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QUASILINEARIZATION FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we develop the method of quasilinearization for fractional differential equations when the right hand side function satisfies a condition weaker than convexity and also in the situation when it is concave.

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1. INTRODUCTION

It has been realized recently that fractional differential equations are appropriate models for studying and describing the memory and hereditary properties of various materials and processes. This lead to a renewed surge of research activity in the area and basic theoretical concepts like differential and integral inequalities [4], existence and uniqueness results [2, 3, 5, 6, 10] were studied. The monotone iterative technique and its generalization for IVPs and PBVPs has been developed in [8, 1] and [9, 11] respectively.

The method of quasilinearization [7] is a very fruitful technique that guarantees the quadratic convergence of a monotone sequence of solutions of linear equations, which are constructed using the method of lower and upper solutions, to a unique solution of the differential equation under certain conditions.

In this paper we develop this useful method of quasilinearization to fractional differential equations which are excellent models for many physical phenomena. We obtain results when the right hand side function of the fractional differential equation satisfies a weaker hypothesis than convexity and also when the right hand side function is concave.

2. PRELIMINARIES

We start with the definitions of Caputo fractional differential equation, Riemann-Liouville fractional differential equation and the relation between the two derivatives.

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The Caputo fractional differential equation is given by

$$ {}^cD^q x = f(t, x),\ \ x(t_0) = x_0. \tag{2.1} $$

and the corresponding Volterra fractional integral equation by

$$ x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}\ f(s, x(s))ds. \tag{2.2} $$

The Riemann-Liouville fractional differential equation is expressed as

$$ D^q x \;=\; f(t, x), \tag{2.3} $$

$$ x(t_0) \;=\; x^0 = x(t)(t-t_0)^{1-q}|_{t=t_0} \tag{2.4} $$

and the corresponding Volterra fractional integral equation is

$$ x(t) = x^0(t) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}\ f(s, x(s))ds, \tag{2.5} $$

where

$$ x^0(t) = \frac{x^0(t-t_0)^{q-1}}{\Gamma(q)}. $$

The relation between the Caputo fractional derivative and the Riemann-Liouville fractional derivative is as follows:

$$ {}^cD^q x(t) = D^q\ [x(t) - x(t_0)]. \tag{2.6} $$

We shall state the needed notation and the required results from [3, 4, 5], without proof, for Riemann-Liouville fractional derivatives below. Let $C_p([t_0, T], R) = \{u \in C((t_0, T],\ R)$ and $(t-t_0)^p\ u(t) \in C([t_0, T],\ R)\}$.

Lemma 2.1. *Let $m \in C_p([t_0, T],\ R)$ be locally Hölder continuous with exponent $\lambda > q$ and for any $t_1 \in (t_0, T]$, we have*

$$ m(t_1) = 0 \textit{ and } m(t) \le 0 \textit{ for } t_0 \le t \le t_1, \tag{2.7} $$

then

$$ D^q m(t_1) \ge 0. \tag{2.8} $$

Lemma 2.2. *Let $\{x_\epsilon(t)\}$ be a family of continuous functions on $[t_0, T]$, for each $\epsilon > 0$, where*

$$ D^q\ x_\epsilon(t) = f(t, x_\epsilon(t)), \quad x_\epsilon^0 = x_\epsilon(t)(t-t_0)^{1-q}|_{t=t_0}, $$

and

$$ |f(t, x_\epsilon(t))| \le M \textit{ for } t_0 \le t \le T. $$

Then the family $\{x_\epsilon(t)\}$ is equicontinuous on $[t_0,\ T]$.

Theorem 2.3. *Let $v, w \in C_p([t_0, T],\ R)$ be locally Hölder continuous for an exponent $0 < \lambda < 1$ and $\lambda > q$, $f \in C([t_0, T] \times R,\ R)$ and*

$$\begin{aligned} (i) \quad & D^q v(t) \le f(t, v(t)), \\ (ii) \quad & D^q w(t) \ge f(t, w(t)),\ t_0 < t \le T, \end{aligned} \tag{2.9}$$

one of the inequalities (i) or (ii) being strict. Then

$$v^0 < w^0 \tag{2.10}$$

where $v^0 = v(t)(t - t_0)^{1-q}|_{t=t_0}$, and $w^0 = w(t)(t - t_0)^{1-q}|_{t=t_0}$, implies

$$v(t) < w(t),\ t_0 \le t \le T. \tag{2.11}$$

We first show that Lemma 2.1 also holds for Caputo derivative.

