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GENERALIZED MONOTONE METHOD FOR PERIODIC BOUNDARY VALUE PROBLEMS OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, the generalized monotone iterative technique is developed to study existence of solutions of PBVP for fractional differential equation, where the function considered is split into two parts, a function that is can be made non-decreasing and a function that is non-increasing.

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1. INTRODUCTION

The derivative of an arbitrary order or fractional derivative has been introduced almost 300 years ago with a query posed by L'Hospital to Leibnitz. The fractional calculus was reasonably developed by 19^{th} century. It was realized, only in the past few decades that these derivatives are better models to study physical phenomenon in transient state. This gave a fresh lease to this field and there is a growing interest to study the theory of fractional differential equations.[1, 3, 4, 5, 6, 8, 9, 10].

The monotone iterative technique [7] is an effective and flexible mechanism that offers theoretical, as well as constructive results in a closed set, namely, the sector. The generalized monotone iterative technique is a generalization and a refinement of the monotone method.

In this paper, the PBVP for Caputo fractional differential equation is considered and the generalized monotone iterative technique is developed to cater to the situation where the function on the righthand side is split into two functions- a function that can be made into a non-decreasing function and a non-increasing function.

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2. PRELIMINARIES

We begin with the definition of the Riemann-Liouville fractional differential equation, the Caputo fractional differential equation and then proceed to give the relation between these derivatives.

The Riemann-Liouville fractional differential equation is given by

$$D^q x = f(t,x),\ x(t_0) = x^0 = x(t)(t-t_0)^{1-q}|_{t=t_0}, \tag{2.1}$$

and the corresponding Volterra fractional integral equation is given by

$$x(t) = x^0(t) + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}x(s)ds$$

where $x^0(t) = \dfrac{x^0(t-t_0)^{q-1}}{\Gamma(q)}$.

The Caputo fractional differential equation is given by

$${}^cD^q x = f(t,x),\ x(t_0) = x_0$$

and the corresponding Volterra fractional integral equation is given by

$$x(t) = x_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}x(s)ds.$$

The relation between the Caputo fractional derivative and the Riemann-Liouville fractional derivative is given by

$${}^cD^q x(t) = D^q[x(t) - x(t_0)].$$

Using this relation we can show that the following results that are true for Riemann Liouville fractional derivative, also hold for Caputo derivative.

We need the following notation before proceeding further.

$$C_p[[t_0,T],R] = [u \in C((t_0,T],R] \text{ and } (t-t_0)^p u(t) \in C[[t_0,T],R].$$

Now we state the following lemmas without proof.

Lemma 2.1. *Let $m \in C_p([t_0,T],R)$ be locally Hölder continuous with exponent $\lambda > q$ and for any $t_1 \in (t_0,T]$, we have*

$$m(t_1) = 0 \quad and \quad m(t) \le 0 \quad for \quad t_0 \le t \le t_1. \tag{2.2}$$

Then it follows that,

$$D^q m(t_1) \ge 0. \tag{2.3}$$

Lemma 2.2. *Let $\{x_\epsilon(t)\}$ be a family of continuous functions on $[t_0,T]$, for each $\epsilon > 0$, where $D^q x_\epsilon(t) = f(t,x_\epsilon(t))$, $x^0_\epsilon = x_\epsilon(t)(t-t_0)^{1-q}|_{t=t_0}$, and $|f(t,x_\epsilon(t))| \le M$ for $t_0 \le t \le T$. Then the family $\{x_\epsilon(t)\}$ is equicontinuous on $[t_0,T]$.*

In order to develop the monotone iterative technique for PBVP of Caputo fractional differential equation we need the explicit solution of the nonhomogeneous linear fractional differential equation of Caputo's type given by

$$^cD^q x = \lambda x + f(t), \quad x(t_0) = x_0, \tag{2.4}$$

where $f \in C_q([t_0, T], R)$, is Hölder continuous with exponent q. Following the method of successive approximations we get the unique solution of (2.4) as

$$x(t) = x_0 E_q(\lambda(t-t_0)^q) + \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) f(s) ds, \quad t \in [t_0, T], \tag{2.5}$$

where

$$E_q(t) = \sum_{k=1}^{\infty} \frac{t^k}{\Gamma(qk+1)} \text{ and } E_{q,q}(t) = \sum_{k=1}^{\infty} \frac{t^k}{\Gamma(qk+q)}$$

are Mittag-Leffler functions of one parameter and two parameters respectively.

