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MULTI-POINT BOUNDARY VALUE PROBLEMS OF FRACTIONAL ORDER

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ABSTRACT. In this paper we study multi-point boundary value problems of fractional order with the Riemann-Liouville and Caputo fractional derivatives. The existence results are obtained using the Schauder fixed point theorem.

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1. PRELIMINARIES

The results of this paper involve both the Riemann-Liouville [14, 16] and the Caputo [2, 8, 14] fractional differential operators. For the recent and classical results in the theory and applications applications of linear and nonlinear differential equations of fractional order we refer the reader to the monographs and works [1, 3, 4, 6, 8, 9, 10, 11, 12, 14, 15, 16]. Observations on the applicability of various fractional order derivatives to the practical scenarios arising in physics and engineering can be found in [13, 14]. In the introductory session we present the preliminaries on the Riemann-Liouville and Caputo fractional derivatives. Among several studies concerning initial and boundary value problems with the Caputo operator we mention [5, 7]. The second section contains the existence criteria obtained using the Schauder fixed point theorem.

The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u \in L^p(0,1)$, $1 \le p < \infty$, is the integral

$$I^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds. \tag{1}$$

The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n = [\alpha] + 1$, is defined by

$$\mathcal{D}^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds. \tag{2}$$

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For $\alpha < 0$, it is convenient to introduce the notation $I^{\alpha}_{0+} = \mathcal{D}^{-\alpha}_{0+}$. It is well-known [14, 16] that if g is an integrable function, then

$$\mathcal{D}^{\alpha}_{0+} I^{\alpha}_{0+} g(t) = g(t). \tag{3}$$

If an integrable function u has the fractional derivative $\mathcal{D}^{\alpha}_{0+}u$, $1 < \alpha < 2$, which is also integrable, then

$$\begin{aligned} I^{\alpha}_{0+}\mathcal{D}^{\alpha}_{0+}u(t) &= u(t) - \mathcal{D}^{\alpha-1}_{0+}u(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)} - \mathcal{D}^{\alpha-2}_{0+}u(0)\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} \\ &= u(t) - \mathcal{D}^{\alpha-1}_{0+}u(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)} - \mathcal{I}^{2-\alpha}_{0+}u(0)\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}. \end{aligned}$$

If, in particular, $u \in C[0,1]$, then

$$u(t) = \mathcal{D}^{\alpha-1}_{0+}u(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)} + I^{\alpha}_{0+}\mathcal{D}^{\alpha}_{0+}u(t). \tag{4}$$

For $1 < \alpha < 2$, we study the integro-differential equation

$$\mathcal{D}^{\alpha}_{0+}u(t) = f(t, u(t), u'(t)), \quad t \in (0,1). \tag{5}$$

We assume throughout the note that

(H_1) $f \in C([0,1] \times \mathbf{R}^2, \mathbf{R})$;
(H_2) $f(s,0,0)$ is not identically zero on $[0,1]$.

We seek solutions of (5) that satisfy the boundary condition

$$\kappa u(\eta) = u(1), \quad \kappa\eta^{\alpha-1} \neq 1, \tag{6}$$

where $0 < \eta < 1$. In addition, we impose the condition

$$\mathcal{I}^{\gamma}_{0+}u(0) = 0, \tag{7}$$

where $\gamma > 1 - \alpha$.

By a solution of the boundary value problem (5)–(7) we understand a function $u \in C[0,1] \cap C^1(0,1]$ with $\mathcal{D}^{\alpha}_{0+}u \in C[0,1]$ satisfying the equation (5) and the conditions (6) and (7).

Since $\alpha + \gamma > 1$, for $g = \mathcal{D}^{\alpha}_{0+}u \in C[0,1]$, we have by the semigroup property

$$\mathcal{I}^{\gamma}_{0+}\mathcal{I}^{\alpha}_{0+}g(t) = \mathcal{I}^{\alpha+\gamma}_{0+}g(t)$$

(see, e.g., [14, 16]) that

$$\mathcal{I}^{\gamma}_{0+}u(t) = \mathcal{D}^{\alpha-1}_{0+}u(0)\frac{t^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} + \mathcal{I}^{\alpha+\gamma}_{0+}g(t).$$

The right side of the above equation vanishes at $t = 0$, which is consistent with (7).