Lemma 2.4. *Let $m \in C_p([t_0, T], R)$ be locally Hölder continuous with exponent $\lambda > q$ and for any $t_1 \in (t_0, T]$ we have*

$$m(t_1) = 0 \text{ and } m(t) \le 0 \text{ for } t_0 \le t \le t_1. \tag{2.12}$$

Then it follows that

$${}^cD^q\ m(t_1) \ge 0. \tag{2.13}$$

Proof. In view of the relation (2.6), we have

$${}^cD^q m(t) = D^q[m(t) - m(t_0)].$$

Noting that $D^q m(t_0) = \dfrac{m(t_0)(t - t_0)^{-q}}{\Gamma(1 - q)}$ and $m(t_0) \le 0$ we get $-D^q m(t_0) \ge 0$.

Hence we obtain ${}^cD^q m(t) \ge D^q m(t)$.

Thus using Lemma 2.1, we conclude

$${}^cD^q m(t_1) \ge D^q\ m(t_1) \ge 0.$$

□

We next claim that both the Lemma 2.2 and the Theorem 2.3 hold for Caputo derivative also. The proofs being similar are omitted.

The explicit solution of the nonhomogeneous linear fractional differential equation of Caputo's type is used to prove the main result. Hence we present it here from [2]. The nonhomogeneous linear fractional differential equation of Caputo's type is given by

$${}^cD^q(x) = \lambda x + f(t), x(t_0) = x_0, \tag{2.14}$$

where $f \in C_q([t_0, T], R)$, is Hölder continuous with exponent q.

Using the method of successive approximations we get the unique solution of (2.14) as

$$x(t) = x_0 E_q(\lambda(t-t_0)^q) + \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) f(s)ds,\ t \in [t_0, T], \tag{2.15}$$

where

$$E_q(t) = \sum_{k=1}^{\infty} \frac{t^k}{\Gamma(qk+1)} \text{ and } E_{q,q}(t) = \sum_{k=1}^{\infty} \frac{t^k}{\Gamma(qk+q)}$$

are Mittag-Leffler functions of one parameter and two parameters respectively.

The next result deals with nonstrict fractional differential inequalities for Caputo derivative.

Theorem 2.5. *Let $v, w \in C_p([t_0, T],\ R)$ be Hölder continuous for an exponent $0 < \lambda < 1$ and $\lambda > q, f \in C([t_0, T] \times R, R)$ and*

$$\begin{array}{ll} (i) & {}^cD^q v(t) \le f(t, v(t)), \\ (ii) & {}^cD^q\ w(t) \ge f(t, w(t)). \end{array} \tag{2.16}$$

Suppose further that the standard Lipschitz condition

$$f(t,x) - f(t,y) \le L(x-y), x \ge y \text{ and } L > 0, \tag{2.17}$$

is satisfied. Then $v(t_0) \le w(t_0)$ implies

$$v(t) \le w(t), t_0 \le t \le T. \tag{2.18}$$

Proof. We set

$$w_\epsilon(t) = w(t) + \epsilon\lambda(t)$$

where

$$\lambda(t) = E_q(2L(t-t_0)^q).$$

Then

$$w_\epsilon(t_0) = w(t_0) + \epsilon > w(t_0)$$

as $\lambda(t_0) = 1$. Thus we get $w_\epsilon(t_0) > w(t_0) \ge v(t_0)$ and $w_\epsilon(t) > w(t)$.

Further using the Lipschitz condition (2.17)

$$\begin{aligned} {}^cD^q\ w_\epsilon(t) &= {}^cD^q w(t) + \epsilon\ {}^cD^q\ \lambda(t) \\ &\ge f(t, w(t)) + 2L\epsilon\ \lambda(t) \\ &\ge f(t,\ w_\epsilon(t)) - L\epsilon\ \lambda(t) + 2L\epsilon\ \lambda(t) \\ &> f(t, w_\epsilon(t)), \quad t_0 < t \le T. \end{aligned}$$