Theorem 2.3. *Let $f \in C([0, 2\pi] \times R, R)$, $v, w \in C([0, 2\pi], R)$, v, w be Hölder continuous for exponent $\lambda > q$, $0 < \lambda < 1$ and for $0 < t \leq 2\pi$,*

$$\left.\begin{array}{rcl} ^cD^q v(t) & \leq & f(t, v(t)),\ v(0) \leq v(2\pi) \\ ^cD^q w(t) & \geq & f(t, w(t)),\ w(0) \geq w(2\pi) \end{array}\right\}. \tag{2.6}$$

Suppose further $f(t, x)$ is strictly decreasing in x for each t. Then

$$v(t) \leq w(t), \qquad 0 \leq t \leq 2\pi. \tag{2.7}$$

Proof. If (2.7) is not true, then there exists an $\epsilon > 0$ and $t_0 \in [0, 2\pi]$ such that

$$v(t_0) = w(t_0) + \epsilon \text{ and } v(t) \leq w(t) + \epsilon, \quad 0 \leq t \leq 2\pi. \tag{2.8}$$

Setting $m(t) = v(t) - w(t) - \epsilon$, we find that, if $t_0 \in (0, 2\pi]$, $m(t_0) = 0$, $m(t) \leq 0$, $0 \leq t \leq t_0 \leq 2\pi$. If $t_0 = 0$, we get, because of (2.6), $v(2\pi) \geq v(0) = w(0) + \epsilon \geq w(2\pi) + \epsilon$, and hence, in all cases, we have

$$m(t_0) \geq 0 \text{ and } m(t) \leq 0 \text{ for } 0 \leq t \leq t_0 \leq 2\pi \tag{2.9}$$

We therefore obtain, using (2.6), strictly decreasing nature of $f(t, x)$ in x and Lemma 2.1,

$$f(t_0, v(t_0)) \geq^c D^q v(t_0) \geq^c D^q w(t_0) \geq f(t_0, w(t_0)) > f(t_0, v(t_0)),$$

which is a contradiction. Hence (2.7) is valid and the proof is complete. □

Corollary 2.4. *Let $m : [0, 2\pi] \to R$ be Hölder continuous and satisfy*

$$^cD^q m(t) \leq -Mm(t),\ 0 \leq t \leq 2\pi, \quad m(0) \leq m(2\pi), \quad M > 0.$$

Then $m(t) \leq 0$, $0 \leq t \leq 2\pi$.

We next consider the PBVP for linear nonhomogeneous fractional differential equation given by

$$^{c}D^{q}u(t) = -Mu(t) + h(t), \quad u(0) = u(2\pi) \tag{2.10}$$

where $M > 0$ and $h \in C([0, 2\pi], R)$.

In order to prove the existence of a solution for the PBVP (2.10), we begin with the solution of the IVP

$$^{c}D^{q}u(t) = -Mu(t) + h(t), \quad u(0) = u_0 \tag{2.11}$$

which is given by

$$u(t) = u_0 E_q(Mt^q) + \int_0^t (t-s)^{q-1} E_{q,q}(M(t-s)^q) h(s) ds.$$

Now setting $t = 2\pi$ and $u(2\pi) = u(0) = u_0$, we get

$$u_0[1 - E_q(Mt^q)] = \int_0^{2\pi} (2\pi - s)^{q-1} E_{q,q}(M(2\pi - s)^q) h(s) ds$$

which give,

$$u_0 = \frac{1}{[1 - E_q(Mt^q)]} \int_0^{2\pi} (2\pi - s)^{q-1} E_{q,q}(M(2\pi - s)^q) h(s) ds.$$

Thus the solution of the PBVP (2.11) is given by

$$\left. \begin{array}{rcl} u(t) & = & \dfrac{E_q(Mt^q)}{1 - E_q(Mt^q)} \int_0^{2\pi} (2\pi - s)^{q-1} E_{q,q}(M(2\pi - s)^q) h(s) ds \\ & & + \int_0^t (t-s)^{q-1} E_{q,q}(M(t-s)^q) h(s) ds. \end{array} \right\}. \tag{2.12}$$

We now proceed to develop the generalized monotone iterative technique.

