\le \left(A_0(x)|_{\lambda_{\alpha_2}=0}\right)\left(\frac{\lambda_\beta}{\lambda_{\alpha_1}+\lambda_\beta}\right)^{(\lambda_\beta/p)}$$
$$\times\, \delta_1 \left[\int_0^x p(s)\left[\left|D^\beta f_j(s)\right|^p + \left|D^\beta f_{j+1}(s)\right|^p\right] ds\right]^{\left(\frac{\lambda_{\alpha_1}+\lambda_\beta}{p}\right)}, \tag{12}$$

$j = 1, 2 \ldots, M-1$, where

$$\delta_1 := \begin{cases} 2^{1-((\lambda_{\alpha_1}+\lambda_\beta)/p)}, & \text{if } \lambda_{\alpha_1}+\lambda_\beta \le p, \\ 1, & \text{if } \lambda_{\alpha_1}+\lambda_\beta \ge p. \end{cases} \tag{13}$$

Hence by adding all the above we find

$$\int_0^x q(s)\left\{\sum_{j=1}^{M-1}\left[|D^{\alpha_1} f_j(s)|^{\lambda_{\alpha_1}}|D^\beta f_j(s)|^{\lambda_\beta} + |D^{\alpha_1} f_{j+1}(s)|^{\lambda_{\alpha_1}}|D^\beta f_{j+1}(s)|^{\lambda_\beta}\right]\right\} ds$$
$$\le \delta_1\varphi_1(x)\left\{\sum_{j=1}^{M-1}\left[\int_0^x p(s)\left[|D^\beta f_j(s)|^p + |D^\beta f_{j+1}(s)|^p\right] ds\right]^{\frac{(\lambda_{\alpha_1}+\lambda_\beta)}{p}}\right\}. \tag{14}$$

Also it holds

$$\int_0^x q(s)\left[|D^{\alpha_1} f_1(s)|^{\lambda_{\alpha_1}}|D^\beta f_1(s)|^{\lambda_\beta} + |D^{\alpha_1} f_M(s)|^{\lambda_{\alpha_1}}|D^\beta f_M(s)|^{\lambda_\beta}\right] ds$$
$$\le \delta_1\varphi_1(x)\left[\int_0^x p(s)\left[|D^\beta f_1(s)|^p + |D^\beta f_M(s)|^p\right] ds\right]^{\frac{(\lambda_{\alpha_1}+\lambda_\beta)}{p}}. \tag{15}$$

Call

$$\varepsilon_1 = \begin{cases} 1, & \text{if } \lambda_{\alpha_1}+\lambda_\beta \ge p, \\ M^{1-\left(\frac{\lambda_{\alpha_1}+\lambda_\beta}{p}\right)}, & \text{if } \lambda_{\alpha_1}+\lambda_\beta \le p. \end{cases} \tag{16}$$

Adding (14) and (15), and using (4) and (5) we have

$$2\int_0^x q(s)\left(\sum_{j=1}^{M}|D^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}|D^{\beta}f_j(s)|^{\lambda_\beta}\right)ds \tag{17}$$

$$\leq \delta_1\varphi_1(x)\left\{\sum_{j=1}^{M-1}\left[\int_0^x p(s)\left[|D^\beta f_j(s)|^p+|D^\beta f_{j+1}(s)|^p\right]ds\right]^{\left(\frac{\lambda_{\alpha_1}+\lambda_\beta}{p}\right)}\right.$$

$$\left.+\left[\int_0^x p(s)\left[|D^\beta f_1(s)|^p+|D^\beta f_M(s)|^p\right]ds\right]^{\left(\frac{\lambda_{\alpha_1}+\lambda_\beta}{p}\right)}\right\}$$

$$\leq \delta_1\varepsilon_1\varphi_1(x)\left\{\int_0^x p(s)\left(2\sum_{j=1}^{M}|D^\beta f_j(s)|^p\right)ds\right\}^{\left(\frac{\lambda_{\alpha_1}+\lambda_\beta}{p}\right)}. \tag{18}$$

We have proved

$$\int_0^x q(s)\left(\sum_{j=1}^{M}|D^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}|D^{\beta}f_j(s)|^{\lambda_\beta}\right)ds$$

$$\leq \delta_1\left(2^{\left(\frac{\lambda_{\alpha_1}+\lambda_\beta}{p}\right)-1}\right)\varepsilon_1\varphi_1(x)\left\{\int_0^x p(s)\left[\sum_{j=1}^{M}|D^\beta f_j(s)|^p\right]ds\right\}^{\left(\frac{\lambda_{\alpha_1}+\lambda_\beta}{p}\right)}. \tag{19}$$

Clearly here we have

$$\delta_1^* = \delta_1\left(2^{\left(\frac{\lambda_{\alpha_1}+\lambda_\beta}{p}\right)-1}\right)\varepsilon_1. \tag{20}$$

From (19) and (20) we derive (11). □

Next we give

Theorem 5. *All here as in Theorem* 4. *Denote*

$$\delta_3 := \begin{cases} 2^{\lambda_{\alpha_2}/\lambda_\beta}-1, & \text{if } \lambda_{\alpha_2}\geq\lambda_\beta,\\ 1, & \text{if } \lambda_{\alpha_2}\leq\lambda_\beta,\end{cases} \tag{21}$$

$$\varepsilon_2 := \begin{cases} 1, & \text{if } \lambda_\beta+\lambda_{\alpha_2}\geq p,\\ M^{1-\left(\frac{\lambda_\beta+\lambda_{\alpha_2}}{p}\right)}, & \text{if } \lambda_\beta+\lambda_{\alpha_2}\leq p,\end{cases} \tag{22}$$

and

$$\varphi_2(x) := \left(A_0(x)\left|_{\lambda_{\alpha_1}=0}\right.\right)2^{\left(\frac{p-\lambda_\beta}{p}\right)}\left(\frac{\lambda_\beta}{\lambda_{\alpha_2}+\lambda_\beta}\right)^{\lambda_\beta/p}\delta_3^{(\lambda_\beta/p)}. \tag{23}$$

If $\lambda_{\alpha_1} = 0$, *then it holds*

$$\int_0^x q(s) \left\{ \left\{ \sum_{j=1}^{M-1} \left[|D^{\alpha_2} f_{j+1}(s)|^{\lambda_{\alpha_2}} |D^{\beta} f_j(s)|^{\lambda_\beta} + |D^{\alpha_2} f_j(s)|^{\lambda_{\alpha_2}} |D^{\beta} f_{j+1}(s)|^{\lambda_\beta} \right] \right\} \right.$$
$$\left. + \left[|D^{\alpha_2} f_M(s)|^{\lambda_{\alpha_2}} |D^{\beta} f_1(s)|^{\lambda_\beta} + |D^{\alpha_2} f_1(s)|^{\lambda_{\alpha_2}} |D^{\beta} f_M(s)|^{\lambda_\beta} \right] \right\} ds$$
$$\leq 2^{\left(\frac{\lambda_\beta + \lambda_{\alpha_2}}{p}\right)} \varepsilon_2 \varphi_2(x) \left\{ \int_0^x p(s) \left[\sum_{j=1}^{M} |D^{\beta} f_j(s)|^p \right] ds \right\}^{\left(\frac{\lambda_\beta + \lambda_{\alpha_2}}{p}\right)}. \tag{24}$$

Proof. From Theorem 5 of [7] we have

$$\int_0^x q(s) \left[|D^{\alpha_2} f_{j+1}(s)|^{\lambda_{\alpha_2}} |D^{\beta} f_j(s)|^{\lambda_\beta} + |D^{\alpha_2} f_j(s)|^{\lambda_{\alpha_2}} |D^{\beta} f_{j+1}(s)|^{\lambda_\beta} \right] ds$$
$$\leq \varphi_2(x) \left(\int_0^x p(s) \left[|D^{\beta} f_j(s)|^p + |D^{\beta} f_{j+1}(s)|^p \right] ds \right)^{\frac{(\lambda_\beta + \lambda_{\alpha_2})}{p}}, \tag{25}$$

for $j = 1, \ldots, M-1$. Hence by adding all of the above we get

$$\int_0^x q(s) \left(\sum_{j=1}^{M-1} \left[|D^{\alpha_2} f_{j+1}(s)|^{\lambda_{\alpha_2}} |D^{\beta} f_j(s)|^{\lambda_\beta} + |D^{\alpha_2} f_j(s)|^{\lambda_{\alpha_2}} |D^{\beta} f_{j+1}(s)|^{\lambda_\beta} \right] \right) ds$$
$$\leq \varphi_2(x) \left(\sum_{j=1}^{M-1} \left(\int_0^x p(s) \left[|D^{\beta} f_j(s)|^p + |D^{\beta} f_{j+1}(s)|^p \right] ds \right)^{\frac{(\lambda_\beta + \lambda_{\alpha_2})}{p}} \right). \tag{26}$$

