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MEASURE OF NONCOMPACTNESS AND FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT. In this paper, the existence of solutions for an initial value problem of a fractional differential equation is obtained by means of Mönch's fixed point theorem and the technique of measures of noncompactness.

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1. INTRODUCTION

In the last few years, many researchers have focused their research on the study of fractional differential and integral equations defined on bounded and unbounded intervals, and various theoretical results have been obtained; see the monographs of Kilbas *et al.* [26], Miller and Ross [30], and the papers of Agarwal *et al.* [1, 2], Belarbi *et al.* [12], Benchohra *et al.* [13, 14], Delboso and Rodino [15], Diethelm and Ford [16], El-Sayed *et al.* [17], Furati and Tatar [19, 20, 21] and Momani *et al.* [31, 32]. These results are applied in different fields of science and engineering such as physics, biology and chemistry, etc. We refer the reader to the papers of Gaul *et al.* [22], Glockle and Nonnenmacher [23], Hilfer [25], Mainardi [28], Metzler *et al.* [29] and Podlubny [35]. Recently Lakshmikantham and Devi [27] studied the existence, uniqueness and continuous dependence on initial data of the solutions of a class of fractional differential equations in Banach spaces involving Riemann-Liouville derivatives. As far as we know there are very few papers related to ordinary fractional differential equations on Banach spaces (see [27]).

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In this paper we consider the existence of solutions of an initial value problem (IVP for short) for a nonlinear fractional differential equation,

$$ {}^c\mathbf{D}^r y(t) = f(t,y), \quad \text{for each } t \in J = [0,T], \quad 1 < r < 2, \tag{1.1} $$

$$ y(0) = y_0, \quad y'(0) = y_1, \tag{1.2} $$

where ${}^c\mathbf{D}^r$ is the Caputo fractional derivative, $f : J \times E \to E$ is a given function satisfying some assumptions that will be specified later, and E is a Banach space with norm $\|\cdot\|$. We will use the technique of measures of noncompactness which is often used in several branches of nonlinear analysis. Especially, that technique turns out to be a very useful tool in existence for several types of integral equations; details are found in Akhmerov *et al.* [4], Alvàrez [5], Banaś *et al.* [6, 7, 8, 9, 10, 11], El-Sayed and Rzepka [18], Guo *et al.* [24], Mönch [33], Mönch and Von Harten [34] and Szufla [36].

The principal goal here is to prove the existence of solutions for the above problem using Mönch's fixed point theorem and its related Kuratowski measure of noncompactness.

2. PRELIMINARIES

We now gather some definitions and preliminary facts which will be used throughout this paper.

Denote by $C(J,E)$ the Banach space of continuous functions $y : J \to E$, with the usual supremum norm

$$ \|y\|_\infty = \sup\{\|y(t)\|, \quad t \in J\}. $$

Let $L^1(J,E)$ be the Banach space of measurable functions $y : J \to E$ which are Bochner integrable, equipped with the norm

$$ \|y\|_{L^1} = \int_J y(t)\,dt. $$

$AC^1(J,E)$ denotes the space of functions $y : J \to E$, whose first derivative is absolutely continuous.

Moreover, for a given set V of functions $v : J \to E$, let us denote by

$$ V(t) = \{v(t) : v \in V\}, \ t \in J, $$

and

$$ V(J) = \{v(t) : v \in V, \ t \in J\}. $$

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.1 ([4, 8])**.** Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty]$ defined by

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_E.$$

This measure of noncompactness satisfies some important properties [4, 8]:

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).
(b) $\alpha(B) = \alpha(\overline{B})$.
(c) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.
(d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$
(e) $\alpha(cB) = |c|\alpha(B)$; $c \in \mathbb{R}$.
(f) $\alpha(convB) = \alpha(B)$.

For our purpose we will need the definition of Caputo derivative of fractional order

Definition 2.2 ([26])**.** The fractional order integral of the function $h \in L^1([a, b])$ of order $r \in \mathbb{R}_+$ is defined by

$$I_a^r h(t) = \frac{1}{\Gamma(r)} \int_a^t \frac{h(s)}{(t-s)^{1-r}} dt,$$

where Γ is the gamma function. When $a = 0$, we write $I^r h(t) = h(t) * \varphi_r(t)$, where $\varphi_r(t) = \frac{t^{r-1}}{\Gamma(r)}$ for $t > 0$, and $\varphi_r(t) = 0$ for $t \leq 0$, and $\varphi_r \to \delta(t)$ as $r \to 0$, where δ is the delta function.