3. GENERALIZED MONOTONE ITERATIVE TECHNIQUE FOR PBVP

In this section, we continue to consider the Caputo fractional differential equation to develop the generalized monotone method for PBVP. We begin by considering the IVP for Caputo fractional differential equation, given by

$$^{c}D^{q}x = f(t, x), \quad x(0) = x_0 \tag{3.1}$$

where $f \in C([0, 2\pi] \times R, R)$, $0 < q < 1$.

The corresponding fractional Volterra integral equation is given by

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds, \quad 0 \le t \le 2\pi. \tag{3.2}$$

We are interested in the situation where we have mixed monotone functions on the right hand side of (3.1) and the problem having periodic boundary conditions. So we consider

$$ {}^cD^q x = f(t,x) + g(t,x), \quad x(0) = x(2\pi) \tag{3.3} $$

where $f, g \in C[J \times R, R]$, $J = [0, 2\pi]$, $x \in C^q[J, R]$.

We note that the coupled lower and upper solutions of type I for (3.3) are as follows.

Definition 3.1. Let $v_0, w_0 \in C^q[J, R]$. Then (v_0, w_0) are said to be coupled lower and upper solutions of type I for (3.3) if,

$$ \begin{aligned} {}^cD^q v_0 &\leq f(t, v_0) + g(t, w_0),\ v_0(0) \leq v_0(2\pi) \\ {}^cD^q w_0 &\geq f(t, w_0) + g(t, v_0),\ w_0(0) \geq w_0(2\pi). \end{aligned} $$

We now proceed to prove existence of solutions for (3.1) using the monotone method.

Theorem 3.2. *Assume that*

(A$_1$) $v_0, w_0 \in C^q[J, R]$ *are coupled lower and upper solutions of type* I *for* (3.1) *with* $v_0(t) \leq w_0(t)$ *on* J.

(A$_2$) $f, g \in C[J \times R, R]$, $g(t, y)$ *is nonincreasing in* y *for each* t *and* $f(t, x) + Mx$ *is nondecreasing in* x *for each* t*. There exist monotone sequences* $\{v_n(t)\}, \{w_n(t)\} \in C^q[J, R]$ *such that* $v_n(t) \to \rho(t)$ *and* $w_n(t) \to r(t)$ *in* $C^q[J, R]$ *and* (ρ, r) *are coupled minimal and maximal solutions of* (3.3) *respectively, that is* (ρ, r) *satisfy*

$$ \begin{aligned} {}^cD^q \rho &= f(t, \rho) + g(t, r) \\ {}^cD^q r &= f(t, r) + g(t, \rho) \qquad \text{on } J. \end{aligned} $$

Proof. Consider the PBVP given by

$$ \left. \begin{aligned} {}^cD^q x + Mx &= f(t, x) + Mx + g(t, x) \\ x(0) &= x(2\pi). \end{aligned} \right\} $$

Since this a resonance problem, we begin by considering the IVP

$$ \left. \begin{aligned} {}^cD^q x + Mx &= f(t, \eta) + M\eta + g(t, \mu) \\ x(0) &= x_0 \end{aligned} \right\} \tag{3.4} $$

where $\eta, \mu \in C^q(J, R]$. The solution of (3.4) is given by

$$ x(t) = x_0 E_q(Mt^q) + \int_0^t (t-s)^{q-1} E_{q,q}(M(t-s)^q) h(s) ds $$

where $h(s) = f(s, \eta(s)) + M\eta(s) + g(t, \mu(s))$. Now setting $t = 2\pi$ and noting that $x(2\pi) = x(0) = u_0$, we get