Similarly it holds

$$\int_0^x q(s) \left[|D^{\alpha_2} f_M(s)|^{\lambda_{\alpha_2}} |D^{\beta} f_1(s)|^{\lambda_\beta} + |D^{\alpha_2} f_1(s)|^{\lambda_{\alpha_2}} |D^{\beta} f_M(s)|^{\lambda_\beta} \right] ds$$
$$\leq \varphi_2(x) \left(\int_0^x p(s) \left[|D^{\beta} f_1(s)|^p + |D^{\beta} f_M(s)|^p \right] ds \right)^{\frac{(\lambda_\beta + \lambda_{\alpha_2})}{p}}. \tag{27}$$

Adding (26) and (27) and using (4), (5) we derive (24). ∎

It follows the general case

Theorem 6. *All here as in Theorem* 4. *Denote*

$$\tilde{\gamma_1} := \begin{cases} 2^{((\lambda_{\alpha_1} + \lambda_{\alpha_2})/\lambda_\beta)} - 1, & \text{if } \lambda_{\alpha_1} + \lambda_{\alpha_2} \geq \lambda_\beta, \\ 1, & \text{if } \lambda_{\alpha_1} + \lambda_{\alpha_2} \leq \lambda_\beta, \end{cases} \tag{28}$$

and

$$\tilde{\gamma_2} := \begin{cases} 1, & \text{if } \lambda_{\alpha_1} + \lambda_{\alpha_2} + \lambda_\beta \geq p, \\ 2^{1-((\lambda_{\alpha_1} + \lambda_{\alpha_2} + \lambda_\beta)/p)}, & \text{if } \lambda_{\alpha_1} + \lambda_{\alpha_2} + \lambda_\beta \leq p. \end{cases} \tag{29}$$

Set

$$\varphi_3(x) := A_0(x)\left(\frac{\lambda_\beta}{(\lambda_{\alpha_1}+\lambda_{\alpha_2})(\lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta)}\right)^{\frac{\lambda_\beta}{p}}\left[\lambda_{\alpha_1}^{\frac{\lambda_\beta}{p}}\,\tilde{\gamma}_2 + 2^{\frac{(p-\lambda_\beta)}{p}}(\tilde{\gamma}_1\lambda_{\alpha_2})^{\frac{\lambda_\beta}{p}}\right], \tag{30}$$

and

$$\varepsilon_3 := \begin{cases} 1, & \text{if } \lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta \geq p, \\ M^{1-(\frac{\lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta}{p})}, & \text{if } \lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta \leq p. \end{cases} \tag{31}$$

Then

$$\begin{aligned}
&\int_0^x q(s)\Bigg[\sum_{j=1}^{M-1}\Big[|D^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}|D^{\alpha_2}f_{j+1}(s)|^{\lambda_{\alpha_2}}|D^\beta f_j(s)|^{\lambda_\beta} \\
&+ |D^{\alpha_2}f_j(s)|^{\lambda_{\alpha_2}}|D^{\alpha_1}f_{j+1}(s)|^{\lambda_{\alpha_1}}|D^\beta f_{j+1}(s)|^{\lambda_\beta}\Big] \\
&+ \Big[|D^{\alpha_1}f_1(s)|^{\lambda_{\alpha_1}}|D^{\alpha_2}f_M(s)|^{\lambda_{\alpha_2}}|D^\beta f_1(s)|^{\lambda_\beta} \\
&+ |D^{\alpha_2}f_1(s)|^{\lambda_{\alpha_2}}|D^{\alpha_1}f_M(s)|^{\lambda_{\alpha_1}}|D^\beta f_M(s)|^{\lambda_\beta}\Big]\Bigg]ds \\
&\leq 2^{(\frac{\lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta}{p})}\varepsilon_3\varphi_3(x)\left\{\int_0^x p(s)\left[\sum_{j=1}^M |D^\beta f_j(s)|^p\right]ds\right\}^{(\frac{\lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta}{p})}.
\end{aligned} \tag{32}$$

Proof. From Theorem 6 of [7] and by adding we get

$$\begin{aligned}
&\sum_{j=1}^{M-1}\int_0^x q(s)\Big[|D^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}|D^{\alpha_2}f_{j+1}(s)|^{\lambda_{\alpha_2}}|D^\beta f_j(s)|^{\lambda_\beta} \\
&+ |D^{\alpha_2}f_j(s)|^{\lambda_{\alpha_2}}|D^{\alpha_1}f_{j+1}(s)|^{\lambda_{\alpha_1}}|D^\beta f_{j+1}(s)|^{\lambda_\beta}\Big]ds \\
&\leq \varphi_3(x)\sum_{j=1}^{M-1}\left(\int_0^x p(s)(|D^\beta f_j(s)|^p + |D^\beta f_{j+1}(s)|^p)ds\right)^{(\lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta)/p}.
\end{aligned} \tag{33}$$

Also it holds

$$\begin{aligned}
&\int_0^x q(s)\Big[|D^{\alpha_1}f_1(s)|^{\lambda_{\alpha_1}}|D^{\alpha_2}f_M(s)|^{\lambda_{\alpha_2}}|D^\beta f_1(s)|^{\lambda_\beta} \\
&+ |D^{\alpha_2}f_1(s)|^{\lambda_{\alpha_2}}|D^{\alpha_1}f_M(s)|^{\lambda_{\alpha_1}}|D^\beta f_M(s)|^{\lambda_\beta}\Big]ds \\
&\leq \varphi_3(x)\left(\int_0^x p(s)\left(|D^\beta f_1(s)|^p + |D^\beta f_M(s)|^p\right)ds\right)^{\frac{(\lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta)}{p}}.
\end{aligned} \tag{34}$$

Adding (33) and (34), along with (4), (5) we derive (32). ∎

We continue with

Theorem 7. *Let $\beta > \alpha_1 + 1$, $\alpha_1 \in \mathbb{R}_+$ and let $f_j \in L_1(0,x)$, $j = 1,\ldots,M \in \mathbb{N}$, $x \in \mathbb{R}_+ - \{0\}$ have, respectively, L_∞ fractional derivatives $D^\beta f_j$, in $[0,x]$, and let $D^{\beta-k} f_j(0) = 0$, for $k = 1,\ldots,[\beta]+1$; $j = 1,\ldots,M$. Consider also $p(t) > 0$ and $q(t) \geq 0$, with $p(t)$, $\frac{1}{p(t)}$, $q(t) \in L_\infty(0,x)$. Let $\lambda_\alpha \geq 0$, $0 < \lambda_{\alpha+1} < 1$, and $p > 1$.*

Denote

$$\theta_3 := \begin{cases} 2^{\lambda_\alpha/(\lambda_{\alpha+1})} - 1, & \text{if } \lambda_\alpha \geq \lambda_{\alpha+1}, \\ 1, & \text{if } \lambda_\alpha \leq \lambda_{\alpha+1}, \end{cases} \tag{35}$$

$$L(x) := \left(2\int_0^x (q(s))^{(1/(1-\lambda_{\alpha+1}))} ds\right)^{(1-\lambda_{\alpha+1})} \left(\frac{\theta_3 \lambda_{\alpha+1}}{\lambda_\alpha + \lambda_{\alpha+1}}\right)^{\lambda_{\alpha+1}}, \tag{36}$$

and

$$P_1(x) := \int_0^x (x-s)^{(\beta-\alpha_1-1)p/(p-1)} (p(s))^{-1/(p-1)} ds, \tag{37}$$

$$T(x) := L(x) \left(\frac{P_1(x)^{(\frac{p-1}{p})}}{\Gamma(\beta-\alpha_1)}\right)^{(\lambda_\alpha+\lambda_{\alpha+1})}, \tag{38}$$

and

$$\omega_1 := 2^{(\frac{p-1}{p})(\lambda_\alpha+\lambda_{\alpha+1})}, \tag{39}$$

$$\Phi(x) := T(x)\,\omega_1. \tag{40}$$

Also put

$$\varepsilon_4 := \begin{cases} 1, & \text{if } \lambda_\alpha + \lambda_{\alpha+1} \geq p, \\ M^{1-(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p})}, & \text{if } \lambda_\alpha + \lambda_{\alpha+1} \leq p. \end{cases} \tag{41}$$

Then

$$\int_0^x q(s) \Bigg\{ \Bigg\{ \sum_{j=1}^{M-1} \left[|D^{\alpha_1} f_j(s)|^{\lambda_\alpha} |D^{\alpha_1+1} f_{j+1}(s)|^{\lambda_{\alpha+1}} + |D^{\alpha_1} f_{j+1}(s)|^{\lambda_\alpha} |D^{\alpha_1+1} f_j(s)|^{\lambda_{\alpha+1}} \right] \Bigg\}$$
$$+ \left[|D^{\alpha_1} f_1(s)|^{\lambda_\alpha} |D^{\alpha_1+1} f_M(s)|^{\lambda_{\alpha+1}} + |D^{\alpha_1} f_M(s)|^{\lambda_\alpha} |D^{\alpha_1+1} f_1(s)|^{\lambda_{\alpha+1}} \right] \Bigg\} ds$$
$$\leq 2^{(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p})} \varepsilon_4 \Phi(x) \left[\int_0^x p(s) \left(\sum_{j=1}^M |D^\beta f_j(s)|^p \right) ds \right]^{(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p})}. \tag{42}$$