Now applying Theorem 2.3 for Caputo differential inequalities to $v(t), w_\epsilon(t)$ we get $v(t) < w_\epsilon(t),\ t_0 \le t \le T$ for every $\epsilon > 0$. As $\epsilon \to 0$ we get $v(t) \le w(t),\ t_0 \le t \le T$ and thus the proof is complete. □

Corollary 2.6. *The function $f(t,v) = \sigma(t)v$ where $\sigma(t) \le L$ is admissible in Theorem 2.5 to yield $v(t) \le 0$ on $t_0 \le t \le T$.*

3. QUASILINEARIZATION

The method of upper and lower solutions coupled with the monotone iterative technique offers monotone sequences that converge to the extremal solutions of the original nonlinear problem. When we employ the technique of lower and upper solutions together with the method of quasilinearization, it is possible to construct monotone sequences which converge quadratically to the solution of the original problem. In this section, we shall extend the method of quasilinearization to the fractional differential equation of Caputo type, namely,

$$ {}^cD^q x = f(t,x),\ x(t_0) = x_0, \tag{3.1}$$

where $f \in C([t_0,T] \times R, R)$, ${}^cD^q$ is Caputo's fractional derivative and

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s,x(s))ds,\ t \in [t_0,T], \tag{3.2}$$

is the equivalent Volterra fractional integral equation.

We shall prove the following simple result concerning quasilinearization, when f satisfies weaker conditions then convexity.

Theorem 3.1. *Assume that*

(i) $f \in C([t_0,T] \times R, R)$, $\alpha_0, \beta_0 \in C^q([t_0,T], R)$ *and* ${}^cD^q\alpha_0 \le f(t,\alpha_0)$, ${}^cD^q\beta_0 \ge f(t,\beta_0)$, $\alpha_0(t) \le \beta_0(t)$, $t \in [t_0,T]$, $\alpha_0(t_0) \le x_0 \le \beta_0(t_0)$;

(ii) $f(t,x) \ge f(t,y) + f_x(t,y)(x-y)$, $x \ge y$, $f_x(t,x)$ *is assumed to exist and satisfy*

$$|f_x(t,x) - f_x(t,y)| \le L|x-y|; \tag{3.3}$$

(iii) $f_x(t,x)$ *is nondecreasing in* x *for each* t.

Then there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ *such that* $\alpha_n \to \rho$, $\beta_n \to r$ *uniformly and monotonically to the unique solution* $\rho = r = x$ *of IVP* (3.1) *on* $[t_0,T]$ *and convergence is quadratic.*

Proof. We consider the following sequences generated by linear fractional differential equations,

$${}^cD^q\alpha_{n+1} = f(t,\alpha_n) + f_x(t,\alpha_n)(\alpha_{n+1} - \alpha_n),\ \alpha_{n+1}(t_0) = x_0, \tag{3.4}$$

$${}^cD^q\beta_{n+1} = f(t,\beta_n) + f_x(t,\alpha_n)(\beta_{n+1} - \beta_n),\ \beta_{n+1}(t_0) = x_0, \tag{3.5}$$

and $\alpha_0(t_0) \le x_0 \le \beta_0(t_0)$. Since the right hand side satisfies a Lipschitz condition, it is clear that unique solutions exist. Our aim is to show that

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \le \beta_n \le \cdots \le \beta_1 \le \beta_0 \ \ on\ [t_0,T]. \tag{3.6}$$

We shall first prove that

$$\alpha_0 \le \alpha_1 \le \beta_1 \le \beta_0 \ \ on\ [t_0,T]. \tag{3.7}$$

Set

$$p = \alpha_1 - \alpha_0$$

so that

$${}^cD^q p = {}^c D^q \alpha_1 - {}^cD^q \alpha_0 \geq f(t, \alpha_0) + f_x(t, \alpha_0)(\alpha_1 - \alpha_0) - f(t, \alpha_0).$$

This implies ${}^cD^q p \geq f_x(t, \alpha_0)p$ and $p(t_0) \geq 0$, which because of Corollary 2.6 yields $p(t) \geq 0$ on $[t_0, T]$. Thus we have $\alpha_0 \leq \alpha_1$. Similarly, we can show $\beta_1 \leq \beta_0$. To show $\alpha_1 \leq \beta_1$, we set $p = \beta_1 - \alpha_1$ so that

$${}^cD^q p = f(t, \beta_0) + f_x(t, \alpha_0)(\beta_1 - \beta_0) - [f(t, \alpha_0) + f_x(t, \alpha_0)(\alpha_1 - \alpha_0)].$$

Using assumption (ii), we get

$${}^cD^q p \geq f_x(t, \alpha_0)[\beta_1 - \beta_0 - (\alpha_1 - \alpha_0) + \beta_0 - \alpha_0] = f_x(t, \alpha_0)p.$$

Since $p(t_0) = 0$, we get $p(t) \geq 0$ on $[t_0, T]$, which implies $\beta_1 \geq \alpha_1$
proving (3.7).

