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NUMERICAL SOLUTION OF HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we study numerical methods for hybrid fractional differential equations. A convergence result is proven and we provide a numerical example called the hybrid relaxation-oscillation equation. The numerical solution is compared to the actual solution.

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1. INTRODUCTION

The origins of fractional calculus go back to 1695 when Leibniz considered the derivative of order 1/2. Miller and Ross [5] and Oldham and Spanier [6] provide historical details on the fractional calculus. Many applications have been found for fractional calculus, some of which are discussed in Debnath [2] and Podlubny [9]. In particular, fractional differential equations have received much attention and a number of recent works concern their numerical solution (see Ford and Connolly [3] and others).

As another development, hybrid systems are dynamical systems that progress continuously in time but have formatting changes called modes at a sequence of discrete times. Some recent papers about hybrid systems include [1, 4, 10]. When the continuous time dynamics of a hybrid system comes from fractional differential equations the system is called a hybrid fractional differential system or a hybrid fractional differential equation. This is one of the first papers to study hybrid fractional differential equations. The aim of this paper is to study their numerical solution.

This paper is organized as follows. In Section 2, we provide some background on fractional differential equations and hybrid fractional differential equations. In Section 3 we discuss the numerical solution of hybrid fractional differential equations by following the results of [8]. The method given uses piecewise application of a numerical method for fractional differential equations. A convergence result is proven when the underlying numerical method for fractional differential equations is one-step

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explicit and numerically stable. In Section 4, as an example, we numerically solve a hybrid relaxation-oscillation equation which is under Grünwald-Letnikov fractional differentiation. The numerical solution is compared to the exact solution.

2. HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS

First consider the fractional differential equation IVP

$$ {}_{t_0}D_t^q x(t) = f(t, x(t)), \;\; x(t_0) = x_0 \in \mathbb{R}, \;\; t_0 < t < T, \tag{2.1} $$

where $q \in (0,1)$, ${}_{t_0}D_t^q$ represents some type of fractional differentiation, and $f : [t_0, T] \times \mathbb{R} \to \mathbb{R}$. The subscripts t_0 and t of ${}_{t_0}D_t^q$ are called terminals. For $q \in (0,1)$, Grünwald-Letnikov fractional differentiation is defined by

$$ {}_aD_t^q x(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{-q} \sum_{r=0}^{n} (-1)^r \binom{q}{r} x(t-rh). \tag{2.2} $$

For $q \in (0,1)$, Riemann-Liouville fractional differentiation is defined by

$$ {}_aD_t^q x(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_a^t (t-\tau)^{-q} x(\tau) d\tau, $$

where Γ is the Gamma function. It is well known that if $x \in C^1$ (and $q \in (0,1)$) then ${}_aD_t^q x(t)$ is the same under both Grünwald-Letnikov and Riemann-Liouville fractional differentiation. For $q \in (0,1)$, Caputo fractional differentiation is defined by

$$ {}_aD_t^q x(t) = \frac{1}{\Gamma(1-q)} \int_a^t (t-\tau)^{-q} x'(\tau) d\tau. $$

Let $\{t_k\}_{k=0}^{\infty}$ be a strictly increasing and unbounded real sequence and let $f : [t_0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. For each $k = 0, 1, 2, \ldots$, let $\lambda_k : \mathbb{R} \to \mathbb{R}$ and let $f_k : [t_k, t_{k+1}] \times \mathbb{R} \to \mathbb{R}$ where $f_k(t, x(t)) = f(t, x(t), \lambda_k(x(t_k)))$. Even though $k \geq 0$, $\{\lambda_k\}_{k=0}^{\infty}$ will be a finite set known as the set of modes. A hybrid fractional differential equation IVP based on (2.1) is a system of fractional differential equation IVPs of the form

$$ \begin{cases} {}_{t_0}D_t^q x_0(t) = f(t, x_0(t), \lambda_0(x_0(t_0))) \equiv f_0(t, x_0(t)), \;\; t \in (t_0, t_1), \;\; x_0(t_0) \in \mathbb{R}, \\ {}_{t_1}D_t^q x_1(t) = f(t, x_1(t), \lambda_1(x_1(t_1))) \equiv f_1(t, x_1(t)), \;\; t \in (t_1, t_2), \;\; x_1(t_1) = x_0(t_1), \\ \quad \vdots \\ {}_{t_k}D_t^q x_k(t) = f(t, x_k(t), \lambda_k(x_k(t_k))) \equiv f_k(t, x_k(t)), \;\; t \in (t_k, t_{k+1}), \;\; x_k(t_k) = x_{k-1}(t_k), \\ \quad \vdots \end{cases} \tag{2.3} $$