Definition 2.3 ([26])**.** For a function h defined on the interval $[a, b]$, the Caputo fractional-order derivative of h, is defined by

$$^c\mathbf{D}_{a^+}^r h(t) = \frac{1}{\Gamma(n-r)} \int_a^t \frac{h^{(n)}(s)ds}{(t-s)^{1-n+r}}$$

Here $n = [\![r]\!] + 1$ and $[\![r]\!]$ denotes the integer part of r.

From the definition of the Caputo derivative, the following auxiliary results have been established in [37].

Lemma 2.4. *Let $r > 0$, then the differential equation*

$$^c\mathbf{D}^r h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1}$, $c_i \in E$, $i = 0, 1, \ldots, n$, $n = [\![r]\!] + 1$.

Lemma 2.5. *Let $r > 0$, then*

$$I^{r\,c}\mathbf{D}^r h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1}$$

for some $c_i \in E$, $i = 0, 1, \ldots, n$, $n = [\![r]\!] + 1$.

Definition 2.6. A map $f : J \times E \to E$ is said to be Carathéodory if

(i) $t \longmapsto f(t,u)$ is measurable for each $u \in E$;
(ii) $u \longmapsto F(t,u)$ is continuous for almost all $t \in J$.

The following theorems will play a major role in our analysis.

Theorem 2.7 ([3, 36])**.** *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Lemma 2.8 ([36])**.** *Let D be a bounded, closed and convex subset of the Banach space $C(J,E)$, G a continuous function on $J \times J$ and f a function from $J \times E \to E$ which satisfies the Carathéodory conditions, and suppose there exists $p \in L^1(J,\mathbb{R}_+)$ such that, for each $t \in J$ and each bounded set $B \subset E$, we have*

$$\lim_{h \to 0^+} \alpha(f(J_{t,h} \times B)) \le p(t)\alpha(B); \text{ here } J_{t,h} = [t-h,t] \cap J.$$

If V is an equicontinuous subset of D, then

$$\alpha\Big(\Big\{ \int_J G(s,t) f(s,y(s))ds : y \in V \Big\}\Big) \le \int_J \|G(t,s)\| p(s) \alpha(V(s))ds.$$

3. MAIN RESULTS

First of all, we define what we mean by a solution of the IVP (1.1)–(1.2).

Definition 3.1. A function $y \in AC^1(J,E)$ is said to be a solution of the problem (1.1)–(1.2) if y satisfies the equation ${}^c\mathbf{D}^r y(t) = f(t,y(t))$ on J, and the conditions $y(0) = y_0$ and $y'(0) = y_1$.

Lemma 3.2. *Let $1 < r < 2$ and let $h : J \to E$ be continuous. A function y is said to be a solution of the fractional integral equation*

$$y(t) = y_0 + y_1 t + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s)ds, \tag{3.1}$$

if and only if y is a solution of the fractional IVP

$${}^c\mathbf{D}^r y(t) = h(t), \quad t \in [0,T], \tag{3.2}$$

$$y(0) = y_0, \quad y'(0) = y_1. \tag{3.3}$$

Proof. By Lemma 2.5 we reduce (3.2)–(3.3) to an equivalent integral equation

$$y(t) = I^r h(t) + c_0 + c_1 t = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s)ds + c_0 + c_1 t$$

for some constants c_0, $c_1 \in E$. Conditions (3.3) give

$$c_0 = y_0, \quad c_1 = y_1.$$

Hence we get (3.1). Conversely, if y satisfies equation (3.1), the equations (3.2)–(3.3) hold. □

To establish our main result concerning existence of solutions of (1.1)–(1.2), we impose suitable conditions on the functions involved in that problem. Namely, we assume that

(H1) $f : J \times E \to E$ satisfies the Carathéodory conditions;

(H2) There exists $p \in L^1(J, \mathbb{R}_+) \cap C(J, \mathbb{R}_+)$, such that,

$$\|f(t,y)\| \le p(t)\|y\|, \quad \text{for } t \in J \text{ and each } y \in E;$$

(H3) For each $t \in J$ and each bounded set $B \subset E$, we have

$$\lim_{h\to 0^+} \alpha(f(J_{t,h} \times B)) \le p(t)\alpha(B); \quad \text{here } J_{t,h} = [t-h, t] \cap J.$$

Theorem 3.3. *Assume that conditions* (H1)–(H3) *hold. Let* $p^* = \sup_{t\in J} p(t)$. *If*

$$\frac{p^* T^r}{\Gamma(r+1)} < 1, \tag{3.4}$$

then the IVP (1.1)–(1.2) *has at least one solution.*

Proof. Transform the problem (1.1)–(1.2) into a fixed point problem. Consider the operator $N : C(J,E) \longrightarrow C(J,E)$ defined by

$$N(y)(t) = y_0 + y_1 t + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s, y(s))ds.$$

Clearly, the fixed points of the operator N are solutions of the problem (1.1)–(1.2). Let

$$r_0 \ge \frac{\|y_0\| + \|y_1\|T}{1 - \frac{p^* T^r}{\Gamma(r+1)}} \tag{3.5}$$

and consider

$$D_{r_0} = \{y \in C(J,E) : \|y\|_\infty \le r_0\}.$$

Clearly, the subset D_{r_0} is closed, bounded and convex. We shall show that N satisfies the assumptions of Theorem 2.7. The proof will be given in three steps.