Proof. From Theorem 8 of [7] we get

$$\int_0^x q(s) \sum_{j=1}^{M-1} \left[|D^{\alpha_1} f_j(s)|^{\lambda_\alpha} |D^{\alpha_1+1} f_{j+1}(s)|^{\lambda_{\alpha+1}} + |D^{\alpha_1} f_{j+1}(s)|^{\lambda_\alpha} |D^{\alpha_1+1} f_j(s)|^{\lambda_{\alpha+1}} \right] ds$$
$$\leq \Phi(x) \sum_{j=1}^{M-1} \left[\int_0^x p(s)(|D^\beta f_j(s)|^p + |D^\beta f_{j+1}(s)|^p) ds \right]^{(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p})}. \tag{43}$$

Similarly it holds

$$\int_0^x q(s)\left[|D^{\alpha_1}f_1(s)|^{\lambda_\alpha}|D^{\alpha_1+1}f_M(s)|^{\lambda_{\alpha+1}}+|D^{\alpha_1}f_M(s)|^{\lambda_\alpha}|D^{\alpha_1+1}f_1(s)|^{\lambda_{\alpha+1}}\right]ds$$

$$\leq \Phi(x)\left[\int_0^x p(s)(|D^\beta f_1(s)|^p+|D^\beta f_M(s)|^p)ds\right]^{\frac{(\lambda_\alpha+\lambda_{\alpha+1})}{p}}. \tag{44}$$

Adding (43) and (44), along with (4), (5) we obtain (42). □

Next it comes

Theorem 8. *All as in Theorem* 4*. Consider the special case of* $\lambda_{\alpha_2}=\lambda_{\alpha_1}+\lambda_\beta$*. Denote*

$$\tilde{T}(x):=A_0(x)\left(\frac{\lambda_\beta}{\lambda_{\alpha_1}+\lambda_\beta}\right)^{\lambda_\beta/p}2^{(p-2\lambda_{\alpha_1}-3\lambda_\beta)/p}, \tag{45}$$

$$\varepsilon_5:=\begin{cases}1, & \text{if } 2(\lambda_{\alpha_1}+\lambda_\beta)\geq p,\\ M^{1-(\frac{2(\lambda_{\alpha_1}+\lambda_\beta)}{p})}, & \text{if } 2(\lambda_{\alpha_1}+\lambda_\beta)\leq p.\end{cases} \tag{46}$$

Then

$$\int_0^x q(s)\Bigg\{\Bigg\{\sum_{j=1}^{M-1}\Bigg[|D^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}|D^{\alpha_2}f_{j+1}(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\left|D^\beta f_j(s)\right|^{\lambda_\beta}$$

$$+|D^{\alpha_2}f_j(s)|^{\lambda_{\alpha_1}+\lambda_\beta}|D^{\alpha_1}f_{j+1}(s)|^{\lambda_{\alpha_1}}\left|D^\beta f_{j+1}(s)\right|^{\lambda_\beta}\Bigg]\Bigg\}$$

$$+\Bigg[|D^{\alpha_1}f_1(s)|^{\lambda_{\alpha_1}}|D^{\alpha_2}f_M(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\left|D^\beta f_1(s)\right|^{\lambda_\beta}$$

$$+|D^{\alpha_2}f_1(s)|^{\lambda_{\alpha_1}+\lambda_\beta}|D^{\alpha_1}f_M(s)|^{\lambda_{\alpha_1}}\left|D^\beta f_M(s)\right|^{\lambda_\beta}\Bigg]\Bigg\}ds$$

$$\leq 2^{\frac{2(\lambda_{\alpha_1}+\lambda_\beta)}{p}}\varepsilon_3\tilde{T}(x)\left[\int_0^x p(s)\left(\sum_{j=1}^M\left|D^\beta f_j(s)\right|^p\right)ds\right]^{\frac{2(\lambda_{\alpha_1}+\lambda_\beta)}{p}}. \tag{47}$$

Proof. Based on Theorem 9 of [7]. The rest as in the proof of Theorem 7. □

Next we give special case of the above theorems.

Corollary 9 (to Theorem 4, $\lambda_{\alpha_2}=0$, $p(t)=q(t)=1$)**.** *It holds*

$$\int_0^x\left(\sum_{j=1}^M|D^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}\left|D^\beta f_j(s)\right|^{\lambda_\beta}\right)ds$$

$$\leq \delta_1^*\varphi_1(x)\left[\int_0^x\left[\sum_{j=1}^M\left|D^\beta f_j(s)\right|^p\right]ds\right]^{(\frac{\lambda_{\alpha_1}+\lambda_\beta}{p})}. \tag{48}$$

In (48)*,* $(A_0(x)|_{\lambda_{\alpha_2}=0})$ *of* $\varphi_1(x)$ *is given in* [7]*, Corollary* 10*, equation* (123)*.*

Corollary 10 (to Theorem 4, $\lambda_{\alpha_2} = 0, p(t) = q(t) = 1, \lambda_{\alpha_1} = \lambda_\beta = 1, p = 2$)**.** *In detail, let* $\alpha_1 \in \mathbb{R}_+$, $\beta > \alpha_1$ *and let* $f_j \in L_1(0, x)$, $j = 1, \ldots, M \in \mathbb{N}$, $x \in \mathbb{R}_+ - \{0\}$, *have, respectively,* L_∞ *fractional derivatives* $D^\beta f_j$ *in* $[0, x]$, *and let* $D^{\beta-k} f_j(0) = 0$ *for* $k = 1, \ldots, [\beta] + 1$; $j = 1, \ldots, M$. *Then*

$$\int_0^x \left(\sum_{j=1}^M |D^{\alpha_1} f_j(s)| \left|D^\beta f_j(s)\right| \right) ds$$
$$\leq \left(\frac{x^{(\beta-\alpha_1)}}{2\Gamma(\beta - \alpha_1)\sqrt{\beta - \alpha_1}\sqrt{2\beta - 2\alpha_1 - 1}} \right) \left\{ \int_0^x \left[\sum_{j=1}^M \left(D^\beta f_j(s)\right)^2 \right] ds \right\}. \tag{49}$$

Proof. Bared on our Corollary 9 and Corollary 11 of [7], see inequality (130) there. □

Corollary 11 (to Theorem 5, $\lambda_{\alpha_1} = 0$, $p(t) = q(t) = 1$)**.** *It holds*

$$\int_0^x \Bigg\{ \left\{ \sum_{j=1}^{M-1} \left[|D^{\alpha_2} f_{j+1}(s)|^{\lambda_{\alpha_2}} \left|D^\beta f_j(s)\right|^{\lambda_\beta} + |D^{\alpha_2} f_j(s)|^{\lambda_{\alpha_2}} \left|D^\beta f_{j+1}(s)\right|^{\lambda_\beta} \right] \right\}$$
$$+ \left[|D^{\alpha_2} f_M(s)|^{\lambda_{\alpha_2}} \left|D^\beta f_1(s)\right|^{\lambda_\beta} + |D^{\alpha_2} f_1(s)|^{\lambda_{\alpha_2}} \left|D^\beta f_M(s)\right|^{\lambda_\beta} \right] \Bigg\} ds$$
$$\leq 2^{\left(\frac{\lambda_\beta + \lambda_{\alpha_2}}{p}\right)} \varepsilon_2 \varphi_2(x) \left\{ \int_0^x \left[\sum_{j=1}^M \left|D^\beta f_j(s)\right|^p \right] ds \right\}^{\left(\frac{\lambda_\beta + \lambda_{\alpha_2}}{p}\right)}. \tag{50}$$

In (50), $(A_0(x) \mid_{\lambda_{\alpha_1}=0})$ *of* $\varphi_2(x)$ *is given in* [7], *Corollary* 12, *equation* (137) *there.*

Corollary 12 (to Theorem 5, $\lambda_{\alpha_1} = 0$, $p(t) = q(t) = 1$, $\lambda_{\alpha_2} = \lambda_\beta = 1$, $p = 2$)**.** *In detail, let* $\alpha_2 \in \mathbb{R}_+$, $\beta > \alpha_2$ *and let* $f_j \in L_1(0, x)$, $j = 1, \ldots, M \in \mathbb{N}$, $x \in \mathbb{R}_+ - \{0\}$, *have, respectively,* L_∞ *fractional derivatives* $D^\beta f_j$ *in* $[0, x]$, *and let* $D^{\beta-k} f_j(0) = 0$, *for* $k = 1, \ldots, [\beta] + 1$; $j = 1, \ldots, M$. *Then*

$$\int_0^x \Bigg\{ \left\{ \sum_{j=1}^{M-1} \left[|D^{\alpha_2} f_{j+1}(s)| \left|D^\beta f_j(s)\right| + |D^{\alpha_2} f_j(s)| \left|D^\beta f_{j+1}(s)\right| \right] \right\}$$
$$+ \left[|D^{\alpha_2} f_M(s)| \left|D^\beta f_1(s)\right| + |D^{\alpha_2} f_1(s)| \left|D^\beta f_M(s)\right| \right] \Bigg\} ds$$
$$\leq \left(\frac{\sqrt{2}\, x^{(\beta-\alpha_2)}}{\Gamma(\beta - \alpha_2)\, \sqrt{\beta - \alpha_2}\, \sqrt{2\beta - 2\alpha_2 - 1}} \right) \left\{ \int_0^x \left[\sum_{j=1}^M (D^\beta f_j(s))^2 \right] ds \right\}. \tag{51}$$