Suppose now that for some $k > 1$,

$$\alpha_0 \leq \alpha_{k-1} \leq \alpha_k \leq \beta_k \leq \beta_{k-1} \leq \beta_0 \ on [t_0, T]. \tag{3.8}$$

and we show that

$$\alpha_k \leq \alpha_{k+1} \leq \beta_{k+1} \leq \beta_k \text{ on } [t_0, T].$$

To prove this, we let $p = \alpha_{k+1} - \alpha_k$ so that we have

$$\begin{aligned}
{}^cD^q p &= f(t, \alpha_k) + f_x(t, \alpha_k)(\alpha_{k+1} - \alpha_k) - [f(t, \alpha_{k-1}) \\
&\quad + f_x(t, \alpha_{k-1})(\alpha_k - \alpha_{k-1})] \\
&\geq f_x(t, \alpha_{k-1})(\alpha_k - \alpha_{k-1}) + f_x(t, \alpha_k)(\alpha_{k+1} - \alpha_k) \\
&\quad - f_x(t, \alpha_{k-1})(\alpha_k - \alpha_{k-1}) \\
&= f_x(t, \alpha_k)p, \ \ p(t_0) = 0.
\end{aligned}$$

Hence we get $p(t) \geq 0$ which yields $\alpha_{k+1} \geq \alpha_k$ on $[t_0, T]$. Similarly we can show $\beta_{k+1} \leq \beta_k$. To prove $\alpha_{k+1} \leq \beta_{k+1}$, we set $p = \beta_{k+1} - \alpha_{k+1}$ so that we obtain

$$\begin{aligned}
{}^cD^q p &= f(t, \beta_k) + f_x(t, \alpha_k)(\beta_{k+1} - \beta_k) \\
&\quad - [f(t, \alpha_k) + f_x(t, \alpha_k)(\alpha_{k+1} - \alpha_k)] \\
&\geq f_x(t, \alpha_k)(\beta_k - \alpha_k) - f_x(t, \alpha_k)(\alpha_{k+1} - \alpha_k) \\
&\quad + f_x(t, \alpha_k)(\beta_{k+1} - \beta_k) \\
&\geq f_x(t, \alpha_k)[\beta_k - \alpha_k - \alpha_{k+1} + \alpha_k] + f_x(t, \alpha_k)(\beta_{k+1} - \beta_k) \\
&= f_x(t, \alpha_k)p \ and \ p(t_0) = 0.
\end{aligned}$$

This proves that $\alpha_{k+1} \leq \beta_{k+1}$ on $[t_0, T]$. Hence by induction (3.8) is valid for all n.

Clearly the sequences are uniformly bounded because of (3.8), which shows that $\{{}^cD^q \alpha_n\}, \{{}^cD^q \beta_n\}$ are also uniformly bounded. By Lemma 2.2 we get the sequences are equicontinuous on $[t_0, T]$ and therefore Ascoli-Arzela Theorem provides subsequences that converge uniformly on $[t_0, T]$. This together with (3.8) gives that the entire sequences $\{\alpha_n\}, \{\beta_n\}$ converge uniformly and monotonically to ρ, r respectively as $n \to \infty$. Using the corresponding Volterra fractional integrals of (3.4), (3.5), one can easily show that ρ, r are solutions of IVP(3.1) and using the fact that f satisfies a Lipschitz condition, since $f_x(t,x)$ is bounded on the sector

$$[\alpha_0, \beta_0] = [x : \alpha_0(t) \leq x \leq \beta_0(t)],$$

we find $\rho = r = x$ is the unique solution of IVP (3.1).