$$x_0 = \frac{1}{1 - E_q(Mt^q)} \int_0^{2\pi} (2\pi - s)^{q-1} E_{q,q}[M(2\pi - s)^q] h(s) ds$$

and the solution of the PBVP is given by

$$\left.\begin{aligned} x(t) \quad = \quad & \frac{E_q(Mt^q)}{1 - E_q(Mt^q)} \int_0^{2\pi} (2\pi - s)^{q-1} E_{q,q}[M(2\pi - s)^q] h(s) ds \\ & + \int_0^t (t - s)^{q-1} E_{q,q}(M(t - s)^q) h(s) ds. \end{aligned}\right\} \tag{3.5}$$

We now claim that the solutions of (3.4) are unique and the proof is as follows. Suppose $x_1(t)$ and $x_2(t)$ be two solutions of (3.4) and set $p = x_2 - x_1$ on J. Then

$${}^cD^q p = {}^cD^q x_2 - {}^cD^q x_1 = -Mp, \; p(0) = p(2\pi).$$

Then from Corollary 2.4, we get that $x_1(t) = x_2(t)$ on J, completing the proof of uniqueness.

We next define the iterates as below.

$${}^cD^q v_{n+1} = f(t, v_n) + g(t, w_n) - M(v_{n+1} - v_n), \; v_{n+1}(0) = v_{n+1}(2\pi) \tag{3.6}$$

and

$${}^cD^q w_{n+1} = f(t, w_n) + g(t, v_n) - M(w_{n+1} - w_n), \; w_{n+1}(0) = w_{n+1}(2\pi). \tag{3.7}$$

Clearly, the above arguments imply the existence of the unique solutions v_{n+1}, w_{n+1} for (3.6), (3.7) respectively. By setting $n = 0$ in (3.6) and (3.7), we get the existence of solutions $v_1(t)$ and $w_1(t)$ respectively. We shall show that $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$. For this, consider $p = v_0 - v_1$ on J. Then,

$${}^cD^q p = {}^cD^q v_0 - {}^cD^q v_1 \leq f(t, v_0) + g(t, w_0) - f(t, v_0) - g(t, w_0) + M(v_1 - v_0).$$

This gives, ${}^cD^q p \leq -Mp$, $p(0) \leq p(2\pi)$. Then using Corollary 2.4, we infer that $p(t) \leq 0$ or $v_0(t) \leq v_1(t)$, $t \in J$. Similarly we can show that $w_1(t) \leq w_0(t)$ and $v_1(t) \leq w_1(t)$, $t \in J$. Assume that for some $k > 1$, $v_{k-1} \leq v_k \leq w_k \leq w_{k-1}$ on J. We claim that $v_k \leq v_{k+1} \leq w_{k+1} \leq w_k$ on J. To prove the claim, set $p = v_k - v_{k+1}$. Then,

$$\begin{aligned} {}^cD^q p &= {}^cD^q v_k - {}^cD^q v_{k+1} \\ &= f(t, v_{k-1}) - M(v_k - v_{k-1}) + g(t, w_{k-1}) - f(t, v_k) + M(v_{k+1} - v_k) - g(t, w_k) \\ &\leq f(t, v_k) + Mv_k + g(t, w_k) - Mv_k - f(t, v_k) + M(v_{k+1} - v_k) - g(t, w_k) \\ &= -Mp, \end{aligned}$$

which holds by the nondecreasing nature of $f(t,x)+Mx$ in x and nonincreasing nature of $g(t,y)$ in y. Also $p(0)=p(2\pi)$. Hence by Corollary 2.4, we deduce that $v_k \le v_{k+1}$ on J. Next consider $p = v_{k+1} - w_{k+1}$. Then

$$\begin{aligned} {}^cD^q p &= {}^cD^q v_{k+1} - {}^cD^q w_{k+1} \\ &= f(t,v_k) + g(t,w_k) - M(v_{k+1} - v_k) - f(t,w_k) - g(t,v_k) + M(w_{k+1} - w_k) \\ &\le f(t,w_k) + g(t,v_k) + Mw_k - Mv_{k+1} - f(t,w_k) - g(t,v_k) + M(w_{k+1} - w_k) \\ &= -Mp \end{aligned}$$