where $q \in (0,1)$ and ${}_{t_k}D_t^q$ represents some type of fractional differentiation (fixed for all k's). A solution to (2.3) will be a sequence of functions

$$\begin{cases} x_0(t), & t_0 \le t \le t_1, \\ x_1(t), & t_1 \le t \le t_2, \\ \quad\vdots & \\ x_k(t), & t_k \le t \le t_{k+1}, \\ \quad\vdots & \end{cases} \tag{2.4}$$

satisfying (2.3) in the sense that for each $k = 0,1,2,\dots$ $x_k(t)$ is a solution of

$$\begin{cases} {}_{t_k}D_t^q x_k(t) = f(t, x_k(t), \lambda_k(x_k(t_k))) \equiv f_k(t, x_k(t)), \quad t_k < t < t_{k+1}, \\ x_k(t_k) = \begin{cases} x_{k-1}(t_k) \text{ if } k \ge 1, \\ x_0(t_0) \text{ if } k = 0. \end{cases} \end{cases} \tag{2.5}$$

We will sometimes write (2.3) in a condensed form as

$${}_{(t_0)}D_t^q x(t) = f(t, x(t), \lambda_k(x(t_k))), \ \ t \in (t_k, t_{k+1}), \ \ k = 0,1,2,\dots, \ \ x(t_0) \in \mathbb{R}, \tag{2.6}$$

where a solution to (2.6) will be a function $x : [t_0, \infty) \to \mathbb{R}$ using (2.4):

$$x(t) = \begin{cases} x_0(t), & t_0 \le t \le t_1, \\ x_1(t), & t_1 \le t \le t_2, \\ \quad\vdots & \\ x_k(t), & t_k \le t \le t_{k+1}, \\ \quad\vdots & \end{cases} . \tag{2.7}$$

We will call the subscripts (t_0) and t of ${}_{(t_0)}D_t^q$ "generalized" terminals. A solution x of (2.6) will be piecewise fractional differentiable of order q over $[t_0, \infty)$ and fractional differentiable of order q in each interval (t_k, t_{k+1}) for $k = 0,1,2,\dots$ To find a solution $x(t)$ of (2.6) we may solve (2.3) piecewise over each $[t_k, t_{k+1}]$ to obtain each x_k. In Section 4, an example of a hybrid fractional differential equation IVP will be given along with its exact solution.

3. NUMERICAL SOLUTION OF HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS

In this section, assuming that (2.6) has a unique solution, we will show that we may numerically approximate the solution $x(t)$ of (2.6) by piecewise application of any one-step explicit numerical method for fractional differential equations which is numerically stable. To numerically integrate the system (2.6) in $[t_0, t_1], [t_1, t_2], \dots, [t_k, t_{k+1}], \dots,$ we will replace each interval $[t_k, t_{k+1}]$ by a set of $N_k + 1$ regularly spaced grid points (including the endpoints) at which the exact solution $x_k(t)$ will be approximated by some $y_k(t)$. For the grid points on $[t_k, t_{k+1}]$ let $t_{k,n} = t_k + nh_k$ where $h_k = (t_{k+1} - t_k)/N_k$ and $0 \le n \le N_k$. We will denote $y_k(t_{k,n})$ by $y_{k,n}$. In the chosen one-step explicit numerically stable method for fractional differential equations we let $y_{0,0} = x(t_0)$ and $y_{k,0} = y_{k-1,N_{k-1}}$ if $k \ge 1$. The next result is parallel to Theorem 3.2 in [8] but the numerical technique is not necessarily the Euler method and the setting is not fuzzy differential equations. The proof of Theorem 3.1 below is basically the same as the proof of Theorem 3.2 in [8].