To prove the quadratic convergence of $\{\alpha_n\}, \{\beta_n\}$ to the unique solution, we consider

$$p_{n+1} = x - \alpha_{n+1},\ r_{n+1} = \beta_{n+1} - x$$

so that $p_{n+1}(t_0) = 0$ and $r_{n+1}(t_0) = 0$. We then have

$$\begin{aligned}
{}^cD^q p_{n+1} &= f(t,x) - [f(t,\alpha_n) + f_x(t,\alpha_n)(\alpha_{n+1} - \alpha_n)] \\
&= f_x(t,\alpha_n)p_n - f_x(t,\alpha_n)\ [p_n - p_{n+1}] \\
&\leq [f_x(t,x) - f_x(t,\alpha_n)]\ p_n + f_x(t,\alpha_n)\ p_{n+1} \\
&\leq L|p_n|^2 + f_x(t,\alpha_n)\ p_{n+1} \leq L|p_n|_0^2 + Np_{n+1},
\end{aligned}$$

where

$$|f_x| \leq N, |p_n|_0 = \sup_{[t_0,T]} |p_n(t)|.$$

This inequality yields the estimate

$$\begin{aligned}
p_{n+1}(t) &\leq L|p_n|_0^2 \int_{t_0}^{t} (t-s)^{q-1}\ E_{q,q}(N(t-s)^q)ds \\
&\leq N_0|p_n|_0^2 \quad \text{where } N_0 = L\tfrac{1}{q}\ (T-t_0)^q E_{q,q}(N(T-t_0)^q).
\end{aligned}$$

Thus we finally have the estimate which shows the quadratic convergence,

$$|p_{n+1}|_0 \leq N_0|p_n|_0^2.$$

A similar computation shows that

$$|r_{n+1}|_0 \leq N_0|r_n|_0^2.$$

The proof is therefore complete. □

4. QUASILINEARIZATION: FUNCTION IS CONCAVE

In this section, we obtain the quadratic convergence to the unique solution for the IVP

$$ {}^cD^q x = g(t,x),\ x(t_0) = x_0, \tag{4.1}$$

where $g \in C[J \times R,\ R]$ and $J = [t_0, T]$ is a concave function. The main result is presented below. ,

Theorem 4.1. *Assume that*

(i) $\alpha_0, \beta_0 \in C^q[J, R]$ *such that for* $t \in J$
${}^cD^q\alpha_0 \leq g(t, \alpha_0), {}^cD^q\beta_0 \geq g(t, \beta_0)$ *and* $\alpha_0(t) \leq \beta_0(t)$ *with*
$\alpha_0(t_0) \leq x_0 \leq \beta_0(t_0)$.

(ii) $g \in C[\Omega, R]$. *Assume* g_x *exists and* $g_{xx} \leq 0$ *on* Ω *where* $\Omega = \{(t, x) : \alpha_0(t) \leq x(t) \leq \beta_0(t), t \in J\}$.

Then there exists monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ *which converge uniformly to the unique solution of the given IVP* (4.1) *and the convergence is quadratic.*

Proof. Using the fact that g_x exists, we get

$$g(t, x) - g(t, y) \leq L(x - y), \tag{4.2}$$

for some $L \geq 0$ and $\alpha_0(t) \leq y \leq x \leq \beta_0(t)$. Further $g_{xx} \leq 0$ implies

$$g(t, x) \geq g(t, y) + g_x(t, x)(x - y), \quad x \geq y. \tag{4.3}$$

Consider the linear Caputo fractional differential equations,

$${}^cD^q\alpha_{k+1} = g(t, \alpha_k) + g_x(t, \beta_k)(\alpha_{k+1} - \alpha_k), \quad \alpha_{k+1}(t_0) = x_0 \tag{4.4}$$

$${}^cD^q\beta_{k+1} = g(t, \beta_k) + g_x(t, \beta_k)(\beta_{k+1} - \beta_k), \quad \beta_{k+1}(t_0) = x_0 \tag{4.5}$$

Taking $k = 0$, we set $p = \alpha_0 - \alpha_1$ and consider

$$\begin{aligned} {}^cD^q p &= {}^cD^q\alpha_0 -, {}^cD^q\alpha_1 \\ &\leq g(t, \alpha_0) - [g(t, \alpha_0) + g_x(t, \beta_0)(\alpha_1 - \alpha_0)] \\ &= g_x(t, \beta_0)(\alpha_0 - \alpha_1) = g_x(t, \beta_0)p. \end{aligned}$$

Further $p(t_0) \leq 0$. Thus using Corollary 2.6, we conclude $p(t) \leq 0$ for $t \in J$, that is

$$\alpha_0(t) \leq \alpha_1(t), \quad t \in J.$$

In a similar manner we can prove that

$$\beta_1(t) \leq \beta_0(t), \quad t \in J.$$

Now set $p = \alpha_1 - \beta_1$, then $p(t_0) = 0$.