From (4) and (6),

$$\frac{1}{\Gamma(\alpha)}\mathcal{D}^{\alpha-1}_{0+}u(0) + \mathcal{I}^{\alpha}_{0+}g(1) = u(1) = \kappa u(\xi) = \frac{\kappa\xi^{\alpha-1}}{\Gamma(\alpha)}\mathcal{D}^{\alpha-1}_{0+}u(0) + \kappa\mathcal{I}^{\alpha}_{0+}g(\xi),$$

so that

$$\mathcal{D}_{0+}^{\alpha-1}u(0) = \frac{\Gamma(\alpha)}{1-\kappa\xi^{\alpha-1}}\left(\kappa\mathcal{I}_{0+}^{\alpha}g(\xi) - \mathcal{I}_{0+}^{\alpha}g(1)\right)$$

and

$$u(t) = \frac{t^{\alpha-1}}{1-\kappa\xi^{\alpha-1}}\left(\kappa\mathcal{I}_{0+}^{\alpha}g(\xi) - \mathcal{I}_{0+}^{\alpha}g(1)\right) + I_{0+}^{\alpha}g(t).$$

Then

$$u'(t) = \frac{(\alpha-1)t^{\alpha-2}}{1-\kappa\xi^{\alpha-1}}\left(\kappa\mathcal{I}_{0+}^{\alpha}g(\xi) - \mathcal{I}_{0+}^{\alpha}g(1)\right) + \mathcal{I}_{0+}^{\alpha-1}g(t).$$

Replacing g with the inhomogeneous term of (5), we obtain that if $u \in C[0,1] \cap C^1(0,1]$ is a solution of the fractional differential equation (5) satisfying (6) and (7), then $u \in C[0,1] \cap C^1(0,1]$ is a solution of the integral equation

$$u(t) = \frac{t^{\alpha-1}}{1-\kappa\xi^{\alpha-1}}\left(\kappa\mathcal{I}_{0+}^{\alpha}f(\cdot,u(\cdot))(\xi) - \mathcal{I}_{0+}^{\alpha}f(\cdot,u(\cdot))(1)\right) + I_{0+}^{\alpha}f(\cdot,u(\cdot))(t). \tag{8}$$

The converse is also true in view of (3).

Since the solvability of the boundary value problem (5)–(7) is equivalent to establishing the existence of a solution of the integral equation (8), we define the mapping

$$\begin{aligned} Tu(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),u'(s))\,ds \\ & + \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\kappa\xi^{\alpha-1})}\Bigg(\kappa\int_0^\xi (\xi-s)^{\alpha-1} f(s,u(s),u'(s))\,ds \\ & - \int_0^1 (1-s)^{\alpha-1} f(s,u(s),u'(s))\,ds\Bigg), \end{aligned}$$

for $t \in [0,1]$. Note that

$$\begin{aligned} (Tu)'(t) = {} & \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} f(s,u(s),u'(s))\,ds \\ & + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)(1-\kappa\xi^{\alpha-1})}\Bigg(\kappa\int_0^\xi (\xi-s)^{\alpha-1} f(s,u(s),u'(s))\,ds \\ & - \int_0^1 (1-s)^{\alpha-1} f(s,u(s),u'(s))\,ds\Bigg), \end{aligned}$$

for $t \in (0,1]$, so that $Tu \in C[0,1] \cap C^1(0,1]$. Also

$$\begin{aligned} \lim_{t\to 0^+} t^{2-\alpha}(Tu)'(t) = {} & \frac{1}{\Gamma(\alpha-1)(1-\kappa\xi^{\alpha-1})}\Bigg(\kappa\int_0^\xi (\xi-s)^{\alpha-1} f(s,u(s),u'(s))\,ds \\ & - \int_0^1 (1-s)^{\alpha-1} f(s,u(s),u'(s))\,ds\Bigg) \end{aligned}$$

exists.

Consider the Banach space

$$X = \{u \in C[0,1] \cap C^1(0,1] : \lim_{t\to 0^+} t^{2-\alpha}u'(t) \text{ exists}\}$$

with the weighted norm $\|u\| = \max\{\|u\|_0, \|t^{2-\alpha}u'\|_0\}$, where $\|\cdot\|_0$ is the sup-norm and $\|t^{2-\alpha}v\|_0 = \sup_{t\in(0,1]} |t^{2-\alpha}v(t)|$.

The Caputo fractional derivative [2, 8, 14] with $n = [\alpha]+1$, is defined in terms of the Riemann-Liouville fractional derivative by

$$\begin{aligned} D^\alpha_{0+}u(t) &= \left(\mathcal{D}^\alpha_{0+}\left[u(s) - \sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{k!}s^k\right]\right)(t) \\ &= \mathcal{D}^\alpha_{0+}u(t) - \sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{\Gamma(k-\alpha+1)}t^{k-\alpha} \\ &= \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^t (t-s)^{n-\alpha-1}\,u^{(n)}(s)\,ds, \quad t\in[0,1], \end{aligned} \tag{9}$$

where the last identity holds if $u \in AC^n[0,1] = \{u : [0,1]\to \mathbf{R} : u^{(n-1)} \in AC[0,1]\}$.