Proof. From Corollary 11 and Corollary 13 of [7], especially equation (146) there. □

Corollary 13 (to Theorem 6, $\lambda_{\alpha_1} = \lambda_{\alpha_2} = \lambda_\beta = 1$, $p = 3$, $p(t) = q(t) = 1$)**.** *It holds*

$$\int_0^x \Bigg[\sum_{j=1}^{M-1} \left[|D^{\alpha_1} f_j(s)|\; |D^{\alpha_2} f_{j+1}(s)|\; \left|D^\beta f_j(s)\right| + |D^{\alpha_2} f_j(s)|\; |D^{\alpha_1} f_{j+1}(s)|\; \left|D^\beta f_{j+1}(s)\right|\right]$$

$$+ \left[|D^{\alpha_1} f_1(s)|\; |D^{\alpha_2} f_M(s)|\; \left|D^\beta f_1(s)\right| + |D^{\alpha_2} f_1(s)|\; |D^{\alpha_1} f_M(s)|\; \left|D^\beta f_M(s)\right|\right]\Bigg] ds$$

$$\leq 2\,\varphi_3^*(x) \left[\int_0^x \left[\sum_{j=1}^{M} \left|D^\beta f_j(s)\right|^3 ds\right]\right]. \tag{52}$$

Here

$$\varphi_3^* := \left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{6}}\right) A_0(x), \tag{53}$$

where in this special case,

$$A_0(x) = \frac{4\, x^{(2\beta-\alpha_1-\alpha_2)}}{\Gamma(\beta-\alpha_1)\Gamma(\beta-\alpha_2)[3(3\beta-3\alpha_1-1)(3\beta-3\alpha_2-1)(2\beta-\alpha_1-\alpha_2)]^{2/3}}. \tag{54}$$

Proof. From Theorem 6 and Corollary 14 of [7], see there equation (151) which is here (54). ☐

Corollary 14 (to Theorem 7, $\lambda_\alpha = 1$, $\lambda_{\alpha+1} = 1/2, p = 3/2, p(t) = q(t) = 1$)**.** *In detail: let $\beta > \alpha_1 + 1$, $\alpha_1 \in \mathbb{R}_+$ and let $f_j \in L_1(0,x)$, $j = 1,\ldots,M \in \mathbb{N}$, $x \in \mathbb{R}_+ - \{0\}$ have, respectively, L_∞ fractional derivatives $D^\beta f_j$ in $[0,x]$, and let $D^{\beta-k} f_j(0) = 0$, for $k = 1,\ldots,[\beta]+1$; $j = 1,\ldots,M$. Set*

$$\Phi^*(x) := \left(\frac{2}{\sqrt{3\beta-3\alpha_1-2}}\right) \frac{x^{\left(\frac{3\beta-3\alpha_1-1}{2}\right)}}{(\Gamma(\beta-\alpha_1))^{3/2}}. \tag{55}$$

Then

$$\int_0^x \Bigg\{\Bigg\{\sum_{j=1}^{M-1} \left[|D^{\alpha_1} f_j(s)|\,\sqrt{|D^{\alpha_1+1} f_{j+1}(s)|} + |D^{\alpha_1} f_{j+1}(s)|\,\sqrt{|D^{\alpha_1+1} f_j(s)|}\right]\Bigg\}$$

$$+ \left[|D^{\alpha_1} f_1(s)|\,\sqrt{|D^{\alpha_1+1} f_M(s)|} + |D^{\alpha_1} f_M(s)|\,\sqrt{|D^{\alpha_1+1} f_1(s)|}\right]\Bigg\} ds$$

$$\leq 2\Phi^*(x) \left[\int_0^x \left(\sum_{j=1}^{M} |D^\beta f_j(s)|^{3/2}\right) ds\right]. \tag{56}$$

Proof. Based on Theorem 7 here, and Corollary 15 of [7], see there equation (161) which is here (55). ☐

Corollary 15 (to Theorem 8, here $p = 2(\lambda_{\alpha_1} + \lambda_\beta) > 1$, $p(t) = q(t) = 1$)**.** *It holds*

$$\int_0^x \Bigg\{ \Bigg\{ \sum_{j=1}^{M-1} \Big[|D^{\alpha_1} f_j(s)|^{\lambda_{\alpha_1}} |D^{\alpha_2} f_{j+1}(s)|^{\lambda_{\alpha_1}+\lambda_\beta} \left|D^\beta f_j(s)\right|^{\lambda_\beta}$$

$$+ |D^{\alpha_2} f_j(s)|^{\lambda_{\alpha_1}+\lambda_\beta} |D^{\alpha_1} f_{j+1}(s)|^{\lambda_{\alpha_1}} \left|D^\beta f_{j+1}(s)\right|^{\lambda_\beta} \Big] \Bigg\}$$

$$+ \Big[|D^{\alpha_1} f_1(s)|^{\lambda_{\alpha_1}} |D^{\alpha_2} f_M(s)|^{\lambda_{\alpha_1}+\lambda_\beta} \left|D^\beta f_1(s)\right|^{\lambda_\beta}$$

$$+ |D^{\alpha_2} f_1(s)|^{\lambda_{\alpha_1}+\lambda_\beta} |D^{\alpha_1} f_M(s)|^{\lambda_{\alpha_1}} \left|D^\beta f_M(s)\right|^{\lambda_\beta} \Big] \Bigg\} ds$$

$$\leq 2\, \tilde{T}(x) \left[\int_0^x \left(\sum_{j=1}^{M} \left|D^\beta f_j(s)\right|^{2(\lambda_{\alpha_1}+\lambda_\beta)} \right) ds \right]. \tag{57}$$

Here $\tilde{T}(x)$ *in* (57) *is given by* (45) *and in detail by* $\tilde{\tilde{T}}(x)$ *of* [7]*, see there Corollary* 16 *and equations* (165)–(169)*.*

Corollary 16 (to Theorem 8, $p = 4$, $\lambda_{\alpha_1} = \lambda_\beta = 1$, $p(t) = q(t) = 1$)**.** *It holds*

$$\int_0^x \Bigg\{ \Bigg\{ \sum_{j=1}^{M-1} \Big[|D^{\alpha_1} f_j(s)| \left(D^{\alpha_2} f_{j+1}(s)\right)^2 \left|D^\beta f_j(s)\right|$$

$$+ \left(D^{\alpha_2} f_j(s)\right)^2 |D^{\alpha_1} f_{j+1}(s)| \left|D^\beta f_{j+1}(s)\right| \Big] \Bigg\} + \Big[|D^{\alpha_1} f_1(s)| \left(D^{\alpha_2} f_M(s)\right)^2 \left|D^\beta f_1(s)\right|$$

$$+ \left(D^{\alpha_2} f_1(s)\right)^2 |D^{\alpha_1} f_M(s)| \left|D^\beta f_M(s)\right| \Big] \Bigg\} ds \leq 2\, \tilde{T}(x) \left[\int_0^x \left(\sum_{j=1}^{M} \left(D^\beta f_j(s)\right)^4 \right) ds \right]. \tag{58}$$

Here in (58) *we have that* $\tilde{T}(x) = T^*(x)$ *of Corollary* 17 *of* [7]*, for that see there equations* (177)–(181)*.*

Next we present the L_∞ case.

Theorem 17. *Let* $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $\beta > \alpha_1, \alpha_2$ *and let* $f_j \in L_1(0,x)$, $j = 1, \ldots, M \in \mathbb{N}$, $x \in \mathbb{R}_+ - \{0\}$, *have, respectively,* L_∞ *fractional derivatives* $D^\beta f_j$ *in* $[0,x]$, *and let* $D^{\beta-k} f_j(0) = 0$ *for* $k = 1, \ldots, [\beta]+1$; $j = 1, \ldots, M$. *Consider* $p(s) \geq 0$, $p(s) \in L_\infty(0,x)$. *Let* $\lambda_{\alpha_1}, \lambda_{\alpha_2}, \lambda_\beta \geq 0$. *Set*

$$\rho(x) := \left\{ \frac{\|p(s)\|_\infty \quad x^{(\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + \beta\lambda_{\alpha_2} - \alpha_2\lambda_{\alpha_2} + 1)}}{(\Gamma(\beta - \alpha_1 + 1))^{\lambda_{\alpha_1}} (\Gamma(\beta - \alpha_2 + 1))^{\lambda_{\alpha_2}} [\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + \beta\lambda_{\alpha_2} - \alpha_2\lambda_{\alpha_2} + 1]} \right\}. \tag{59}$$