Also

$$\begin{aligned} {}^cD^q p &= {}^cD^q\alpha_1 - {}^cD^q\beta_1 \\ &= g(t, \alpha_0) + g_x(t, \beta_0)(\alpha_1 - \alpha_0) - [g((t, \beta_0) + g_x(t, \beta_0)(\beta_1 - \beta_0)]. \end{aligned}$$

Using relation (4.3) we arrive at,

$${}^cD^q p \leq g_x(t, \beta_0)[(\alpha_1 - \alpha_0) - (\beta_1 - \beta_0) - (\beta_0 - \alpha_0)] = g_x(t, \beta_0)p.$$

Again Corollary 2.6 yields, $p(t) \leq 0$ on J which implies $\alpha_1 \leq \beta_1$ on J. Thus

$$\alpha_0(t) \leq \alpha_1(t) \leq \beta_1(t) \leq \beta_0(t), \ t \in J. \tag{4.6}$$

Suppose for $k > 1$,

$$\alpha_0(t) \leq \alpha_{k-1}(t) \leq \alpha_k(t) \leq \beta_k(t) \leq \beta_{k-1}(t) \leq \beta_0(t) \ \textit{on} \ J. \tag{4.7}$$

We now show that

$$\alpha_k(t) \le \alpha_{k+1}(t) \le \beta_{k+1}(t) \le \beta_k(t) \text{ on } J \tag{4.8}$$

where α_{k+1} and β_{k+1} are the solutions of the linear fractional differential equations of Caputo's type (4.4) and (4.5) respectively.

To prove the relation (4.8) we set $p = \alpha_k - \alpha_{k+1}$, so that $p(t_0) = 0$ and

$$\begin{aligned} {}^cD^q p &= g(t, \alpha_{k-1}) + g_x(t, \beta_{k-1})(\alpha_k - \alpha_{k-1}) \\ &\quad -[g(t, \alpha_k) + g_x(t, \beta_k)(\alpha_{k+1} - \alpha_k)] \\ &\le -g_x(t, \beta_{k-1})(\alpha_k - \alpha_{k-1}) \\ &\quad +g_x(t, \beta_{k-1})(\alpha_k - \alpha_{k-1}) + g_x(t, \beta_k)(\alpha_k - \alpha_{k+1}) \\ &= g_x(t, \beta_k)p. \end{aligned}$$

Applying Corollary 2.6 we obtain $\alpha_k(t) \le \alpha_{k+1}(t)$ on J.

Similarly we can show $\beta_{k+1}(t) \le \beta_k(t), t \in J$. Finally consider $p = \alpha_{k+1} - \beta_{k+1}$.

$$\begin{aligned} {}^cD^q p &= g(t, \alpha_k) + g_x(t, \beta_k)(\alpha_{k+1} - \alpha_k) - [g(t, \beta_k) + g_x(t, \beta_k)(\beta_{k+1-}\beta_k)] \\ &\le g_x(t, \beta_k)(\alpha_k - \beta_k) + g_x(t, \beta_k)(\alpha_{k+1} - \alpha_k) - g_x(t, \beta_k)(\beta_{k+1} - \beta_k) \\ &\le g_x(t, \beta_k)[(\alpha_k - \beta_k) - (\beta_{k+1} - \beta_k)] + g_x(t, \beta_k)(\alpha_{k+1} - \alpha_k) \\ &= g_x(t, \beta_k)p. \end{aligned}$$

Also $p(t_0) = 0$. Again using Corollary 2.6 we conclude that $\alpha_{k+1}(t) \le \beta_{k+1}(t)$ on J. Thus using the principle of mathematical induction, we infer that

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \beta_n \le \beta_{n-1} \le \cdots \le \beta_1 \le \beta_0 \quad \text{on} \quad J. \tag{4.9}$$

From (4.9) we observe that the sequences are uniformly bounded which also yields that $\{{}^cD^q\alpha_n\}, \{{}^cD^q\beta_n\}$ are uniformly bounded.