In addition to the differential equation (5), we investigate the equation

$$D^\alpha_{0+}u(t) = f(t, u(t), u'(t)), \quad t \in (0,1), \tag{10}$$

where $1 < \alpha < 2$, subject to the boundary conditions

$$\kappa u(\xi) = u(1), \quad \kappa\xi \neq 1, \tag{11}$$

and

$$u(0) = 0. \tag{12}$$

The operator (9) has the following properties (see [8, 14]).

Theorem 1.1. *Let $u \in AC^n[0,1]$ (or $u \in C^n[0,1]$), $\alpha \in (n-1,n)$ and $v \in C[0,1]$. Then, for $t \in [0,1]$,*

(a) $I^\alpha_{0+}D^\alpha_{0+}u(t) = u(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}u^{(k)}(0)$;
(b) $D^\alpha_{0+}I^\alpha_{0+}v(t) = v(t)$.

For $g \in C[0,1]$ and $D^\alpha_{0+}u = g$, from Lemma 1.1 (a) it follows that

$$I^\alpha_{0+}g(t) = u(t) - u(0) - tu'(0).$$

Using (11),

$$u'(0) = \frac{1}{1-\kappa\xi}\left(\kappa I^\alpha_{0+}g(\xi) - I^\alpha_{0+}g(1)\right),$$

so that

$$u(t) = I^\alpha_{0+}g(t) + \frac{t}{1-\kappa\xi}\left(\kappa I^\alpha_{0+}g(\xi) - I^\alpha_{0+}g(1)\right), \quad t\in[0,1]. \tag{13}$$

Using Lemma 1.1 and the equation (13) we can show that the solvability of boundary value problem (10)–(12) in the class $AC^2[0,1]$ is equivalent the the existence of a fixed point of the operator $T : C^1[0,1] \to C^1[0,1]$ defined by

$$\begin{aligned} Su(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s),u'(s))\, ds \\ &\quad + \frac{t}{\Gamma(\alpha)(1-\kappa\xi)} \left(\kappa \int_0^\xi (\xi-s)^{\alpha-1} f(s,u(s),u'(s))\, ds \right. \\ &\quad \left. - \int_0^1 (1-s)^{\alpha-1} f(s,u(s),u'(s))\, ds \right), \end{aligned} \tag{14}$$

for $t \in [0,1]$. To this end, note that

$$\begin{aligned} (Su)'(t) &= \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s,u(s),u'(s))\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)(1-\kappa\xi)} \left(\kappa \int_0^\xi (\xi-s)^{\alpha-1} f(s,u(s),u'(s))\, ds \right. \\ &\quad \left. - \int_0^1 (1-s)^{\alpha-1} f(s,u(s),u'(s))\, ds \right), \end{aligned}$$

for $t \in [0,1]$, so that $Su \in C^1[0,1]$. Moreover, given $t_1, t_2 \in [0,1]$ with $t_1 < t_2$,

$$\begin{aligned} &|(Su)'(t_2) - (Su)'(t_1)| \\ &\le \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} \left((t_1-s)^{\alpha-2} - (t_2-s)^{\alpha-2} \right) |f(s,u(s),u'(s))|\, ds \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-2} |f(s,u(s),u'(s))|\, ds \\ &\le C \left(t_1^{\alpha-1} + 2(t_2-t_1)^{\alpha-1} - t_2^{\alpha-1} \right) < 2C(t_2-t_1)^{\alpha-1}, \end{aligned}$$

which shows that $(Su)' \in AC[0,1]$.

Define $Y = C^1[0,1]$ with the norm $\|u\| = \max\{\|u\|_0, \|u'\|_0\}$, a Banach space. The next lemma is easy to prove.

Lemma 1.2. *The mappings $T : X \to X$ and $S : Y \to Y$ are completely continuous.*

2. MAIN RESULTS

The first existence result is obtained for the boundary value problem (5)–(7).

By $(H1)$ for $A, B > 0$, $C(A,B) = \sup\{|f(t,x,y)| : 0 \le t \le 1, |x| \le A, t^{2-\alpha}|y| \le B\} < \infty$.

Theorem 2.1. *Let the assumptions* (H_1) *and* (H_2) *be satisfied. Assume that there exist constants* $A, B > 0$ *such that*

$$\frac{1}{\Gamma(\alpha+1)}\left(1+\frac{1+|\kappa|\xi^\alpha}{|1-\kappa\xi^{\alpha-1}|}\right)C(A,B) \le A$$

and

$$\frac{1}{\Gamma(\alpha+1)}\left(\alpha+\frac{(\alpha-1)(1+|\kappa|\xi^\alpha)}{|1-\kappa\xi^{\alpha-1}|}\right)C(A,B) \le B.$$

Then the boundary value problem (5)–(7) *has a nontrivial solution.*

Proof. Let $A, B > 0$ and define

$$\mathcal{D} = \{u \in X : |u(t)| \le A, t^{2-\alpha}|u'(t)| \le B \quad \text{for all } t \in [0,1]\}.$$

Then $\mathcal{D}$ is a closed and convex subset of X.