Then

$$\int_0^x p(s)\Bigg\{\Bigg\{\sum_{j=1}^{M-1}\Big[|D^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}\,|D^{\alpha_2}f_{j+1}(s)|^{\lambda_{\alpha_2}}\,\left|D^{\beta}f_j(s)\right|^{\lambda_\beta}$$

$$+|D^{\alpha_2}f_j(s)|^{\lambda_{\alpha_2}}\,|D^{\alpha_1}f_{j+1}(s)|^{\lambda_{\alpha_1}}\,\left|D^{\beta}f_{j+1}(s)\right|^{\lambda_\beta}\Big]\Bigg\}+\Big[|D^{\alpha_1}f_1(s)|^{\lambda_{\alpha_1}}\,|D^{\alpha_2}f_M(s)|^{\lambda_{\alpha_2}}\,\left|D^{\beta}f_1(s)\right|^{\lambda_\beta}$$

$$+|D^{\alpha_2}f_1(s)|^{\lambda_{\alpha_2}}\,|D^{\alpha_1}f_M(s)|^{\lambda_{\alpha_1}}\,\left|D^{\beta}f_M(s)\right|^{\lambda_\beta}\Big]\Bigg\}ds$$

$$\le \rho(x)\left\{\sum_{j=1}^{M}\left\{\left\|D^{\beta}f_j\right\|_\infty^{2(\lambda_{\alpha_1}+\lambda_\beta)}+\left\|D^{\beta}f_j\right\|_\infty^{2\lambda_{\alpha_2}}\right\}\right\}. \tag{60}$$

Proof. Based on Theorem 18 of [7]. □

Similarly we give

Theorem 18 (as in Theorem 17, $\lambda_{\alpha_2}=0$)**.** *It holds*

$$\int_0^x p(s)\left(\sum_{j=1}^{M}|D^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}|D^{\beta}f_j(s)|^{\lambda_\beta}\right)ds$$

$$\le\left(\frac{x^{(\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}+1)}\quad\|p(s)\|_\infty}{(\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}+1)(\Gamma(\beta-\alpha_1+1))^{\lambda_{\alpha_1}}}\right)\left(\sum_{j=1}^{M}\left\|D^{\beta}f_j\right\|_\infty^{\lambda_{\alpha_1}+\lambda_\beta}\right). \tag{61}$$

Proof. Based on Theorem 19 of [7]. □

It follows

Theorem 19 (as in Theorem 17, $\lambda_{\alpha_2}=\lambda_{\alpha_1}+\lambda_\beta$)**.** *It holds*

$$\int_0^x p(s)\Bigg\{\Bigg\{\sum_{j=1}^{M-1}\Big[|D^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}\,|D^{\alpha_2}f_{j+1}(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\,\left|D^{\beta}f_j(s)\right|^{\lambda_\beta}$$

$$+|D^{\alpha_2}f_j(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\,|D^{\alpha_1}f_{j+1}(s)|^{\lambda_{\alpha_1}}\,\left|D^{\beta}f_{j+1}(s)\right|^{\lambda_\beta}\Big]\Bigg\}$$

$$+\Big[|D^{\alpha_1}f_1(s)|^{\lambda_{\alpha_1}}\,|D^{\alpha_2}f_M(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\,\left|D^{\beta}f_1(s)\right|^{\lambda_\beta}$$

$$+|D^{\alpha_2}f_1(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\,|D^{\alpha_1}f_M(s)|^{\lambda_{\alpha_1}}\,\left|D^{\beta}f_M(s)\right|^{\lambda_\beta}\Big]\Bigg\}ds$$

$$\le\left(\frac{2\,x^{(2\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}+\beta\lambda_\beta-\alpha_2\lambda_{\alpha_1}-\alpha_2\lambda_\beta+1)}\quad\|p(s)\|_\infty}{(2\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}+\beta\lambda_\beta-\alpha_2\lambda_{\alpha_1}-\alpha_2\lambda_\beta+1)(\Gamma(\beta-\alpha_1+1))^{\lambda_{\alpha_1}}(\Gamma(\beta-\alpha_2+1))^{(\lambda_{\alpha_1}+\lambda_\beta)}}\right)$$

$$\left(\sum_{j=1}^{M}\left\|D^{\beta}f_j\right\|_\infty^{2(\lambda_{\alpha_1}+\lambda_\beta)}\right). \tag{62}$$

Proof. By Theorem 20 of [7]. □

We continue with

Theorem 20 (as in Theorem 17, $\lambda_\beta = 0, \lambda_{\alpha_1} = \lambda_{\alpha_2}$)**.** *It holds*

$$\int_0^x p(s)\left\{\left\{\sum_{j=1}^{M-1}\left[|D^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}|D^{\alpha_2}f_{j+1}(s)|^{\lambda_{\alpha_1}}+|D^{\alpha_2}f_j(s)|^{\lambda_{\alpha_1}}|D^{\alpha_1}f_{j+1}(s)|^{\lambda_{\alpha_1}}\right]\right\}\right.$$
$$\left.+\left[|D^{\alpha_1}f_1(s)|^{\lambda_{\alpha_1}}|D^{\alpha_2}f_M(s)|^{\lambda_{\alpha_1}}+|D^{\alpha_2}f_1(s)|^{\lambda_{\alpha_1}}|D^{\alpha_1}f_M(s)|^{\lambda_{\alpha_1}}\right]\right\}ds$$
$$\leq 2\,\rho^*(x)\left[\sum_{j=1}^{M}\left\|D^\beta f_j\right\|_\infty^{2\lambda_{\alpha_1}}\right]. \tag{63}$$

Here we have

$$\rho^*(x) := \left(\frac{x^{(2\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}-\alpha_2\lambda_{\alpha_1}+1)}\quad\|p(s)\|_\infty}{(2\beta\,\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}-\alpha_2\lambda_{\alpha_1}+1)(\Gamma(\beta-\alpha_1+1))^{\lambda_{\alpha_1}}(\Gamma(\beta-\alpha_2+1))^{\lambda_{\alpha_1}}}\right). \tag{64}$$

Proof. Based on Theorem 21 of [7]. □

Next we give

Theorem 21 (as in Theorem 17, $\lambda_{\alpha_1} = 0, \lambda_{\alpha_2} = \lambda_\beta$)**.** *It holds*

$$\int_0^x p(s)\left\{\left\{\sum_{j=1}^{M-1}\left[|D^{\alpha_2}f_{j+1}(s)|^{\lambda_{\alpha_2}}\left|D^{\beta}f_j(s)\right|^{\lambda_{\alpha_2}}+|D^{\alpha_2}f_j(s)|^{\lambda_{\alpha_2}}\left|D^{\beta}f_{j+1}(s)\right|^{\lambda_{\alpha_2}}\right]\right\}\right.$$
$$\left.+\left[|D^{\alpha_2}f_M(s)|^{\lambda_{\alpha_2}}\left|D^{\beta}f_1(s)\right|^{\lambda_{\alpha_2}}+|D^{\alpha_2}f_1(s)|^{\lambda_{\alpha_2}}\left|D^{\beta}f_M(s)\right|^{\lambda_{\alpha_2}}\right]\right\}ds$$
$$\leq 2\left(\frac{x^{(\beta\lambda_{\alpha_2}-\alpha_2\lambda_{\alpha_2}+1)}\quad\|p(s)\|_\infty}{(\beta\lambda_{\alpha_2}-\alpha_2\lambda_{\alpha_2}+1)(\Gamma(\beta-\alpha_2+1))^{\lambda_{\alpha_2}}}\right)\left(\sum_{j=1}^{M}\left\|D^\beta f_j\right\|_\infty^{2\lambda_{\alpha_2}}\right). \tag{65}$$

Proof. Based on Theorem 22 of [7]. □

Some special cases follow.

Corollary 22 (to Theorem 20, all as in Theorem 17, $\lambda_\beta = 0$, $\lambda_{\alpha_1} = \lambda_{\alpha_2}, \alpha_2 = \alpha_1 + 1$)**.** *It holds*

$$\int_0^x p(s)\left\{\left\{\sum_{j=1}^{M-1}\left[|D^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}\left|D^{\alpha_1+1}f_{j+1}(s)\right|^{\lambda_{\alpha_1}}+\left|D^{\alpha_1+1}f_j(s)\right|^{\lambda_{\alpha_1}}|D^{\alpha_1}f_{j+1}(s)|^{\lambda_{\alpha_1}}\right]\right\}\right.$$
$$\left.+\left[|D^{\alpha_1}f_1(s)|^{\lambda_{\alpha_1}}\left|D^{\alpha_1+1}f_M(s)\right|^{\lambda_{\alpha_1}}+\left|D^{\alpha_1+1}f_1(s)\right|^{\lambda_{\alpha_1}}|D^{\alpha_1}f_M(s)|^{\lambda_{\alpha_1}}\right]\right\}ds$$
$$\leq 2\left(\frac{x^{(2\beta\lambda_{\alpha_1}-2\alpha_1\lambda_{\alpha_1}-\lambda_{\alpha_1}+1)}\quad\|p(s)\|_\infty}{(2\beta\lambda_{\alpha_1}-2\alpha_1\lambda_{\alpha_1}-\lambda_{\alpha_1}+1)(\beta-\alpha_1)^{\lambda_{\alpha_1}}(\Gamma(\beta-\alpha_1))^{2\lambda_{\alpha_1}}}\right)\left[\sum_{j=1}^{M}\left\|D^\beta f_j\right\|_\infty^{2\lambda_{\alpha_1}}\right]. \tag{66}$$