Theorem 3.1. *Consider the system* (2.6) *assuming it has a unique solution. Suppose for some fixed* $k \in \mathbb{Z}^+$ *that* $\{\{y_{i,j}\}_{j=0}^{N_i}\}_{i=0}^{k}$ *is obtained by some one-step explicit numerically stable method for fractional differential equations with* $y_{0,0} = x(t_0)$ *and* $y_{k,0} = y_{k-1,N_{k-1}}$ *if* $k \ge 1$. *Then*

$$\lim_{h_0,\dots,h_k \to 0} y_{k,N_k} = x(t_{k+1}). \tag{3.1}$$

Proof. Fix $k \in \mathbb{Z}^+$. For each $i = 0, 1, \dots, k$, let $\{z_{i,j}\}_{j=0}^{N_i}$ be the numerical approximation to the fractional differential equation IVP

$$\begin{cases} {}_{t_i}D_t^q x_i(t) = f_i(t, x_i(t)) \\ x_i(t_i) = x(t_i) \end{cases} \tag{3.2}$$

generated by the one-step explicit numerically stable method for fractional differential equations where $z_{i,0} = x(t_i)$. Note that the initial condition of (3.2) is the actual value of $x(t_i)$. Choose $\epsilon > 0$. For each $i = 0, 1, \dots, k$ we will find a $\delta_i^* > 0$ such that $h_i < \delta_i^*$ implies

$$|x(t_{k+1}) - y_{k,N_k}| < \epsilon,$$

where the h_i values are allowable by regular partition of the $[t_i, t_{i+1}]$'s. By convergence of the numerical method over $[t_k, t_{k+1}]$ for (3.2), there exists a $\delta_k^* > 0$ such that if $h_k < \delta_k^*$ then

$$|z_{k,N_k} - x(t_{k+1})| < \frac{\epsilon}{2}.$$

By numerical stability there exists a $\delta_k > 0$ such that

$$|z_{k,0} - y_{k,0}| < \delta_k \tag{3.3}$$

implies

$$|z_{k,N_k} - y_{k,N_k}| < \frac{\epsilon}{2}.$$

Therefore if $h_k < \delta_k^*$ and (3.3) holds then

$$|x(t_{k+1}) - y_{k,N_k}| \le |x(t_{k+1}) - z_{k,N_k}| + |z_{k,N_k} - y_{k,N_k}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{3.4}$$

By convergence of the numerical method over $[t_{k-1}, t_k]$ for (3.2), there exists a $\delta^*_{k-1} > 0$ such that if $h_{k-1} < \delta^*_{k-1}$ then

$$|z_{k-1,N_{k-1}} - x(t_k)| < \delta_k/2.$$

By numerical stability there exists a $\delta_{k-1} > 0$ such that

$$|z_{k-1,0} - y_{k-1,0}| < \delta_{k-1} \tag{3.5}$$

implies

$$|z_{k-1,N_{k-1}} - y_{k-1,N_{k-1}}| < \delta_k/2.$$

Therefore if $h_{k-1} < \delta^*_{k-1}$ and (3.5) holds then

$$|x(t_k) - y_{k-1,N_{k-1}}| \leq |x(t_k) - z_{k-1,N_{k-1}}| + |z_{k-1,N_{k-1}} - y_{k-1,N_{k-1}}| < \delta_k/2 + \delta_k/2 = \delta_k. \tag{3.6}$$

Continue inductively for each $i = k-2, \ldots, 2, 1$ to find a $\delta^*_i > 0$ such that if $h_i < \delta^*_i$ then

$$|z_{i,N_i} - x(t_{i+1})| < \delta_{i+1}/2.$$

By numerical stability there exists a $\delta_i > 0$ such that

$$|z_{i,0} - y_{i,0}| < \delta_i \tag{3.7}$$

implies

$$|z_{i,N_i} - y_{i,N_i}| < \delta_{i+1}/2. \tag{3.8}$$

Therefore if $h_i < \delta^*_i$ and (3.7) holds then

$$|x(t_{i+1}) - y_{i,N_i}| \leq |x(t_{i+1}) - z_{i,N_i}| + |z_{i,N_i} - y_{i,N_i}| < \delta_{i+1}/2 + \delta_{i+1}/2 = \delta_{i+1}.$$

In particular, there exists a $\delta^*_1 > 0$ such that if $h_1 < \delta^*_1$ and (3.7) holds with $i = 1$ then

$$|x(t_2) - y_{1,N_1}| < \delta_2.$$

By convergence of the numerical method over $[t_0, t_1]$ for (3.2), we may choose $\delta^*_0 > 0$ such that $h_0 < \delta^*_0$ implies

$$|x(t_1) - y_{0,N_0}| < \delta_1. \tag{3.9}$$

Suppose for each $i = 0, \ldots, k$ that $h_i < \delta^*_i$. Since (3.9) is the same as (3.7) with $i = 1$, we obtain (3.8) with $i = 1$. Since (3.8) with $i = 1$ implies (3.7) with $i = 2$, we obtain (3.8) with $i = 2$. Continue inductively to obtain (3.3), and (3.4), proving (3.1). □