Step 1: N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J,E)$. Then for each $t \in J$,

$$\begin{aligned} \|N(y_n)(t) - N(y)(t)\| &= \Big\|\frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} [f(s, y_n(s)) - f(s, y(s))]ds\Big\| \\ &\le \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \|f(s, y_n(s)) - f(s, y(s))\|ds. \end{aligned}$$

Since f is of Carathéodory type, then by the Lebesgue dominated convergence theorem we have

$$\|N(y_n) - N(y)\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Step 2: N maps D_{r_0} into itself.

For each $y \in D_{r_0}$, by $(H2)$ and (3.5), we have, for each $t \in J$,

$$\begin{aligned}
\|N(y)(t)\| &\leq \|y_0 + y_1 t\| + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \|f(s, y(s))\| ds \\
&\leq \|y_0\| + \|y_1\| T + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} p(s) \|y(s)\| ds \\
&\leq \|y_0\| + \|y_1\| T + \frac{r_0}{\Gamma(r)} \int_0^t (t-s)^{r-1} p(s) ds \\
&\leq \|y_0\| + \|y_1\| T + \frac{r_0 p^* T^r}{\Gamma(r+1)} \\
&\leq r_0.
\end{aligned}$$

Step 3: $N(D_{r_0})$ is bounded and equicontinuous.

By Step 2, it is obvious that $N(D_{r_0}) \subset C(J, E)$ is bounded. For the equicontinuity of $N(D_{r_0})$, let $t_1, t_2 \in J$, $t_1 < t_2$ and $y \in D_{r_0}$. Then

$$\begin{aligned}
\|N(y)(t_2) - N(y)(t_1)\| &\leq \|y_1 t_2 - y_1 t_1\| \\
&+ \frac{1}{\Gamma(r)} \Big\| \int_0^{t_2} (t_2 - s)^{r-1} f(s, y(s)) \\
&- \int_0^{t_1} (t_1 - s)^{r-1} f(s, y(s)) \Big\| ds \\
&\leq \|y_1\|(t_2 - t_1) \\
&+ \frac{1}{\Gamma(r)} \int_0^{t_1} [(t_2 - s)^{r-1} - (t_1 - s)^{r-1}] \|f(s, y(s))\| ds \\
&+ \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} (t_2 - s)^{r-1} \|f(s, y(s))\| ds \\
&\leq \|y_1\|(t_2 - t_1) \\
&+ \frac{r_0}{\Gamma(r)} \int_0^{t_1} [(t_2 - s)^{r-1} - (t_1 - s)^{r-1}] p(s) ds \\
&+ \frac{r_0}{\Gamma(r)} \int_{t_1}^{t_2} (t_2 - s)^{r-1} p(s) ds.
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero.

Now let V be a subset of D_{r_0} such that $V \subset \overline{conv}(N(V) \cup \{0\})$.

From Step 3, the subset V is bounded and equicontinuous and therefore the function $v \longmapsto v(t) = \alpha(V(t))$ is continuous on J. Since the function $t \longmapsto y_0 + y_1 t$ is continuous on J, the set $\overline{\{y_0 + y_1 t, t \in J\}} \subset E$ is compact. Using this fact, (H3), Lemma 2.8 and the properties of the measure α, we have, for each $t \in J$,

$$\begin{aligned}
v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\
&\leq \alpha(N(V)(t)) \\
&\leq \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} p(s) \alpha(V(s)) ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}p(s)v(s)ds \\
&\leq \|v\|_\infty \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}p(s)ds \\
&\leq \|v\|_\infty \frac{p^* T^r}{\Gamma(r+1)}.
\end{aligned}
$$

This means that

$$
\|v\|_\infty \leq \|v\|_\infty \frac{p^* T^r}{\Gamma(r+1)}.
$$

By (3.4) it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in D_{r_0}. Applying now Theorem 2.7, we conclude that N has a fixed point which is a solution of the problem (1.1)–(1.2). □

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