Proof. Based on Corollary 23 of [7]. □

Corollary 23 (to Corollary 22)**.** *In detail: Let* $\alpha_1 \in \mathbb{R}_+$, $\beta > \alpha_1 + 1$ *and let* $f_j \in L_1(0,x)$, $j = 1, \ldots, M \in \mathbb{N}$, $x \in \mathbb{R}_+ - \{0\}$, *have, respectively,* L_∞ *fractional derivatives* $D^\beta f_j$ *in* $[0,x]$, *and let* $D^{\beta-k} f_j(0) = 0$, *for* $k = 1, \ldots, [\beta] + 1$; $j = 1, \ldots, M$. *Then*

$$\int_0^x \left\{ \left\{ \sum_{j=1}^{M-1} \left[|D^{\alpha_1} f_j(s)|\, |D^{\alpha_1+1} f_{j+1}(s)| + |D^{\alpha_1+1} f_j(s)|\, |D^{\alpha_1} f_{j+1}(s)| \right] \right\} \right.$$
$$\left. + \left[|D^{\alpha_1} f_1(s)|\, |D^{\alpha_1+1} f_M(s)| + |D^{\alpha_1+1} f_1(s)|\, |D^{\alpha_1} f_M(s)| \right] \right\} ds$$
$$\leq \frac{x^{2(\beta-\alpha_1)}}{(\beta-\alpha_1)^2 (\Gamma(\beta-\alpha_1))^2} \left(\sum_{j=1}^{M} \left\| D^\beta f_j \right\|_\infty^2 \right). \tag{67}$$

Proof. Based on Corollary 24 of [7]. □

Corollary 24 (to Corollary 23)**.** *It holds*

$$\int_0^x \left(\sum_{j=1}^{M} |D^{\alpha_1} f_j(s)|\, |D^{\alpha_1+1} f_j(s)| \right) ds$$
$$\leq \left(\frac{x^{2(\beta-\alpha_1)}}{2(\beta-\alpha_1)^2 (\Gamma(\beta-\alpha_1))^2} \right) \left(\sum_{j=1}^{M} \left\| D^\beta f_j \right\|_\infty^2 \right). \tag{68}$$

Proof. Based on inequality (207) of [7]. □

3. APPLICATIONS

We present our first application.

Theorem 25. *Let* $\alpha_i \in \mathbb{R}_+$, $\beta > \alpha_i$, $i = 1, \ldots, r \in \mathbb{N}$, *and let* $f_j \in L_1(0,x)$, $j = 1, \ldots, M \in \mathbb{N}$; $x \in \mathbb{R}_+ - \{0\}$, *have, respectively,* L_∞ *fractional derivatives* $D^\beta f_j$ *in* $[0,x]$, *and let* $D^{\beta-k} f_j(0) = \alpha_{kj} \in \mathbb{R}$ *for* $k = 1, \ldots, [\beta] + 1$; $j = 1, \ldots, M$. *Furthermore for* $j = 1, \ldots, M$, *we have that*

$$D^\beta f_j(s) = F_j\left(s, \{D^{\alpha_i} f_j(s)\}_{i=1,j=1}^{r,M} \right), \tag{69}$$

all $s \in [0,x]$.

Here $F_j(s, \overrightarrow{z_1}, \overrightarrow{z_2}, \ldots, \overrightarrow{z_M})$ *are continuous for* $(\overrightarrow{z_1}, \overrightarrow{z_2}, \ldots, \overrightarrow{z_M}) \in (\mathbb{R}^r)^M$, *bounded for* $s \in [0,x]$, *and satisfy the Lipschitz condition*

$$|F_j(t; x_{11}, x_{12}, \ldots, x_{1r}; x_{21}, x_{22}, \ldots, x_{2r}; x_{31}, \ldots, x_{3r}; \ldots, x_{M1}, \ldots, x_{Mr})$$
$$- F_j(t, x'_{11}, x'_{12}, \ldots, x'_{1r}; x'_{21}, x'_{22}, \ldots, x'_{2r}; x'_{31}, \ldots, x'_{3r}; \ldots, x'_{M1}, \ldots, x'_{Mr})|$$
$$\leq \sum_{i=1}^{r} \left(\sum_{\ell=1}^{M} q_{\ell,i,j}(t) |x_{\ell i} - x'_{\ell i}| \right), \tag{70}$$

$j=1,2,\dots,M$, where all $q_{\ell,i,j}(s)\geq 0$ are bounded on $[0,x]$, $1\leq i\leq r$, $\ell=1,\dots,M$. Call

$$W:=\max\left\{\|q_{\ell,i,j}\|_{\infty},|\ \ell,j=1,2,\dots,M;i=1,\dots,r\right\}. \tag{71}$$

Assume here that

$$\phi^{*}(x):=W\left(\frac{1}{2}+\frac{M-1}{\sqrt{2}}\right)\left(\sum_{i=1}^{r}\left(\frac{x^{\beta-\alpha_i}}{\Gamma(\beta-\alpha_i)\sqrt{\beta-\alpha_i}\sqrt{2\beta-2\alpha_i-1}}\right)\right)<1. \tag{72}$$

Then, if the system (69) has two pairs of solutions $(f_1,f_2,\dots,f_M)$ and $(f_1^{*},f_2^{*},\dots,f_M^{*})$, we prove that $f_j=f_j^{*}$, $j=1,2,\dots,M$; that is we have uniqueness of solution for the fractional system (69).

Proof. Assume that there are two M-tuples of solutions $(f_1,\dots,f_M)$ and, $(f_1^{*},\dots,f_M^{*})$ satisfying the system (69). Set $g_j:=f_j-f_j^{*}$, $j=1,2,\dots,M$. Then $D^{\beta-k}g_j(0)=0$, $k=1,\dots,[\beta]+1$; $j=1,\dots,M$. It holds

$$D^{\beta}g_j(s)=F_j\left(s,\{D^{\alpha_i}f_j(s)\}_{i=1,j=1}^{r,M}\right)-F_j\left(s,\left\{D^{\alpha_i}f_j^{*}(s)\right\}_{i=1,j=1}^{r,M}\right). \tag{73}$$

Hence by (70) we have

$$|D^{\beta}g_j(s)|\leq\sum_{i=1}^{r}\left(\sum_{\ell=1}^{M}g_{\ell,i,j}(t)|D^{\alpha_i}g_\ell(s)|\right). \tag{74}$$

Thus

$$|D^{\beta}g_j(s)|\leq W\sum_{i=1}^{r}\left(\sum_{\ell=1}^{M}|D^{\alpha_i}g_\ell(s)|\right). \tag{75}$$

The last implies

$$(D^{\beta}g_j(s))^2\leq W\sum_{i=1}^{r}\left(\sum_{\ell=1}^{M}|D^{\beta}g_j(s)|\,|D^{\alpha_i}g_\ell(s)|\right), \tag{76}$$

and

$$\sum_{j=1}^{M}(D^{\beta}g_j(s))^2\leq\ W\sum_{i=1}^{r}\sum_{j=1}^{M}\sum_{\ell=1}^{M}|D^{\beta}g_j(s)|\,|D^{\alpha_i}g_\ell(s)|. \tag{77}$$

Integrating (77) we observe

$$\begin{aligned}I&:=\int_0^x\left(\sum_{j=1}^{M}(D^{\beta}g_j(s))^2\right)ds\\&\leq W\left\{\sum_{i=1}^{r}\sum_{j=1}^{M}\left(\sum_{\ell=1}^{M}\left(\int_0^x|D^{\beta}g_j(s)|\,|D^{\alpha_i}g_\ell(s)|ds\right)\right)\right\}.\end{aligned} \tag{78}$$

That is

$$I \le W\Bigg\{\sum_{i=1}^{r}\Bigg[\Bigg(\int_0^x\Bigg(\sum_{\lambda=1}^{M}\left|D^{\alpha_i}g_\lambda(s)\right|\left|D^{\beta}g_\lambda(s)\right|\Bigg)ds\Bigg)$$
$$+\sum_{\substack{\tau,m\in\{1,\ldots,M\}\\ \tau\neq m}}\Bigg(\int_0^x\left(\left|D^{\alpha_i}g_m(s)\right|\left|D^{\beta}g_\tau(s)\right|+\left|D^{\alpha_i}g_\tau(s)\right|\left|D^{\beta}g_m(s)\right|\right)ds\Bigg)\Bigg]\Bigg\}. \quad (79)$$

Using Corollary 10 from here, and Corollary 13 of [7], we find

$$I \le W\Bigg\{\sum_{i=1}^{r}\Bigg[\Bigg(\frac{x^{(\beta-\alpha_i)}\ \ I}{2\,\Gamma(\beta-\alpha_i)\sqrt{\beta-\alpha_i}\sqrt{2\beta-2\alpha_i-1}}\Bigg)$$
$$+\Bigg(\frac{x^{(\beta-\alpha_i)}\ (M-1)I}{\sqrt{2}\ \Gamma(\beta-\alpha_i)\sqrt{\beta-\alpha_i}\sqrt{2\beta-2\alpha_i-1}}\Bigg)\Bigg]\Bigg\}, \quad (80)$$

i.e we got that

$$I \le \phi^*(x)\ I. \quad (81)$$

If $I \neq 0$ then $\phi^*(x) \ge 1$, a contradiction by assumption that $\phi^*(x) < 1$, see (72). Therefore $I = 0$, implying that

$$\sum_{j=1}^{M}(D^{\beta}g_j(s))^2 = 0, \quad a.e. \text{ in } [0,x], \quad (82)$$

i.e.

$$(D^{\beta}g_j(s))^2 = 0, \quad a.e. \text{ in } [0,x], \quad j=1,\ldots,M, \quad (83)$$

and

$$D^{\beta}g_j(s) = 0, \quad a.e. \text{ in } [0,x], \quad j=1,\ldots,M. \quad (84)$$

By $D^{\beta-k}g_j(0) = 0$, $k = 1,\ldots,[\beta]+1$; $j = 1,\ldots,M$, and Lemma 3, apply (3) for $\alpha = 0$, we find $g_j(s) \equiv 0$, all $s \in [0,x]$, all $j = 1,\ldots,M$.