4. EXAMPLE- A HYBRID RELAXATION-OSCILLATION EQUATION

For $q \in (0,1)$ and $A \in \mathbb{R}$, we will study the nonhomogeneous linear hybrid fractional differential equation

$$\begin{cases} {}_{(t_0)}D_t^q x(t) = \lambda_k(x(t_k)) - Ax(t), \ \ t \in (t_k, t_{k+1}), t_k = k+1, \lambda_k(x) = x, \ \ k = 0,1,2,\dots \\ x(1) \in \mathbb{R}, \end{cases} \tag{4.1}$$

where D^q is Grünwald-Letnikov fractional differentiation. (4.1) may be called a hybrid relaxation-oscillation equation and it is based on (8.2) in Podlubny [9]. As an example we will numerically solve (4.1) by piecewise application of a method motivated by Section 8.3.1 of Podlubny [9]. The method will use changes of variable as well as changes in time scale. Moreover, we will compare the numerical solution of (4.1) to the exact solution.

First consider the nonhomogeneous linear fractional differential equation

$$\begin{cases} {}_0D_t^q z(t) = f(t) - Az(t), \ \ t > 0, \\ z(0) = 0, \ \ q \in (0,1), \ \ A \in \mathbb{R}, \end{cases} \tag{4.2}$$

where D^q is Grünwald-Letnikov fractional differentiation. (4.2) is called the relaxation-oscillation equation. Applications are discussed in [7]. By the method given in (8.4) of Podlubny [9], if h is an allowable step size over $[0,T]$ then a numerical solution of (4.2) over $[0,T]$ is given by

$$\begin{cases} z_m = -Ah^q z_{m-1} - \sum_{j=1}^m (-1)^j \binom{q}{j} z_{m-j} + h^q f(mh), \quad 1 \le m \le T/h, \\ z_0 = 0, \end{cases} \tag{4.3}$$

where the actual value $z(mh)$ is approximated by z_m in (4.3). As a slight generalization of (4.2) consider the equation

$$\begin{cases} {}_{t_0}D_t^q z(t) = f(t) - Az(t), \ \ t > t_0, \\ z(t_0) = 0, \ \ q \in (0,1), \ \ A \in \mathbb{R}. \end{cases} \tag{4.4}$$

Let $w(t) = z(t+t_0)$ for $t \ge 0$. Then $w(0) = z(t_0) = 0$ and $w(t-t_0) = z(t)$ for $t \ge t_0$. By (2.2),

$${}_0D_{t-t_0}^q w(t-t_0) = {}_{t_0} D_t^q z(t).$$

Then (4.4) is equivalent to

$$\begin{cases} {}_0D_{t-t_0}^q w(t-t_0) = f(t) - Aw(t-t_0), \ \ t - t_0 > 0, \\ w(0) = 0, \ \ q \in (0,1), \ \ A \in \mathbb{R}. \end{cases} \tag{4.5}$$

Let $F(t-t_0)=f(t)$ and $s=t-t_0$. Then (4.5) is equivalent to

$$\begin{cases} {}_0D_s^q w(s)=F(s)-Aw(s), \;\; s>0, \\ w(0)=0, \;\; q\in(0,1), \;\; A\in\mathbb{R}, \end{cases} \tag{4.6}$$

which is basically (4.2). A numerical solution of (4.6) may be found by adapting (4.3). Therefore if h is an allowable step size over $[t_0,T]$, then a numerical solution of (4.4) over $[t_0,T]$ is given by

$$\begin{cases} z_m=-Ah^q z_{m-1}-\sum_{j=1}^m(-1)^j\binom{q}{j}z_{m-j}+h^q f(t_0+mh), 1\le m\le (T-t_0)/h, \\ z_0=0, \end{cases} \tag{4.7}$$

where the actual value $z(t_0+mh)$ is approximated by z_m in (4.7). To solve (4.1) over each $[t_k,t_{k+1}]$ we convert (4.1) over each $[t_k,t_{k+1}]$ to an equivalent equation of the form (4.4) by letting $z(t)=x(t)-x(t_k)$. By this change of variable over each $[t_k,t_{k+1}]$, (4.1) becomes the equation

$$\begin{cases} {}_{t_k}D_t^q z(t)=x(t_k)\Big(1-A-\frac{t^{-q}}{\Gamma(1-q)}\Big)-Az(t), \;\; t_k<t<t_{k+1}, \;\; t_k=k+1, \\ z(t_k)=0, \;\; q\in(0,1), \;\; A\in\mathbb{R}. \end{cases} \tag{4.8}$$