By Lemma 2.2 we can conclude that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are equicontinuous on J and therefore Ascoli Arzela Theorem provides subsequences that converge uniformly on J. This together with (4.9) guarantees that the entire sequences $\{\alpha_n\}, \{\beta_n\}$ converge uniformly and monotonically to p and r respectively as $n \to \infty$. Using the corresponding Volterra fractional integrals of

$$\begin{aligned} {}^cD^q\alpha_k &= g(t, \alpha_{k-1}) + g_x(t, \beta_{k-1})(\alpha_k - \alpha_{k-1}), \alpha_k(t_0) = x_0 \\ {}^cD^q\beta_k &= g(t, \beta_{k-1}) + g_x(t, \beta_{k-1})(\beta_k - \beta_{k-1}), \beta_k(t_0) = x_0 \end{aligned}$$

one can easily show that p and r are solutions of IVP (4.1). Since g satisfies Lipschitz condition we find $p = r = x$ is the unique solution of IVP (4.1).

To prove the quadratic convergence of $\{\alpha_n\}, \{\beta_n\}$ to the unique solution we consider $p_{n+1} = x - \alpha_{n+1}, q_{n+1} = \beta_{n+1} - x$. Clearly $p_{n+1}(t_0) = 0$ and $q_{n+1}(t_0) = 0$. Using the relation (4.4) and the mean value theorem successively we get

$$\begin{aligned} {}^cD^q p_{n+1} &= g(t,x) - [g(t,\alpha_n) + g_x(t,\beta_n)(\alpha_{n+1}-\alpha_n)] \\ &= g(t,x) - g(t,\alpha_n) - g_x(t,\beta_n)(\alpha_{n+1}-\alpha_n) \\ &= g_x(t,\eta)p_n - g_x(t,\beta_n)p_n + g_x(t,\beta_n)p_{n+1} \\ &\le [g_x(t,\alpha_n) - g_x(t,\beta_n)]p_n + g_x(t,\beta_n)p_{n+1} \\ &= -g_{xx}(t,\xi)(\beta_n-\alpha_n)p_n + g_x(t,\beta_n)p_{n+1} \\ &= -g_{xx}(t,\xi)q_np_n - g_{xx}(t,\xi)p_n^2 + g_x(t,\beta_n)p_{n+1} \end{aligned}$$

where $\alpha_n < \eta < x; \alpha_n < \xi < \beta_n$.

Note that $g_{xx} \le 0$.

Let

$$|g_{xx}(t,x)| \le M_1, |g_x(t,x) \le L, (t,x) \in \Omega.$$

Hence

$$\begin{aligned} {}^cD^q p_{n+1} &\le M_1[p_nq_n + p_n^2] + Lp_{n+1} \\ &\le \tfrac{M_1}{2}[3p_n^2 + q_n^2] + Lp_{n+1} \; with \; p_{n+1}(t_0) = 0. \end{aligned}$$

Thus

$${}^cD^q p_{n+1} \le \frac{M_1}{2}\, [3|p_n|_0^2 + |q_n|_0^2] + Lp_{n+1}.$$

where

$$|p_n|_0 = \sup_{t\in J} |p_n(t)|, |q_n|_0 = \sup_{t\in J} |q_n(t)|.$$

This inequality gives the estimate

$$\begin{aligned} p_{n+1}(t) &\le \frac{M_1}{2}\, [3|p_n|_0^2 + |q_n|_0^2] \int_{t_0}^{t} (t-s)^{q-1}\, E_{q,q}(L(t-s)^q)ds \\ &\le L_0[3|p_n|_0^2 + |q_n|_0^2] \end{aligned}$$

$$\text{where } L_0 = \frac{M_1}{2q}\, (T-t_0)^q\, E_{q,q}(L(T-t_0)^q).$$

Thus we have the estimate $|p_{n+1}|_0 \le L_0[3|p_n|_0^2 + |q_n|_0^2]$, which implies the quadratic convergence of the sequence. A similar computation slows that
$|q_{n+1}|_0 \le L_0[|p_n|_0^2 + 3|q_n|_0^2]$. The proof is complete. □

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