Let $u \in \mathcal{D}$, then

$$\begin{aligned} |Tu(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u(s),u'(s))|\,ds \\ &+ \frac{t^{\alpha-1}}{\Gamma(\alpha)|1-\kappa\xi^{\alpha-1}|}\left(|\kappa|\int_0^\xi (\xi-s)^{\alpha-1}|f(s,u(s),u'(s))|\,ds\right. \\ &\qquad \left.+\int_0^1 (1-s)^{\alpha-1}|f(s,u(s),u'(s))|\,ds\right) \\ &\le \frac{1}{\Gamma(\alpha+1)}\left(1+\frac{1+|\kappa|\xi^\alpha}{|1-\kappa\xi^{\alpha-1}|}\right)C(A,B) \\ &\le A, \end{aligned}$$

and

$$\begin{aligned} t^{2-\alpha}|(Tu)'(t)| &\le \frac{t^{2-\alpha}}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}|f(s,u(s),u'(s))|\,ds \\ &+ \frac{1}{\Gamma(\alpha-1)|1-\kappa\xi^{\alpha-1}|}\left(|\kappa|\int_0^\xi (\xi-s)^{\alpha-1}|f(s,u(s),u'(s))|\,ds\right. \\ &\qquad \left.+\int_0^1 (1-s)^{\alpha-1}|f(s,u(s),u'(s))|\,ds\right) \\ &\le \frac{1}{\Gamma(\alpha)}\left(1+\frac{(\alpha-1)(1+|\kappa|\xi^\alpha)}{\alpha|1-\kappa\xi^{\alpha-1}|}\right)C(A,B) \\ &\le B, \end{aligned}$$

for all $t \in [0,1]$.

It follows from the inequalities above that $Tu \in \mathcal{D}$. Hence $T : \mathcal{D} \to \mathcal{D}$. Since $\mathcal{D}$ is convex and T is completely continuous by Lemma 1.2, by the Schauder fixed point theorem T has a fixed point. By the arguments in Section 1, the fixed point is a solution of the boundary value problem (5)–(7). By (H_2) the solution is nontrivial. □

The next existence result concerns the boundary value problem (10), (6), and (11).

By (C_1), for $A, B > 0$, $C(A,B) = \sup\{|f(t,x,y)| : 0 \le t \le 1, |x| \le A, |y| \le B\} < \infty$.

Theorem 2.2. *Let the assumptions (H_1) and (H_2) be satisfied. Assume that there exist constants $A, B > 0$ such that*

$$\frac{1}{\Gamma(\alpha)}\left(1 + \frac{1+|\kappa|\xi^\alpha}{|1-\kappa\xi|}\right)C(A,B) < A \quad \textit{and} \quad \frac{1}{\Gamma(\alpha+1)}\left(\alpha + \frac{1+|\kappa|\xi^\alpha}{|1-\kappa\xi|}\right)C(A,B) \le B. \tag{15}$$

Then the boundary value problem (5)–(7) has a nontrivial solution.

Proof. Let $A, B > 0$ and define is a closed and convex subset of Y, $\mathcal{D}$, by

$$\mathcal{D} = \{u \in Y : |u(t)| \le A, |u'(t)| \le B \text{ for all } t \in [0,1]\}.$$

Let $u \in \mathcal{D}$, then

$$\begin{aligned}
|Su(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u(s),u'(s))|\,ds \\
&\quad + \frac{t}{\Gamma(\alpha)|1-\kappa\xi|}\Bigg(|\kappa|\int_0^\xi (\xi-s)^{\alpha-1}|f(s,u(s),u'(s))|\,ds \\
&\qquad + \int_0^1 (1-s)^{\alpha-1}|f(s,u(s),u'(s))|\,ds\Bigg) \\
&\le \frac{1}{\Gamma(\alpha+1)}\left(1 + \frac{1+|\kappa|\xi^\alpha}{|1-\kappa\xi|}\right)C(A,B) \\
&\le A
\end{aligned}$$

and

$$\begin{aligned}
|(Su)'(t)| &\le \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}|f(s,u(s),u'(s))|\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)|1-\kappa\xi|}\Bigg(|\kappa|\int_0^\xi (\xi-s)^{\alpha-1}|f(s,u(s),u'(s))|\,ds \\
&\qquad + \int_0^1 (1-s)^{\alpha-1}|f(s,u(s),u'(s))|\,ds\Bigg) \\
&\le \frac{1}{\Gamma(\alpha+1)}\left(\alpha + \frac{1+|\kappa|\xi^\alpha}{|1-\kappa\xi|}\right)C(A,B) \\
&\le B.
\end{aligned}$$

The rest of the proof is identical to that of Theorem 2.1. □

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