To obtain (4.8) we used that

$${}_{t_k}D_t^q x(t_k)=x(t_k)\cdot\frac{t^{-q}}{\Gamma(1-q)}.$$

Therefore if h_k is an allowable step size over $[t_k,t_{k+1}]$ then a numerical solution of (4.8) over $[t_k,t_{k+1}]$ is given by

$$\begin{cases} z_{k,m}=-Ah_k^q z_{k,m-1}-\sum_{j=1}^m(-1)^j\binom{q}{j}z_{k,m-j}+h_k^q y_{k,0}\Big(1-A-\frac{(t_k+mh_k)^{-q}}{\Gamma(1-q)}\Big), \\ \qquad 1\le m\le (t_{k+1}-t_k)/h_k, \\ y_{0,0}=x(t_0), \\ y_{k,0}=z_{k-1,h_{k-1}^{-1}}+y_{k-1,0} \text{ if } k\ge 1, \\ z_{k,0}=0, \end{cases} \tag{4.9}$$

where $z_{k,m}$ approximates $z(t_k+mh_k)$. Lastly, to find the approximate solution of (4.1) we use (4.9) and

$$x(t_{k,m})=z(t_{k,m})+x(t_k)\approx z_{k,m}+y_{k,0}$$

over each $[t_k,t_{k+1}]$. The numerical solution of (4.1) using (4.9) with $x(1)=0.2$, $A=1$, each $h_k=0.01$, and $q=0.5$ is shown in Figure 1. Figure 3 show the numerical solutions of (4.1) using (4.9) with $x(1)=0.2$, $A=1$, each $h_k=0.01$, and $q=0.1$ to $q=0.9$ in steps of 0.1. The numerical solution of (4.1) using (4.9) with $x(1)=0.2$, $A=-0.1$, each $h_k=0.01$, and $q=0.5$ is shown in Figure 2. Figure 4 show the numerical solutions of (4.1) using (4.9) with $x(1) = 0.2$, $A = -0.1$, each $h_k = 0.01$, and $q = 0.1$ to $q = 0.9$ in steps of 0.1.

The exact solution of (4.2) is given by (8.5) in Podlubny [9] as

$$z(t) = \int_0^t (t-\tau)^{q-1} E_{q,q}(-A(t-\tau)^q) f(\tau) d\tau, \tag{4.10}$$

where $E_{u,v}$ is the two-parameter Mittag-Leffler function. We will use (4.10) to find the exact solution of (4.4). By applying (4.10) to (4.6) we get

$$w(s) = \int_0^s (s-\tau)^{q-1} E_{q,q}(-A(s-\tau)^q) F(\tau) d\tau. \tag{4.11}$$

Since $s = t - t_0$, (4.11) becomes

$$w(t-t_0) = \int_0^{t-t_0} (t-t_0-\tau)^{q-1} E_{q,q}(-A(t-t_0-\tau)^q) F(\tau) d\tau. \tag{4.12}$$

Since $w(t-t_0) = z(t)$, we get

$$z(t) = \int_0^{t-t_0} (t-t_0-\tau)^{q-1} E_{q,q}(-A(t-t_0-\tau)^q) F(\tau) d\tau. \tag{4.13}$$

By letting $U = t_0 + \tau$, (4.13) becomes

$$z(t) = \int_{t_0}^t (t-U)^{q-1} E_{q,q}(-A(t-U)^q) F(U-t_0) dU = \int_{t_0}^t (t-U)^{q-1} E_{q,q}(-A(t-U)^q) f(U) dU. \tag{4.14}$$

Therefore, the exact solution of (4.4) is given by

$$z(t) = \int_{t_0}^t (t-\tau)^{q-1} E_{q,q}(-A(t-\tau)^q) f(\tau) d\tau. \tag{4.15}$$