The last implies $f_j = f_j^*$, $j = 1,\ldots,M$, over $[0,x]$, that is proving the uniqueness of solution to the initial value problem of this theorem. □

It follows another related application.

Theorem 26. *Let $\alpha_i \in \mathbb{R}_+, \beta > \alpha_i$, $i = 1,\ldots,r \in \mathbb{N}$ and let $f_j \in L_1(0,x)$, $j = 1,\ldots,M \in \mathbb{N}$; $x \in \mathbb{R}_+ - \{0\}$, have, respectively, fractional derivatives $D^{\beta}f_j$ in $[0,x]$, that are absolutely continuous on $[0,x]$, and let $D^{\beta-k}f_j(0) = 0$ for $k = 1,\ldots,[\beta]+1$; $j = 1,\ldots,M$. Furthermore $(D^{\beta}f_j)(0) = A_j \in \mathbb{R}$. Furthermore for $0 \le s \le x$ we have holding the system of fractional differential equations*

$$\left(D^{\beta}f_j\right)'(s) = F_j\left(s, \{D^{\alpha_i}f_j(s)\}_{i=1,j=1}^{r,M}, \left\{D^{\beta}f_j(s)\right\}_{j=1}^{M}\right), \quad (85)$$

for $j = 1, \dots, M$. *Here* F_j *is Lebesgue measurable on* $[0, x] \times (\mathbb{R}^{r+1})^M$ *such that*

$$|F_j\Big(t; x_{11}, x_{12}, \dots, x_{1r}, x_{1,r+1}; x_{21}, x_{22}, \dots, x_{2r}, x_{2,r+1};$$
$$\dots x_{M1}, x_{M2}, \dots, x_{M,r+1}\Big)| \le \sum_{i=1}^{r} \left(\sum_{\ell=1}^{M} q_{\ell,i,j}(t) |x_{\ell i}| \right), \tag{86}$$

where $q_{\ell,i,j} \ge 0$, $1 \le i \le r$; $\ell, j = 1, \dots, M$, *are bounded on* $[0, x]$. *Call*

$$W := \max\{\|q_{\ell,i,j}\|_\infty;\ \ell, j = 1, \dots, M;\ i = 1, \dots, r\}. \tag{87}$$

Also we set $(0 \le s \le x)$

$$\theta(s) := \sum_{\lambda=1}^{M} (D^\beta f_\lambda(s))^2, \tag{88}$$

$$\rho := \sum_{\lambda=1}^{M} A_\lambda^2, \tag{89}$$

$$Q(s) := W(1 + \sqrt{2}(M-1)) \left(\sum_{i=1}^{r} \left(\frac{s^{\beta-\alpha_i}}{\Gamma(\beta-\alpha_i)\sqrt{\beta-\alpha_i}\sqrt{2\beta-2\alpha_i-1}} \right) \right), \tag{90}$$

and

$$\chi(s) := \sqrt{\rho} \left\{ 1 + Q(s) \cdot e^{(\int_0^s Q(t)dt)} \cdot \left[\int_0^s \left(e^{-(\int_0^t Q(y)dy)} \right) dt \right] \right\}^{1/2}. \tag{91}$$

Then

$$\sqrt{\theta(s)} \le \chi(s), \quad 0 \le s \le x. \tag{92}$$

Consequently we get

$$|D^\beta f_j(s)| \le \chi(s), \tag{93}$$

$$|f_j(s)| \le \frac{1}{\Gamma(\beta)} \int_0^s (s-t)^{\beta-1} \chi(t) dt, \tag{94}$$

all $0 \le s \le x$, $j = 1, \dots, M$. *Also it holds*

$$|D^{\alpha_i} f_j(s)| \le \frac{1}{\Gamma(\beta-\alpha_i)} \int_0^s (s-t)^{\beta-\alpha_i-1} \chi(t) dt, \tag{95}$$

all $0 \le s \le x$, $j = 1, \dots, M$, $i = 1, \dots, r$.

Proof. We observe that

$$(D^\beta f_j)(s)(D^\beta f_j)'(s) = (D^\beta f_j)(s)\, F_j \left(s, \{D^{\alpha_i} f_j(s)\}_{i=1,j=1}^{r,M}, \{D^\beta f_j(s)\}_{j=1}^{M} \right), \tag{96}$$

$j = 1, \dots, M$, all $0 \le s \le x$.

We then integrate (96),

$$\int_0^y (D^\beta f_j)(s)(D^\beta f_j)'(s) ds$$
$$= \int_0^y (D^\beta f_j)(s)\ F_j \left(s, \{D^{\alpha_i} f_j(s)\}_{i=1,j=1}^{r,M}, \{D^\beta f_j(s)\}_{j=1}^{M} \right) ds, \tag{97}$$

$0 \le y \le x$.

Hence we obtain

$$\frac{((D^{\beta}f_j)(s))^2}{2}\Bigg|_0^y \leq \int_0^y |D^{\beta}f_j(s)|\,|F_j\cdots|ds \overset{(86)}{\underset{(87)}{\leq}}$$

$$W\int_0^y |D^{\beta}f_j(s)|\left[\sum_{i=1}^r\left(\sum_{\ell=1}^M |D^{\alpha_i}f_\ell(s)|\right)\right]ds$$

$$= W\left(\sum_{i=1}^r\left(\sum_{\ell=1}^M\left(\int_0^y |D^{\alpha_i}f_\ell(s)|\,|D^{\beta}f_j(s)|ds\right)\right)\right). \quad (98)$$

Thus we have for $j = 1,\ldots,M$ that

$$(D^{\beta}f_j(y))^2 \leq A_j^2 + 2W\left(\sum_{i=1}^r\left(\sum_{\ell=1}^M\left(\int_0^y |D^{\alpha_i}f_\ell(s)|\,|D^{\beta}f_j(s)|ds\right)\right)\right). \quad (99)$$

Consequently it holds

$$\theta(y) \leq \rho + 2W\left(\sum_{i=1}^r\left(\sum_{j=1}^M\left(\sum_{\ell=1}^M\left(\int_0^y |D^{\alpha_i}f_\ell(s)|\,|D^{\beta}f_j(s)|ds\right)\right)\right)\right) \quad (100)$$

$$= \rho + 2W\left\{\sum_{i=1}^r\left\{\left(\int_0^y\left(\sum_{\lambda=1}^M |D^{\alpha_i}f_\lambda(t)|\,|D^{\beta}f_\lambda(t)|\right)dt\right)\right.\right.$$

$$\left.\left.+\sum_{\substack{\tau,m\in\{1,\ldots,M\}\\ \tau\neq m}}\left(\int_0^y \left(|D^{\alpha_i}f_m(t)|\,|D^{\beta}f_\tau(t)| + |D^{\alpha_i}f_\tau(t)|\,|D^{\beta}f_m(t)|\right)dt\right)\right\}\right\}. \quad (101)$$

Using Corollary 10 from here, and Corollary 13 of [7], we find

$$\theta(y) \leq \rho + 2\,W\left\{\sum_{i=1}^r\left\{\left(\frac{y^{(\beta-\alpha_i)}}{2\,\Gamma(\beta-\alpha_i)\sqrt{\beta-\alpha_i}\sqrt{2\beta-2\alpha_i-1}}\right)\left(\int_0^y \theta(t)dt\right)\right.\right.$$

$$\left.\left.+\left(\frac{y^{(\beta-\alpha_i)}}{\sqrt{2}\,\Gamma(\beta-\alpha_i)\sqrt{\beta-\alpha_i}\sqrt{2\beta-2\alpha_i-1}}\right)(M-1)\left(\int_0^y \theta(t)dt\right)\right\}\right\}. \quad (102)$$

Hence we have

$$\theta(y) \leq \rho + Q(y)\int_0^y \theta(t)dt, \qquad \text{all } \ 0\leq y\leq x. \quad (103)$$

Here $\rho \geq 0$, $Q(y) \geq 0$, $Q(0) = 0$, $\theta(y) \geq 0$, all $0 \leq y \leq x$. As in the proof of Theorem 27 of [7], see also [5], we get (92), (93). Using Lemma 3, see (3), for $\alpha = 0$, and (93) we obtain (94). Using again (3) and (93) we get (95). □

Finally we give a specialized application.