and $p(0)=p(2\pi)$. Hence as earlier, by Corollary 2.4, we conclude that $v_{k+1} \le w_{k+1}$ on J. Similarly, we can show that $w_{k+1} \le w_k$ on J. Hence we have from the principle of mathematical induction,

$$v_0 \le v_1 \le v_2 \le \cdots \le v_k \le w_k \le \cdots \le w_2 \le w_1 \le w_0 \ \text{ on } \ J. \tag{3.8}$$

Since the sequences $\{v_n\}, \{w_n\}$ are uniformly bounded by (3.8), we observe that $\{{}^cD^q v_n\}$ and $\{{}^cD^q w_n\}$ are also uniformly bounded on $[0, 2\pi]$, in view of the relations (3.6) and (3.7). Then utilizing Lemma 2.2 we can conclude the equicontinuity of the sequences $\{v_n\}, \{w_n\}$. Now from Ascoli-Arzela Theorem we obtain that $v_n \to \rho$ and $w_n \to r$ as $n \to \infty$ uniformly on J. Clearly we have $v_0 \le \rho \le r \le w_0$ on J. Now to show that (ρ, r) are coupled solutions of (3.3), we consider the fractional Volterra integral equations corresponding to the equations (3.6) and (3.7) for v_n and w_n respectively and observe that as $n \to \infty$, we obtain that ρ and r satisfy the equation (3.5) with the corresponding $h(s)$, proving the required result.

To prove that (ρ, r) are coupled minimal and maximal solutions of PBVP (3.3), we first suppose that for some $k > 0$,

$$v_{k-1} \le x \le w_{k-1} \text{ on } J,$$

where x is any solution of the PBVP (3.3) such that $v_0 \le x \le w_0$. Setting $p = v_k - x$, we have

$$\begin{aligned} {}^cD^q p &= {}^cD^q v_k - {}^cD^q x \\ &= f(t,v_{k-1}) + g(t,w_{k-1}) - M(v_k - v_{k-1}) - f(t,x) - g(t,x) \\ &\le f(t,x) + Mx + g(t,\, w_{k-1}) - Mv_k - f(t,x) - g(t,\, w_{k-1}) \\ &= -Mp, \end{aligned}$$

by the nondecreasing nature of $f(t,x)+Mx$ in x and the nonincreasing nature of $g(t,x)$ in x for each t. Thus,

$${}^cD^q p \le -Mp, \quad p(0) = p(2\pi).$$

Again Corollary 2.4 yields $v_k \le x$ on J. Similar arguments allow us to conclude that $x \le w_k$ on J. Then it follows by induction that

$$v_n \le x \le w_n \text{ for all } n \text{ on } J.$$

Hence $\rho \le x \le r$ on J, proving the theorem. □

Corollary 3.3. *If, in addition to the assumption of Theorem* 3.2*, we assume that for* $x_1 \ge x_2$*,* f *and* g *satisfy*

$$f(t, x_1) - f(t, x_2) \le -N_1(x_1 - x_2)$$

and

$$g(t, x_1) - g(t, x_2) \ge -N_2(x_1 - x_2)$$

where $0 < N_1 < M$*, and* $N_2 > 0$*. Then* $\rho = x = r$ *is the unique solution of* (3.3)*.*

Proof. Since $\rho \le r$ it is enough to show that $r \le \rho$. Consider $p = r - \rho$. Then

$$\begin{aligned} {}^cD^q p &= {}^cD^q r - {}^cD^q \rho \\ &= f(t, r) - f(t, \rho) + g(t, \rho) - g(t, r) \\ &\le -(N_1 + N_2)(r - \rho) \\ \text{and} \qquad p(0) &= p(2\pi). \end{aligned}$$

Hence, by Corollary 2.4, we arrive at the required conclusion that $r \le \rho$, which means $\rho = x = r$ is the unique solution. □

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