As a result, the exact solution of (4.8) over $[t_k, t_{k+1}]$ is given by

$$z(t) = \int_{t_k}^t (t-\tau)^{q-1} E_{q,q}(-A(t-\tau)^q) x(t_k) \left(1 - A - \frac{\tau^{-q}}{\Gamma(1-q)}\right) d\tau. \tag{4.16}$$

Since $z(t) = x(t) - x(t_k)$, we obtain the exact solution of (4.1) over each $[t_k, t_{k+1}]$ as

$$x(t) = x(t_k) + \int_{t_k}^t (t-\tau)^{q-1} E_{q,q}(-A(t-\tau)^q) x(t_k) \left(1 - A - \frac{\tau^{-q}}{\Gamma(1-q)}\right) d\tau. \tag{4.17}$$

The graphs of the exact solution (4.17) over $[1, 5]$ appear identical to the graphs of the numerical solutions of (4.1) in Figures 1 and 2.

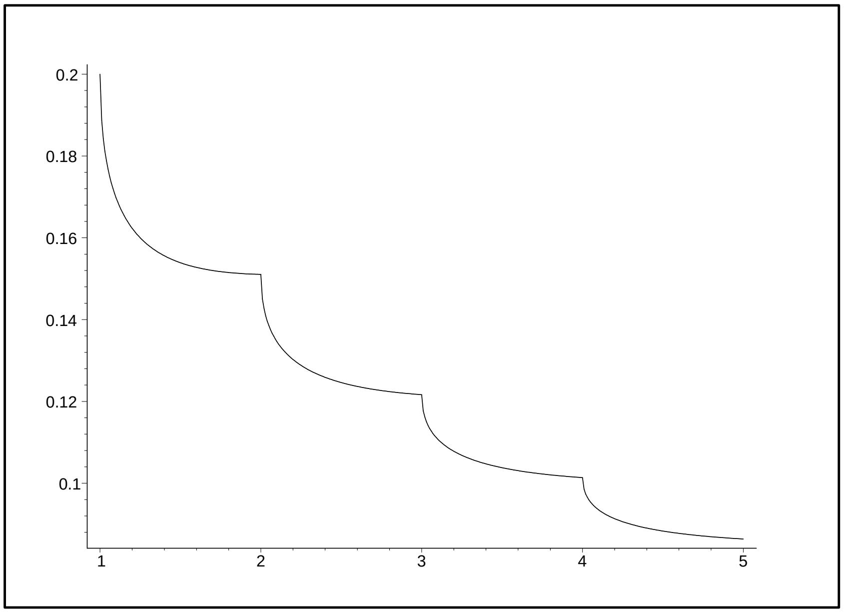

FIGURE 1. Numerical solution of (4.1) using (4.9) with $x(1) = 0.2$, $A = 1$, each $h_k = 0.01$, and $q = 0.5$. The graph of the exact solution of (4.1) appears identical.

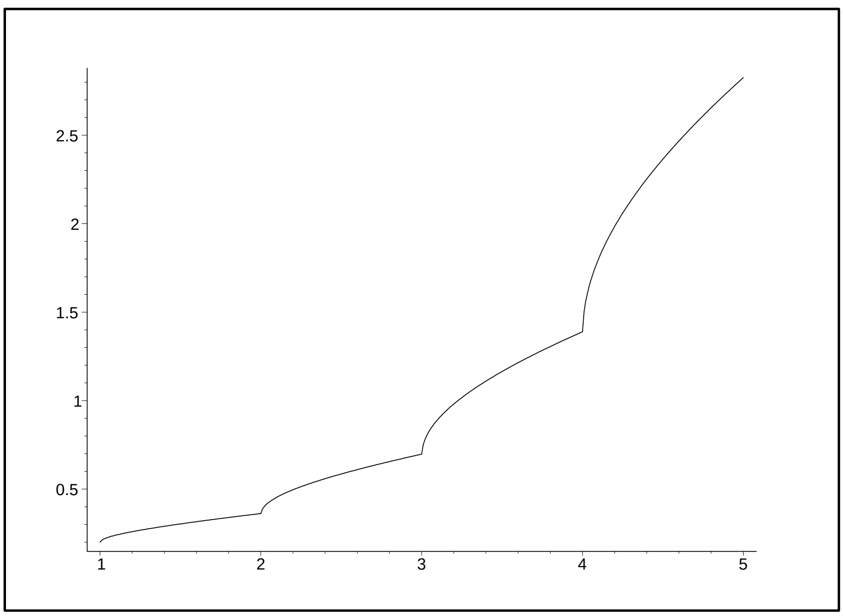

FIGURE 2. Numerical solution of (4.1) using (4.9) with $x(1) = 0.2$, $A = -0.1$, each $h_k = 0.01$, and $q = 0.5$. The graph of the exact solution of (4.1) appears identical.