Theorem 27. *Let* $\alpha_i \in \mathbb{R}_+$, $\beta > \alpha_i$, $i = 1,\ldots,r \in \mathbb{N}$ *and* $f_j \in L_1(0,x)$, $j = 1,\ldots,M \in \mathbb{N}$; $x \in \mathbb{R}_+ - \{0\}$, *have, respectively, fractional derivatives* $D^{\beta}f_j$ *in* $[0,x]$, *that are absolutely continuous on* $[0,x]$, *and let* $D^{\beta-\mu}f_j(0) = 0$ *for* $\mu = 1,\ldots,[\beta]+1$;

$j = 1, \ldots, M$. *Furthermore* $(D^{\beta} f_j)(0) = A_j \in \mathbb{R}$. *Furthermore for* $0 \leq s \leq x$ *we have holding the system of fractional differential equations*

$$\left(D^{\beta} f_j\right)'(s) = F_j\left(s, \left\{D^{\alpha_i} f_j(s)\right\}_{i=1,j=1}^{r,M}, \left\{D^{\beta} f_j(s)\right\}_{j=1}^{M}\right), \tag{104}$$

for $j = 1, \ldots, M$.

For fixed $i_* \in \{1, \ldots, r\}$ *we assume that* $\alpha_{i_*+1} = \alpha_{i_*} + 1$, *where* $\alpha_{i_*}, \alpha_{i_*+1} \in \{\alpha_1, \ldots, \alpha_r\}$. *Call* $k := \alpha_{i_*}$, $\gamma := \alpha_{i_*} + 1$, *i.e.* $\gamma = k + 1$. *Here* F_j *is Lebesgue measurable on* $[0, x] \times (\mathbb{R}^{r+1})^M$ *such that*

$$\begin{aligned}
&|F_j(t, x_{11}, x_{12}, \ldots, x_{1r}, x_{1,r+1}; x_{21}, x_{22}, \ldots, x_{2r}, x_{2,r+1}; \\
&x_{31}, x_{32}, \ldots, x_{3r}, x_{3,r+1}; \ldots; x_{M1}, x_{M2}, \ldots, x_{Mr}, x_{M,r+1})| \\
&\leq \left\{\left\{\sum_{\ell=1}^{M-1}\left(q_{\ell,1,j}(t)|x_{\ell_{i_*}}|\sqrt{|x_{\ell+1,i_*+1}|} + q_{\ell,2,j}(t)|x_{\ell+1,i_*}|\sqrt{|x_{\ell,i_*+1}|}\right)\right\}\right. \\
&\left. + \left(q_{M,1,j}(t)|x_{1i_*}|\sqrt{|x_{M,i_*+1}|} + q_{M,2,j}(t)|x_{Mi_*}|\sqrt{|x_{1,i_*+1}|}\right)\right\},
\end{aligned} \tag{105}$$

where all $0 \leq q_{\ell,1,j}$, $q_{\ell,2,j} \not\equiv 0$, *are bounded over* $[0, x]$. *Put*

$$W := \max\left\{\|q_{\ell,1,j}\|_\infty, \|q_{\ell,2,j}\|_\infty\right\}_{\ell,j=1}^{M}. \tag{106}$$

Also set

$$\theta(s) := \sum_{j=1}^{M} |(D^{\beta} f_j)(s)|, \quad 0 \leq s \leq x, \tag{107}$$

$$\rho := \sum_{j=1}^{M} |A_j|,$$

$$\Phi^*(s) := \left(\frac{2}{\sqrt{3\beta - 3k - 2}}\right) \frac{s^{\left(\frac{3\beta - 3k - 1}{2}\right)}}{(\Gamma(\beta - k))^{3/2}}, \tag{108}$$

all $0 \leq s \leq x$, *and*

$$Q(s) := 2MW\Phi^*(s), \quad 0 \leq s \leq x, \tag{109}$$

$$\sigma := \|Q(s)\|_\infty = \frac{4MW\, x^{\left(\frac{3\beta - 3k - 1}{2}\right)}}{\sqrt{3\beta - 3k - 2}\, (\Gamma(\beta - k))^{3/2}}. \tag{110}$$

We assume that

$$x\sigma\sqrt{\rho} < 2. \tag{111}$$

Call

$$\tilde{\varphi}(s) := \rho + Q(s)\left[\frac{4\rho^{3/2} s - \sigma\rho^2 s^2}{(2 - \sigma\sqrt{\rho}\, s)^2}\right], \quad \text{all } 0 \leq s \leq x. \tag{112}$$

Then

$$\theta(s) \leq \tilde{\varphi}(s), \quad \text{all } 0 \leq s \leq x, \tag{113}$$

in particular we have

$$|D^{\beta} f_j(s)| \leq \tilde{\varphi}(s), \quad j = 1, \ldots, M, \quad \text{for all } 0 \leq s \leq x. \tag{114}$$

Furthermore we get

$$|f_j(s)| \leq \frac{1}{\Gamma(\beta)} \int_0^s (s-t)^{\beta-1} \tilde{\varphi}(t) dt, \tag{115}$$

and,

$$|D^{\alpha_i} f_j(s)| \leq \frac{1}{\Gamma(\beta-\alpha_i)} \int_0^s (s-t)^{\beta-\alpha_i-1} \tilde{\varphi}(t) dt, \tag{116}$$

$j = 1, \ldots, M$*;* $i = 1, \ldots, r$*, all* $0 \leq s \leq x$*.*

Proof. Notice here that $W > 0$ and $\sigma > 0$. Clearly, here $D^\beta f_j$ are L_∞ fractional derivatives in $[0, x]$ For $0 \leq s \leq y \leq x$, by (104) we get that

$$\int_0^y \left(D^\beta f_j\right)'(s) ds = \int_0^y F_j \left(s, \{D^{\alpha_i} f_j(s)\}_{i=1,j=1}^{r,M}, \{D^\beta f_j(s)\}_{j=1}^M\right) ds. \tag{117}$$

That is

$$(D^\beta f_j)(y) = A_j + \int_0^y F_j(s, \ldots) ds. \tag{118}$$

Then we observe that

$$\begin{aligned} |D^\beta f_j(y)| &\leq |A_j| + \int_0^y |F_j(s, \ldots)| ds \leq |A_j| \\ &+ W \int_0^y \Bigg\{ \Bigg\{ \sum_{\ell=1}^{M-1} (|D^{\alpha_{i_*}} f_\ell(s)| \sqrt{|D^{\alpha_{i_*}+1} f_{\ell+1}(s)|} + |D^{\alpha_{i_*}} f_{\ell+1}(s)| \sqrt{|D^{\alpha_{i_*}+1} f_\ell(s)|}) \Bigg\} \\ &+ (|D^{\alpha_{i_*}} f_1(s)| \sqrt{|D^{\alpha_{i_*}+1} f_M(s)|} + |D^{\alpha_{i_*}} f_M(s)| \sqrt{|D^{\alpha_{i_*}+1} f_1(s)|}) \Bigg\} ds. \end{aligned} \tag{119}$$

That is

$$\begin{aligned} |D^\beta f_j(y)| &\leq |A_j| + W \Bigg(\int_0^y \Bigg\{ \Bigg\{ \sum_{\ell=1}^{M-1} \Bigg(|D^k f_\ell(s)| \sqrt{|D^{k+1} f_{\ell+1}(s)|} \\ &+ |D^k f_{\ell+1}(s)| \sqrt{|D^{k+1} f_\ell(s)|} \Bigg) \Bigg\} \Bigg) \\ &+ \left(|D^k f_1(s)| \sqrt{|D^{k+1} f_M(s)|} + |D^k f_M(s)| \sqrt{|D^{k+1} f_1(s)|} \right) \Bigg\} ds. \end{aligned} \tag{120}$$

By Corollary 14 we obtain

$$|D^\beta f_j(s)| \leq |A_j| + 2W \phi^*(s) \left(\int_0^s \left(\sum_{\ell=1}^M |D^\beta f_\ell(t)|^{3/2} \right) dt \right), \tag{121}$$

$j = 1, \ldots, M$, all $0 \leq s \leq x$.

Therefore by adding all of inequalities (121) we get

$$\theta(s) \leq \rho + 2MW\phi^*(s) \left(\int_0^s \left(\sum_{\ell=1}^M |D^\beta f_\ell(t)|^{3/2} \right) dt \right), \tag{122}$$

$$\overset{(4)}{\leq} \rho + 2MW\phi^*(s) \left(\int_0^s \left(\sum_{\ell=1}^M |D^\beta f_\ell(t)| \right)^{3/2} dt \right), \tag{123}$$

i.e.

$$\theta(s) \leq \rho + (2MW\phi^*(s)) \left(\int_0^s (\theta(t))^{3/2} dt \right), \tag{124}$$

all $0 \leq s \leq x$.

More precisely we get that

$$\theta(s) \leq \rho + Q(y) \left(\int_0^s (\theta(t))^{3/2} dt \right), \quad \text{all } 0 \leq s \leq x. \tag{125}$$

Notice that $\theta(s) \geq 0$, $\rho \geq 0$, $Q(s) \geq 0$, and $Q(0) = 0$ by $\Phi^*(0) = 0$. Acting here as in the proof of Theorem 28 of [7] we derive (113) and (114).

Using Lemma 3, see (3), along with (114), we obtain (115) and (116). □

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