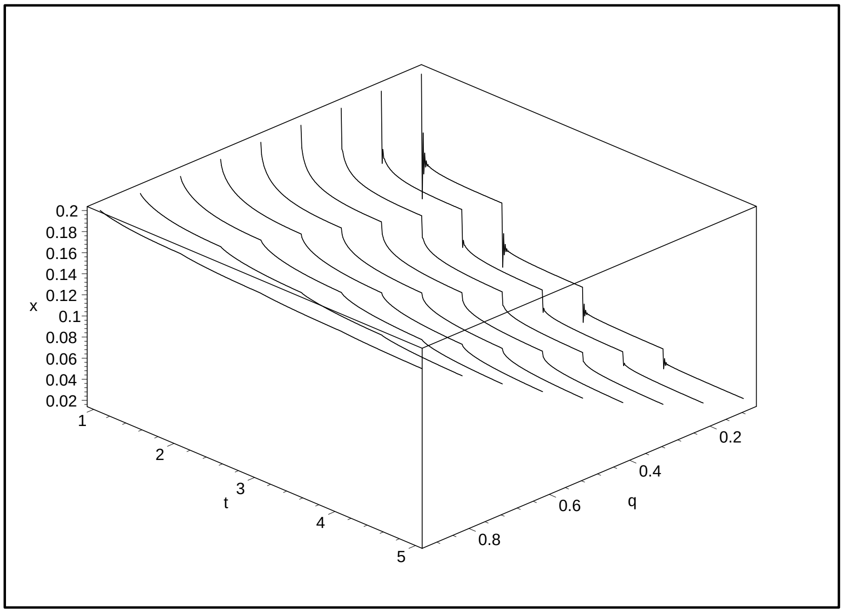

FIGURE 3. Numerical solutions of (4.1) using (4.9) with $x(1) = 0.2$, $A = 1$, each $h_k = 0.01$, and $q = 0.1$ to $q = 0.9$ in steps of 0.1.

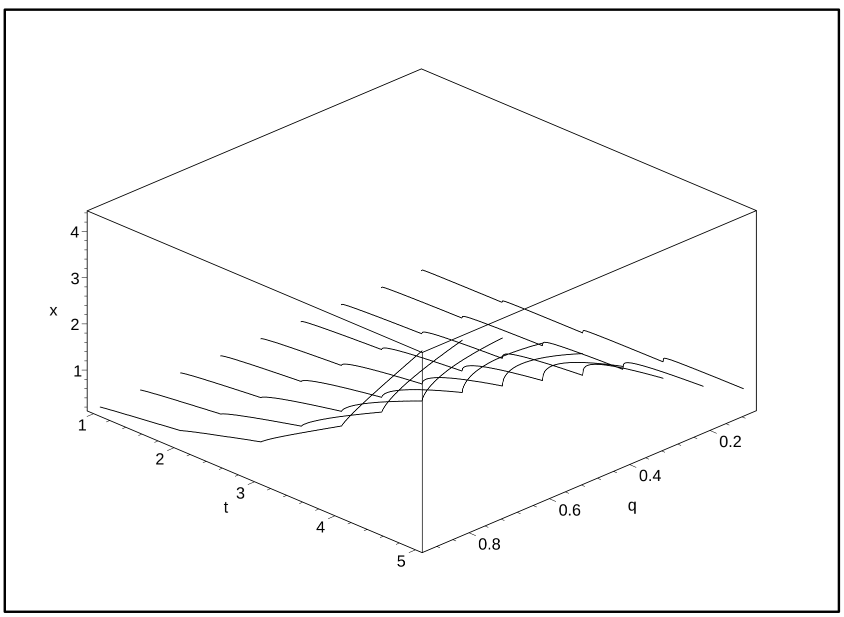

FIGURE 4. Numerical solutions of (4.1) using (4.9) with $x(1) = 0.2$, $A = -0.1$, each $h_k = 0.01$, and $q = 0.1$ to $q = 0.9$ in steps of 0.1.

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